WEIERSTRASS HIRONAKA DIVISION THEOREM FOR ANALYTIC AND SMOOTH FUNCTIONS. STRONG HIRONAKA RESOLUTION FOR ANALYTIC SPACES

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Abstract. We give a proof of the strong Hironaka desingularization of analytic spaces by blow-ups of smooth normally flat centers. The method relies on a weak desingularization algorithm (simplified in works of Bierstone-Milman, Villamayor, Wlodarczyk and Kollar), an a proven here generalized Weierstrass-Hironaka division theorem. We also show some other applications of the theorem for analytic and smooth functions extending the results on Weierstarss Hironaka and Malgrange-Mather division.

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0. Introduction

One of the main purposes of the paper is to give a simplified proof the strong Hironaka resolution of analytic spaces. The strong resolution uses only smooth normally flat centers, and thus is more geometric. The blow-ups replace smooth subvarieties with the projectivization of normal cone bundle.

The method relies on existing algortithm for the (weak Hironaka desingularization) and a version of Weierstrass-Hironaka division with respect to norms. We show various applications of the theorem. The approach allows to consider seemingly different cases of polynomial, analytic, formal analytic, and certain classes of smooth functions from the same perspective. The introduced notions of dominating weights with respect to norm is coherent and can be used for studyding invariants like Hilbert-Samuel functions on anlytic spaces. The punctual constructions of Hironaka standard basis extend, in the new setting, along Samuel stratum. This reduces the proof of the strong version of Hironaka desingularization of analytic spaces to resolution of marked ideals as in the weak Hironaka resolution.

The constructions here are conceptually much simpler and more general than in algebraic situation. Unfortunately they are no longer valid in algebraic case. The latter case was adressed in a separate paper.

The problem of reduction of strong resolution to marked ideals (idealistic exponents) was approached by Hironaka introducing his distinguished data. The construction is carried via his Henselian division Theorem

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which relies on Weirstrass-Hironaka division for formal analytic functions. A different method was used in the Bierstone-Milman paper which require studying of formal analytic functions along Samuel stratum and uses relaxed version of the above mentionned Hironaka-Weierstrass division for . In the recent paper the canonical reduction to the weak resolution is obtained by the construction of the (algebraic) standard basis along Samuel stratum, and is a consequence of (proven also in smooth and analytic cases) singular implicit function theorem.(see [25]).

Recall that Hironaka in his proof of desingularization theorem and the other mathematicians like Briancon, in the subsequent papers studied a version of Weierstrass-division for multiple generators (respectively ideals). A so called **Weierstrass-Hironaka division theorem**, says that for a set of analytic functions $f_1, \ldots, f_k$ and the corresponding set of leading exponents there exist a division with remainder

$$g = \sum h_i f_i + r,$$

where h_i, r satisfy some combinatorial conditions. The theorem is a starting point for the rather complex, and quite special constructions of distuinguished data.

On the other hand the analog of the classical Weierstrass division/prepartion theorem for smooth real functions was conjectured by R. Thom and proved by Malgrange (Weierstrass preparation) and Mather (Weierstrass division). The smooth case is much more difficult and deeper Malgrange-Mather preparation/division theorem is of fundamental importance in the theory of singularities. Note that Malgrange-Mather division for smooth functions as long as its Weierstrass-Hironaka generalization is not unique. We formulate and prove the Weierstrass-Hironaka extension of this theorem in the case of smooth convergent functions consisting roughly of smooth functions with convergent Taylor expansions. Moreover Weierstrass-Hironaka division for smooth convergent fuctions is unique, and it holds in a version similar as in the analytic case. Note that a weaker version of Weierstrass-Hironaka division for all smooth functions is proven in [25].

The elementary and simple methods rely on the ideas of Hironaka, Aroca, Vincente, Briancon, Bierstone-Milman and many others. There are some overlaps of this paper with the previous manuscript which addresses more general and difficult cases of algebraic and smooth functions.

1. Weierstrass-Hironaka division over Banach rings

Let K denote a Banach ring, that is a ring which is equipped with a submultiplicative norm and which is complete with respect to this norm.

For any n-tuple of real positive (respectively nonnegative) numbers $\rho = (\rho_1, \ldots, \rho_n) \in \mathcal{R}^n_{\geq 0}$ define the *monomial* norm (respectively seminorm) on $K[[u]] = K[[u_1, \ldots, u_n]]$ over K in the variables $u_1, \ldots, u_n$

$$||f||_\rho := \sum_\alpha ||c_\alpha|| \rho^\alpha,$$

where $\rho^\alpha = \rho_1^{\alpha_1}, \ldots, \rho_n^{\alpha_n}$. Let $K\{u\}_\rho$ defines a K-subalgebra of all functions with finite ρ- norm. It is immediate that the ρ-norm (or seminorm) is submultiplicative and $K\{u\}_\rho$ is a Banach ring for any $\rho > 0$. Moreover it satisfies the following condition: $||u^\alpha||_\rho \cdot ||f||_\rho = ||u^\alpha \cdot f||_\rho$.

The Banach ring $K\{u\}_\rho$ is the completion of the ring of polynomials $K[u_1, \ldots, u_n]$ with respect to the norm ρ. It defines all functions which are convergent on the closed polydisc $D(0, \rho)$.

The ring of convergent power series $K\{u\} = K\{u_1, \ldots, u_n\}$ is defined as

$$K\{u\} := \bigcup_{\rho > 0} K\{u\}_\rho.$$

Example 1.0.1. This approach allows to consider analytic, formal analytic functions, and polynomial functions in a unified way: If $K = \mathbb{R}, \mathbb{C}, \mathbb{Q}_p$ with the standard norms then $K\{u\}$ is the local ring of analytic functions, while $K\{u\}_\rho$ is the ring of analytic functions on the closed polydiscs $D(0, \rho)$ in K^n .

If K is any field with the trivial norm (that is $|x| = 1$ for $x \in K \setminus \{0\}$, $|0| = 0$) then for any positive ρ with $0 < \rho_i < 1$,

$$K\{u\}_\rho = K\{u\} = K[[u_1, \ldots, u_n]]$$

is a ring of formal analytic functions. On the other hand for $\rho_i \geq 1$ we have that

$$K\{u\}_\rho = \bigcap_{\rho > 0} K\{u\}_\rho = K[u_1, \ldots, u_n].$$

Thus theorem is also applicable in this situation

Remark. Observe that the construction can be carried differently in nonarchimedian situation. Let K be a non archimedian Banach ring. Then with any ρ one can associate the norarchimedian norm

$$||f||_\rho^{nar} := \sup_\alpha ||c_\alpha||\rho^\alpha.$$

Since K is nonarchimedian one deduces immediately that it is a submultiplicative nonarchimedian norm

Then $K\{u\}_\rho^{nar}$ is the completion of the ring of polynomials $K[u_1,\ldots,u_n]$ with respect to the norm ρ^{nar} and it is a nonarcimedian Banach ring. We are not going to pursue this approach, except of Theorem 1.0.3.

The monomials u^α are naturally identified with elements of $\mathbb{N}^n$, where $\mathbb{N}$ denotes the set of natural numbers and zero. For any nonzero function $f \in K\{u\}$, $f = \sum c_\alpha u^\alpha$ define the the *support* of f to be

$$\mathrm{supp}(f) := \{\alpha \in \mathbb{N}^n \mid c_\alpha \neq 0\}.$$

By the *differential support* of f we mean

$$\mathrm{supd}(f) := \{\alpha \mid D_{u^\alpha}(f) \neq 0\},$$

where $D_{u^\alpha} = \frac{1}{\alpha!}\frac{\partial}{\partial u^\alpha}$.

The notion of the differential support better reflects properties of the analytic functions and can be also extended to smooth functions. It follows immediately from the definition that $\mathrm{supd}(f) \supseteq \mathrm{supp}(f)$.

For any n-tuple of nonnegative integers $\alpha = (a_1,\ldots,a_n)$ set $|\alpha| := a_1 + \ldots + a_n$. Then the multiplicity of $f = \sum c_\alpha u^\alpha$ is defined as

$$\mathrm{ord}(f) = \min\{|\alpha|, c_\alpha \neq 0\}.$$

For any set of exponents $\alpha^1, \alpha^2, \ldots, \alpha^k \in \mathbb{N}^n$ set

$$\Delta := \bigcup_i \alpha^i + \mathbb{N}^n, \quad \Gamma_0 = \Gamma := \mathbb{N}^n \setminus \Delta$$

Consider any partition $\{\Delta_i\}$ of Δ such that $\Delta_i \subseteq \alpha_i + \mathbb{N}^n$.

For $i = 1,\ldots,k$, let Γ_i be the unique set such that $\Delta_i = a^i + \Gamma_i$. Then $\{\Delta_i\}$ (or Δ) will be called a *diagram of initial exponents* for the exponents $\alpha^1, \alpha^2, \ldots, \alpha^k$.

Definition 1.0.2. We say that the weight $\alpha_0 \in \mathbb{N}^r$ is *dominating* for $f = \sum c_\alpha u^\alpha \in K\{u\}$ with respect to a seminorm ρ and write

$$\exp_\rho(f) = \alpha_0$$

if

(1) $f \in K\{u\}_\rho$.
(2) $||f - c_{\alpha_0} u^{\alpha_0}||_\rho < ||c_{\alpha_0} u^{\alpha_0}||_\rho$.
(3) The coefficient c_{α_0} is invertible.

One of the main theorem in the paper is the following extension of Briancon-Aroca-Hironaka-Vincente Weierstrass division theorem.

Theorem 1.0.3. *Let $f_1 = \sum c_{1\alpha} x^\alpha, \ldots, f_r = \sum c_{r\alpha} u^\alpha \in K\{u\}_\rho$ be analytic functions such that $\exp_\rho(f_i) = \alpha_i$ for a certain norm $\rho := (\rho_1,\ldots,\rho_n)$ and let Δ be a diagram of initial exponents for $\alpha_1,\ldots,\alpha_r$.*

Then for every $g \in K\{u\}_\rho$, there exist unique $h_i \in K\{u\}_\rho$ and $r(g) \in K\{u\}_\rho$ such that $\mathrm{supp}(h_i) \subset \Gamma_i$, $\mathrm{supp}(r) \subset \Gamma_0$, *and*

$$g = \sum h_i f_i + r(g)$$

Moreover,if $\mathrm{ord}(\alpha_i) = \mathrm{ord}(f_i)$ *then* $\mathrm{ord}(r) \geq \mathrm{ord}_(g)$ *,and* $\mathrm{ord}(h_i) \geq \mathrm{ord}(g) - \mathrm{ord}(f_i)$.

Proof. We can assume that the coefficient $c_{i\alpha_i}$ of f_i is equal 1, by replacing f_i with $c_{i\alpha_i}^{-1} f_i$, if necessary.

Observe that any function $g \in K\{u\}_\rho$, can be written uniquely as $g = \sum h_i u^{\alpha_i} + r(f)$, where $\mathrm{supp}(h_i) \subset \Gamma_i, \mathrm{supp}(r(f)) \subset \Gamma_i$. Moreover

$$||g||_\rho = \sum ||h_i||_\rho \cdot \rho^{\alpha_i} + ||r(f)||_\rho$$

Define the linear transformation $T : K\{u\}_\rho \to K\{u\}_\rho$, as

$$g = \sum h_i u^{\alpha_i} + r(g) \mapsto T(g) = \sum h_i c_{i\alpha_i}^{-1} f_i + r(g)$$

We show that T is invertible. The transformation can be written as $T = I + U$, where

$$U(g) = \sum h_i(f_i - u_i^\alpha).$$

Denote by

$$s := \sup_i \frac{||f_i - c_{i\alpha_i}u^{\alpha_i}||_\rho}{||c_{i\alpha_i}u^{\alpha_i}||_\rho} < 1.$$

Then

$$||U(g)||_\rho \leq \sum ||h_i||_\rho \cdot ||{c_{i\alpha_i}}^{-1} \cdot f_i - u^{\alpha_i}||_\rho \leq s \cdot \sum ||h_i||_\rho \cdot ||u^{\alpha_i}||_\rho \leq s \cdot ||g||_\rho.$$

This implies that $||U||_\rho \leq s < 1$. Thus T is invertible with inverse given by

$$\mathcal{R}^{-1} = I - U + U^2 - \ldots$$

(The latter formula makes sense, since $||U^i|| \leq s^i$). Now if we write $\mathcal{R}^{-1}(g) = \sum \overline{h_i} u^{\alpha_i}$ then we exactly get the unique presentation $g = \sum \overline{h_i} f_i$ satisfying the desired properties. For the last part observe that if $g = \sum h_i u^{\alpha_i} + r(f)$ then $\mathrm{ord}(g) \leq \mathrm{ord}(h_i u^{\alpha_i}) \quad \mathrm{ord}(g) \leq \mathrm{ord}(r(f))$

This implies that $\mathrm{ord}(T(g) \geq \mathrm{ord}(g)$ and consequently $\mathrm{ord}(U(g) \geq \mathrm{ord}(g)$, and finally $\mathrm{ord}(\mathcal{R}^{-1}(g) \geq (g)$.

□

Remark. The theorem is still valid in the nonarchimedian situation when replacing $K\{u\}_\rho$ with $K\{u\}_\rho^{nar}$ with the same proof and nearly identical estimations for $||U(g)||_\rho \leq s \cdot ||g||_\rho$. .

Note that the unique presentation (2) is independent of the norm ρ which allows to consider different convenient norms ρ' and they will give the same decomposition as long as $\exp_{\rho'}(f_i) = \alpha_i$.

Remark. Unlike the other versions of the Weierstrass-Hironaka division with respect to the (linear or monomial) orders the approach with norms have some critical advantages. First of all, as it follows from Example 1.0.1, this approach is far more flexible, and it can be applied to seemingly different situations of analytic, formal analytic or polynomial settings, and finally some classes of (real) smooth functions. Secondly, unlike the linear order it allows to control functions (or singularities) in a neighborhood. Thus it is very suitable for studying Samuel stratum in the strong Hironaka desingularization.

Lemma 1.0.4. *Given a finite set of functions $f_i \in K\{u\}$ such that $\mathrm{ord}(f_i) = |\alpha_i|$, and $\exp_\rho(f_i)$ for a certain seminorm $\rho := (\rho_1, \ldots, \rho_k, 0, \ldots, 0)$, with $\rho_i > 0$ one can find find a norm $\rho_\epsilon := (\epsilon\rho_1, \ldots, \epsilon\rho_k, \epsilon^2, \ldots, \epsilon^2)$ such that $f_1, \ldots, f_r \in K\{u\}_{\rho_\epsilon}$, and $\exp_{\rho_\epsilon}(f_i) = \alpha_i$ for sufficiently small $\epsilon > 0$*

Proof. First observe that $f_1, \ldots, f_r \in K\{u\}_{\rho_\epsilon}$ for sufficiently small $\epsilon > 0$. Since $\mathrm{ord}(f_i - c_{i\alpha_i}u^{\alpha_i}) \leq \mathrm{ord}(u^{\alpha_i})$, we get

$$\lim_{\epsilon \to 0} \frac{||f_i - c_{i\alpha_i}u^{\alpha_i}||_{\rho_\epsilon}}{||c_{i\alpha_i}u^{\alpha_i}||_{\rho_\epsilon}} \leq \lim_{\epsilon \to 0} \frac{||f_i - \mathrm{in}(f_i)||_{\rho_\epsilon}}{||c_{i\alpha_i}u^{\alpha_i}||_{\rho_\epsilon}} + \lim_{\epsilon \to 0} \frac{||\mathrm{in}(f_i) - c_{i\alpha_i}u^{\alpha_i}||_{\rho_\epsilon}}{||c_{i\alpha_i}u^{\alpha_i}||_{\rho_\epsilon}} = 0 + \lim_{\epsilon \to 0} \frac{||\mathrm{in}(f_i) - c_{i\alpha_i}u^{\alpha_i}||_{\rho_\epsilon}}{||c_{i\alpha_i}u^{\alpha_i}||_{\rho_\epsilon}} =$$

$$= \frac{||\mathrm{in}(f_i) - c_{i\alpha_i}u^{\alpha_i}||_\rho}{||c_{i\alpha_i}u^{\alpha_i}||_\rho} \leq \frac{||f_i - c_{i\alpha_i}u^{\alpha_i}||_\rho}{||c_{i\alpha_i}u^{\alpha_i}||_\rho} < 1,$$

and thus $\exp_{\rho_\epsilon}(f_i) = \alpha_i$ for small ϵ.

□

One can extend the theorem to seminorms and the ring of all convergent functions.

Corollary 1.0.5. *Let $f_1 = \sum c_{1\alpha}x^\alpha, \ldots, f_r = \sum c_{r\alpha}u^\alpha \in K\{u\}$ be analytic functions such that $\exp_\rho(f_i) = \alpha_i$ for a certain seminorm $\rho := (\rho_1, \ldots, \rho_k, 0, \ldots, 0)$ and assume that $\mathrm{ord}(\alpha_i) = \mathrm{ord}(f_i)$. Let Δ be the diagram of initial exponents for $\alpha_1, \ldots, \alpha_k$.*

Then for every $g \in K\{u\}$, there exist unique $h_i \in K\{u\}$ and $r(g) \in K\{u\}$ such that $\mathrm{supp}(h_i) \subset \Gamma_i$, $\mathrm{supp}(r) \subset \Gamma_0$, and

$$g = \sum h_i f_i + r(g)$$

$\mathrm{ord}(r) \geq \mathrm{ord}(g)$ *,and* $\mathrm{ord}(h_i) \geq \mathrm{ord}_(g) - \mathrm{ord}(f_i)$.

Proof. By the previous Lemma for any $g, f_1, \ldots, f_r$ we can find a norm ρ_ϵ such that $g, f_1, \ldots, f_r \in K\{u\}_{\rho'}$, and $\exp_{\rho_\epsilon}(f_i) = \alpha_i$ for small ϵ. It suffices to apply the Theorem 1.0.3.

□

2. Dominating weights in a neighborhood.

One of the important feature of the notion of the dominating weight with respect to the norm is that the notion extends to a neighborhood.

Any series $f=\sum_{\alpha\in\mathrm{supd}(f)} c_\alpha u^\alpha \in K\{u_1,\ldots,u_n\}_\rho$ with finite norm ρ defines an analytic function which is absolutely convergent on the polydisc $D(0,\rho)\subset K$. We can consider the supremum norm on $K\{u_1,\ldots,u_n\}_\rho$, that is

$$||f||_{\sup,\rho} := \sup\{||f(v)|| \mid v\in D(0,\rho)\}.$$

Then it follows that

$$||f||_{\sup,\rho}\le ||f||_\rho$$

Lemma 2.0.1. *Let ρ be a norm and assume that*

$$f := \sum_{\alpha\in\mathrm{supd}(f)} c_\alpha u^\alpha \in K\{u_1,\ldots,u_n\}_\rho$$

with $\exp_\rho(f)=\alpha$. Then we can write the induced function $f(u+v)$ as

$$f(u+v)=\sum \frac{1}{\alpha!}\frac{\partial^\alpha f(v)}{\partial v^\alpha}u^\alpha = \sum C_\alpha u^\alpha,$$

where $C_\alpha := \frac{1}{\alpha!}\frac{\partial^\alpha f(v)}{\partial v^\alpha}\in K\{v_1,\ldots,v_n\}_\rho$. Then there exists a sufficiently small $0<\epsilon<\rho$ such that

1. *$f(u+v)=\sum c_{\alpha,\beta}u^\alpha v^\beta \in K\{v_1,\ldots,v_n,u_1,\ldots,u_n\}_{\epsilon,\rho-\epsilon}$, and $\exp_{\epsilon,\rho-\epsilon}(f(u+v))=\alpha$.*
2. *$f(u+v)=\sum C_\alpha u^\alpha \in K\{v_1,\ldots,v_n\}_{\sup,\epsilon}\{u_1,\ldots,u_n\}_{\rho-\epsilon}$, and $\exp_{\rho-\epsilon}(f(u+v))=\alpha$.*
3. *For any fixed $v\in D(0,\epsilon)$ the function $f_v(u)=f(u+v)\in K\{u_1,\ldots,u_n\}_{\rho-\epsilon}$, and $\exp_{\rho-\epsilon}(f_v)=\alpha$.*
4. *Let $\overline{u}=: u-v$. For any fixed $v\in D(0,\epsilon)$ the function $f(u)=f_v(\overline{u})\in K\{\overline{u}_1,\ldots,\overline{u}_n\}_{\rho-\epsilon}$, and $\exp_{\rho-\epsilon}(f_v)=\alpha$.*

Proof. We can associate with the K-analytic function

$$f := \sum_{\alpha\in\mathrm{supd}(f)} c_\alpha u^\alpha \in K\{u_1,\ldots,u_n\}_\rho$$

the real analytic function

$$\overline{f} = \sum_{\alpha\in\mathrm{supd}(f)} ||c_\alpha|| x^\alpha \in \mathbb{R}\{u_1,\ldots,u_n\}$$

which is convergent in the polydisc $D_R(0,\rho):=\{x\in\mathbb{R}^n \mid |x_i|\le\rho_i\}$.

Then

$$||f(u+v)||_{\rho-\epsilon,\epsilon} = \overline{f}(\rho-\epsilon+\epsilon)=\overline{f}(\rho)=||f(u)||_\rho.$$

The function

$$f(u+v)=\sum\frac{1}{\alpha!}\frac{\partial^\alpha f(v)}{\partial v^\alpha}u^\alpha$$

which is convergent for $(u,v)\in D(0,\epsilon)\times D(0,\rho-\epsilon)$, and thus $f(u+v)\in K\{v_1,\ldots,v_n,u_1,\ldots,u_n\}_{\epsilon,\rho-\epsilon}$

One estimates the coefficients for $v\in D(0,\epsilon)$ as

$$\frac{\partial^\alpha f}{\partial v^\alpha}(v)\le \frac{\partial^\alpha \overline{f}}{\partial X^\alpha}(||v||)\le \frac{\partial^\alpha \overline{f}}{\partial X^\alpha}(\epsilon)$$

This shows that $f(u+v)\in K\{v_1,\ldots,v_n\}_{\sup,\epsilon}\{u_1,\ldots,u_n\}_{\rho-\epsilon}$, and at any point $v\in D(0,\epsilon)$ with $\epsilon<\rho$ the function $f_v(u)\in K\{u_1,\ldots,u_n\}_{\rho-\epsilon}$, . Moreover one can assume that $\exp_{\rho-\epsilon}(f)=\alpha_0$ for sufficiently small ϵ. Then

$$||f(u+v)-C_{\alpha_0}u^{\alpha_0}||_{\rho-\epsilon}\le ||f(u)-c_{\alpha_0}\cdot u^{\alpha_0}||_{\rho-\epsilon}+||(C_{\alpha_0}-c_{\alpha_0})(u)^{\alpha_0}||_{\rho-\epsilon}+||f(u+v)-f(u)||_{\rho-\epsilon}$$

converges to $||f(u)-c_{\alpha_0}\cdot u^{\alpha_0}||_{\rho-\epsilon}$ as $\epsilon\to 0$. Similarly $||C_{\alpha_0}u^{\alpha_0}||_{\rho-\epsilon}$ converges to $||c_{\alpha_0}u^{\alpha_0}||_{\rho-\epsilon}$.

Thus for sufficiently small $\epsilon>0$,

$$||f(u+v)-C_{\alpha_0}(u)^{\alpha_0}||_{\rho-\epsilon,\epsilon} < ||C_{\alpha_0}u^{\alpha_0}||_{\rho-\epsilon},$$

and for $v \in D(0, \epsilon)$ we have
$$||f_v - c_{\alpha_0} u^{\alpha_0}||_{\rho-\epsilon} < ||c_{\alpha_0}|| \cdot ||u^{\alpha_0}||_{\rho-\epsilon}.$$
□

Corollary 2.0.2. *Let $\rho = (\rho_1, \ldots, \rho_n)$, where $\rho_i > 0$ be a norm on $K\{u_1, \ldots, u_n\}_\rho$. For $i = 1, \ldots, r$ consider a finite set of functions*
$$f_i := \sum c_{i\alpha} u^\alpha \in K\{u_1, \ldots, u_n\}_\rho$$
with $\exp_\rho(f_i) = \alpha_{i0}$, and $\alpha_{i0} = (\alpha_{i01}, \ldots, \alpha_{i0k}, 0, \ldots, 0)$. Then there exists an open neighborhood $U \subset K^n$ of O and a seminorm $\overline{\rho} = (\overline{\rho}_1, \ldots, \overline{\rho}_k, 0, \ldots, 0)$ such that such that for any $v \in U$ we have $\exp_{\overline{\rho}}(f) = \alpha$, with respect to the coordinate system $u_1 - u_1(v), \ldots, u_n - u_n(v)$.

Proof. By Lemma 2.0.1(4), for a sufficiently small $\epsilon > 0$, and $v \in D(0, \epsilon)$ we have $\exp_{\rho-\epsilon}(f_i) = \alpha$. Put $\overline{\rho} := (\rho - \epsilon, \ldots, \rho - \epsilon, 0, \ldots, 0)$. Then for any $v \in D(0, \epsilon)$
$$||f_{iv} - c_{\alpha_{i0}} u^{\alpha_{i0}}||_{\overline{\rho}} \leq ||f_v - c_{i\alpha_0} u^{\alpha_{i0}}||_{\rho\delta-\epsilon} < ||c_{\alpha_{i0}}|| \cdot ||u^{\alpha_{i0}}||_{\rho\delta-\epsilon} = ||c_{\alpha_{i0}}|| \cdot ||u^{\alpha_{i0}}||_{\overline{\rho}}.$$
The latter equality follows from the condition $\alpha_{i0} = (\alpha_{i01}, \ldots, \alpha_{i0k}, 0, \ldots, 0)$. □

Remark. The properties described by the lemma, and corollary are no longer valid for linear orders described in the following section.

3. Division with respect to linear forms

Definition 3.0.1. We call a linear form
$$L = a_1 x_1 + \ldots a_n x_n : \mathbb{R}^n \to \mathbb{R}$$
a positive (respectively *nonnegative*) if all $a_i > 0$ (resp. $a_i \geq 0$). More generally any k-tuple $\overline{T} = (T_1, \ldots, T_k)$ of nonnegative linear forms is *a positive* if $T_1 + \ldots + T_k$ is positive.

The form L and likewise k-tuple $\overline{T}$ define the order on $\mathbb{N}^n$:
$$\alpha \leq_{\overline{T}} \beta \quad \text{if} \quad \overline{T}(\alpha) \leq_{\text{lex}} \overline{T}(\beta).$$
We shall call it *linear order* defined by $\overline{T}$. (Here lex stands for the lexicographic order)

This allows to consider the initial or dominating exponents of the convergent power series.

Definition 3.0.2. Let $f = \sum c_\alpha u^\alpha \in K\{u\}$. Let $\overline{T} = (T_1, \ldots, T_k)$ be a k-tuple of nonnegative linear forms. A weight $\alpha_0 \in \text{supd}(f)$ is *dominating* for $f = \sum c_\alpha u^\alpha \in K\{u\}$ with respect to $\overline{T}$, shortly
$$\alpha_0 = \exp_{\overline{T}}(f)$$
if $c_{\alpha_0} \neq 0$, and $\overline{T}(\alpha_0) <_{\text{lex}} \overline{T}(\alpha)$ for each $\alpha \in \text{supd}(f) \setminus \{\alpha_0\}$.

Any positive or nonnegative form $L = a_1 x_1 + \ldots + a_n x_n$ defines an n-tuple $\rho(L, d) = (d^{a_1}, \ldots, d^{a_n})$ where $0 < d < 1$. The relevant norm is then given by
$$||f||_{L,d} := \sum ||c_\alpha|| d^{L(\alpha)}$$
Observe that if d is sufficiently small and L is positive then $||f||_{L,d}$ is finite for any $f = \sum c_\alpha u^\alpha \in K\{u_1, \ldots, u_n\}$.

Then we get that dominating weights with respect to a linear order can be represented by certain norms.

Lemma 3.0.3. *If $\exp_L(f) = \alpha_0$ then $\alpha_0 = \exp_{\rho_{L,d}}(f)$ for sufficiently small d.*

Proof. Suppose $\alpha_0 = \exp_L(f)$. We can assume that d is sufficiently small so that is convergent in $\rho(L, d)$. Then the quotient
$$||f - c_{\alpha_0} u^{\alpha_0}||_{\rho(L,d)} / ||c_{\alpha_0} u^{\alpha_0}||_{\rho(L,d)} = \sum_{L(\alpha) > L(\alpha_0)} ||c_\alpha|| d^{L(\alpha)} /// ||c_{\alpha_0} d^{L(\alpha_0)} = \sum_{L(\alpha) > L(\alpha_0)} ||c_\alpha|| d^{L(\alpha) - L(\alpha_0)} / ||c_{\alpha_0}$$
The latter converges to 0 as $d \to 0$.

Then for sufficiently small d, we have $\alpha_0 = \exp_{\rho_{L,d}}(f)$. Applying this to finitely many function f_i we get $\alpha_i = \exp_{\rho_{L,d}}(f)$.

□

Lemma 3.0.4. *Let $\Delta \subset \mathbb{N}^n$ be any subset which is $\mathbb{N}^n$-invariant, that is:*

If $\alpha \in \Delta$, and $\beta \in \mathbb{N}^n$ then $\alpha + \beta \in \Delta$.

Then it is finitely generated, that is, there exists a finite subset $A \subset \Delta$ such that

$$\Delta = \bigcup_{\alpha \in A} \{\alpha\} + \mathbb{N}^n$$

Proof. Δ defines a monomial $(\mathbb{C}^*)^n$-stable ideal in a Noetherian ring $|CC[x_1, \ldots, x_n$. Thus it has a finite $\mathbb{C}^*$-semistable basis consisting of monomials corresponding to the exponents in a certain finite set A. □

Lemma 3.0.5. *Consider any positive r-tuple of nonnegative functionals $\overline{T} = (T_1, \ldots, T_r)$ defined on $\mathbb{N}^n$, and finite set of elements $\{\alpha_i = (\alpha_{i1}, \ldots, \alpha_{in}) \mid i = 0, \ldots, k\}$ in $\mathbb{N}^n$. There exists a positive linear form L_0 such that $L(\beta) > L(\alpha_i)$ for any $\beta \in \mathbb{N}^n$ with $\beta >_{\overline{T}} \alpha_i$.*

Proof. Consider an $\mathbb{N}^n$-invariant subsets

$$\Delta(\alpha_i) := \{\beta \in \mathbb{N}^n \mid \overline{T}(\beta)) >_{lex} \overline{T}(\alpha_i)\}.$$

For each i denote by A_i the set of vertices (generators) of $\Delta(\alpha_i)$.

Set

$$m_i := \min\{|T_i(\alpha)| \mid \quad \alpha \in A_i - \alpha_i, \quad i = 1, \ldots, k\}$$

$$M_i := \max\{|T_i(\alpha)| \mid \quad \alpha \in A_i - \alpha_i, \quad i = 1, \ldots, k\}$$

Put $m := \min\{m_i\}$, $M := \max\{m_i\}$, and $p := \frac{2k \cdot m}{M}$.

Then consider the linear form

$$L_0(\alpha) := p^{k-1} T_1(\alpha) + p^{k-2} T_2(\alpha) + \ldots + T_k(\alpha).$$

For any $\beta \in A_i$ write

$$\overline{T}(\beta - \alpha_i) = (0, \ldots, 0, c_j, c_{j+1}, \ldots, c_k),$$

where $c_j = T_j(\beta - \alpha_i) > 0$ is the first nonzero entry from the left.

By definition $c_j \geq m$ is positive, while all the other coefficients $c_\ell = T_\ell(\beta - \alpha_i)$ satisfy

$$-M \leq c_\ell \leq M$$

$$L_0(\beta) - L_0(\alpha_i) > p^{k-j} c_j + p^{k-j-1} c_{j+1} + \ldots c_k > p^{k-j} m - k \cdot p^{k-j-1} \cdot M =$$

$$p^{k-i} m (1 - \frac{k \cdot m}{p \cdot M}) = \frac{p^{k-i} m}{2} > 0$$

Now for

$$b := \max\{|\alpha| \mid \quad \alpha \in \bigcup A_i - \alpha_i\}, \quad \text{and} \quad L := L_0 + p/4b \cdot (x_1 + \ldots + x_n)$$

we get $L(\beta - \alpha_i) > p/2 - p/4 = p/4 > 0$ for any $\beta \in A_i$. This implies that $L(\beta) > L(\alpha)$ for any $\beta \in \mathbb{N}^n$, $\beta >_{\overline{T}} \alpha_0$, and L is positive.

□

Lemma 3.0.6. *Let $\overline{T} = (T_1, \ldots, T_r)$ be defined on $\mathbb{N}^n$. Then for any $\alpha \in \mathbb{N}^n$ there exists a positive linear form L_0 such that $L_0(\beta) < L_0(\alpha_i)$ for any $\beta \in \mathbb{N}^n$ with $\beta <_{\overline{T}} \alpha_0$.*

Proof. Let $V \subset \mathbb{N}^n$ will be the maximal coordinate subspace such that T is positive on V. We can assume that $V = \mathbb{N}^k \times \{0\}$. Then $\mathbb{N}^n$ is naturally isomorphic to $V \times \mathbb{N}^{n-k}$, where T vanishes on $\{0\} \times \mathbb{N}^{n-k}$

Then the set

$$\Delta(\alpha_i)^- := \{\beta \in \mathbb{N}^n \mid \overline{T}(\beta)) <_{lex} \overline{T}(\alpha_0)\}.$$

can be written as $A_i \times \mathbb{N}^{n-k}$, where $A_i := \{\beta \in \mathbb{N}^k \mid \overline{T}(\beta)) <_{lex} \overline{T}(\alpha_0)\}$ is finite since the restriction of $\overline{T}$ to $\mathbb{N}^k$ is positive on $\mathbb{N}^k$. The remaining constructions and estimations are the same as before. □

The following theorem slightly generalizes WHDT of Aroca, Hironaka, Vincente and others:

Theorem 3.0.7. *Consider an r-tuple of nonnegative functionals $\overline{T} = (T_1, \dots, T_r)$ defined on $\mathbb{N}^n$.*

Let $f_1, \dots, f_k \in K\{u, v\} = K\{u_1, \dots, u_n, v_1, \dots, v_m\}$ be convergent power series, where K is any Banach ring. Assume

$$\alpha_1 := \exp_{\overline{T}}(f_1(u, 0)), \dots, \alpha_r := \exp_{\overline{T}}(f_r(u, 0)) \in \mathbb{N}^n.$$

Consider a diagram Δ defined for $\alpha_1, \dots, \alpha_k \in \mathbb{N}^n$.

Then for every $g \in K\{u, v\}$, there exist unique $h_i \in K\{u, v\}$ and $r(g) \in K\{u, v\}$ such that $\operatorname{supp}(h_i) \subset \Gamma_i$, $\operatorname{supp}(r(g)) \subset \Gamma_0$, and

$$g = \sum h_i f_i + r(g)$$

Moreover, if $\operatorname{ord}(f_i(u, 0)) = |\alpha_i|$ for any i then

$$\operatorname{ord}(r(g)(u, 0)) \geq \operatorname{ord}(g(u, 0)), \quad \operatorname{ord}(h_i(u, 0)) \geq \operatorname{ord}(g(u, 0)) - |\alpha_i|$$

Proof. By Lemmas 3.0.5 there is a positive linear form L such that $\exp_\rho(f_i) = \alpha_i$. Then by Lemma 3.0.3 there exists a norm $\rho = \rho_{L,d}$ such that $g, f_1, \dots, f_k \in K\{u, v\}_\rho$, and $\exp_\rho(f_i) = \alpha_i$. It suffices to apply Theorem 1.0.3. □

In view of the previous remarks, and Example 1.0.1 the theorem can be applied in the analytic case or in the formal analytic case, where K is condidered with the trivial norm and thus $K\{u, v\} = k[[u, v]]$. The theorem is still valid in the polynomial setting for the ring $K[u, v] = K\{u, v\}_\rho$, where K any ring with trivial norm and $\rho \geq 1$.

Definition 3.0.8. For any $f = \sum c_\alpha u^\alpha \in K[u]$. and $\overline{T} = (T_1, \dots, T_k)$ we write

$$\alpha_0 := \operatorname{Exp}_{\overline{T}}(f)$$

if $c_{\alpha_0} \neq 0$, and $\overline{T}(\alpha_0) >_{\text{lex}} \overline{T}(\alpha)$ for each $\alpha \in \operatorname{supd}(f) \setminus \{\alpha_0\}$.

Lemma 3.0.9. *If $\operatorname{Exp}_L(f) = \alpha_0$ for $f \in K[u]$ then $\alpha_0 = \exp_{\rho_{L,d}}(f)$ for sufficiently large $d >> 0$.*

Proof. Suppose $\alpha_0 = \operatorname{Exp}_L(f)$. The estimation is identical as before

$$||f - c_{\alpha_0} u^{\alpha_0}||_{\rho(L,d)} / ||c_{\alpha_0} u^{\alpha_0}||_{\rho(L,d)} = \sum_{L(\alpha) < L(\alpha_0)} ||c_\alpha|| d^{L(\alpha) - L(\alpha_0)} / ||c_{\alpha_0}$$

The latter converges to 0 as $d \to \infty$.

Then for sufficiently small d, we have $\alpha_0 = \exp_{\rho_{L,d}}(f)$. Applying this to finitely many function f_i we get $\alpha_i = \exp_{\rho_{L,d}}(f)$.

□

Theorem 3.0.10. *Consider an r-tuple of nonnegative functionals $\overline{T} = (T_1, \dots, T_r)$ defined on $\mathbb{N}^n$.*

Let $f_1, \dots, f_k \in K[u, v]$ where K is any ring. Assume

$$\alpha_1 := \operatorname{Exp}_{\overline{T}}(f_1(u, 0)), \dots, \alpha_r := \operatorname{Exp}_{\overline{T}}(f_r(u, 0)) \in \mathbb{N}^n.$$

Consider the diagram Δ defined for $\alpha_1, \dots, \alpha_k) \in \mathbb{N}^n$.

Then for every $g \in K[u, v]$, there exist unique $h_i \in K[u, v]$ and $r(g) \in K[u, v]$ such that $\operatorname{supp}(h_i) \subset \Gamma_i$, $\operatorname{supp}(r(g)) \subset \Gamma_0$, and

$$g = \sum h_i f_i + r(g)$$

Moreover, if $\operatorname{ord}(f_i(u, 0)) = |\alpha_i|$ for any i then

$$\operatorname{ord}(r(g)(u, 0)) \geq \operatorname{ord}(g(u, 0)), \quad \operatorname{ord}(h_i(u, 0)) \geq \operatorname{ord}(g(u, 0)) - |\alpha_i|$$

Proof. As before by Lemma 3.0.6 we can find a positive linear form L, such that $\alpha_i := \operatorname{Exp}_L(f_i)$. Then by Lemma 3.0.9 there is a norm $\rho = \rho_{L,d}$ on $K[u, v]$ such that $\alpha_i = \exp_\rho(f_i)$ for sufficiently large d . It suffices to apply Theorem 1.0.3. □

Example 3.0.11. If $f(u, v)$ is k-regular, that is $f(u, 0) = c \cdot u^k$ then we get Weierstrass division theorem (for a total functional $\overline{T} = x_1$ on $\mathbb{N}$)

4. Hironaka standard basis for local analytic rings

Let K denote a complete real valued field. Let $\overline{T} = (T_1, \dots, T_r)$ be a total r-tuple of noonegative functionals on $\mathbb{N}^n$ and let $\mathcal{I}$ be an ideal in $K\{u\}$ then *the diagram (or ideal) of initial exponents* is defined as

$$\exp_{\overline{T}}(\mathcal{I}) = \{\exp_{\overline{T}}(f) \mid f \in \mathcal{I}\}.$$

Since $\exp_{\overline{T}}(\mathcal{I}) + \mathbb{N}^n \subseteq \exp(\mathcal{I})$ and $\exp_{\overline{T}}(\mathcal{I})$ is finitely generated there exists a unique set of its *vertices* $\{\alpha_1, \dots, \alpha_s\}$ satisfying the conditions

(1) $\exp_{\overline{T}}(\mathcal{I}) := \bigcup \alpha_i + \mathbb{N}^n$.
(2) $\alpha_i \notin \bigcup_{j \neq i} \alpha_j + \mathbb{N}^n$

Definition 4.0.1. The r-tuple $\overline{T}$ of noonegative functionals on $\mathbb{N}^n$ is *normalized* if $T_1 = x_1 + \ldots + x_k$.

Lemma 4.0.2. *If T is normalized then* $\operatorname{ord}(\exp_{\overline{T}}(f)) = \operatorname{ord}(f)$.

Proof. $\operatorname{ord}(\exp_{\overline{T}}(f)) = T_1(\exp_{\overline{T}}(f)) = \min\{T_1(\alpha) \mid \alpha \in \operatorname{supd}(f)\} = \operatorname{ord}(f)$. □

For any function $f = \sum c_\alpha u^\alpha$ we define its initial form as the homogenous polynomial $\operatorname{in}(f) = \sum_{|\alpha|=\operatorname{ord}(f)} c_\alpha u^\alpha$.

Proposition-Definition 4.0.3. Let $\overline{T}$ be a total normalized order on $\mathbb{N}^n$. Let $\mathcal{I} \subset K\{u\}$ be any ideal. There is a *standard basis of $\mathcal{I}$ with respect to T* that is, a uniquely determined set of functions $\{f_i \in \mathcal{I}\}$ such that:

(1) $\exp_{\overline{T}}(f_i) = \alpha_i$ are vertices of $\Delta := \exp_{\overline{T}}(\mathcal{I})$. In particular, $\operatorname{ord}(f_i) = |\alpha_i|$.
(2) $f_i := u^{\alpha^i} + r_i$ such that $\operatorname{supd}(r_i)_{i=0}^n$ contained in Γ.

Proof. (1) By WHDT , for any $g \in \mathcal{I}$, we can write $g = \sum h_i f_i + r(g)$. Since $g, f_i \in \mathcal{I}$ we get that $r(g) \in \mathcal{I}$ and consequently $\operatorname{supp}(r(g) \in \exp(I) = \bigcup \Delta_i$. But again by the Theorem $\operatorname{supp}(r(g) \in \Gamma_0$. Both conditions imply that $\operatorname{supp}(r(g) \in \Gamma_0 \cap (\bigcup \Delta_i) =$. That is $r(g)$ is constant and eventually 0, and $g = \sum h_i f_i$. (2) If T is normalized then $\exp(\operatorname{in}(f) = \exp(f)$, and

$$\exp_{\overline{T}}(\mathcal{I}) = \exp_{\overline{T}}(\operatorname{in}(\mathcal{I}))$$

So (2) foolows from (1) applied to $\operatorname{in}(f_i) \in (\operatorname{in}(\mathcal{I}))$. (3) Write $u^{\alpha_i} = \sum_j h^i_j f_j + r(u^{\alpha^i})$. Then put $\overline{f}_i := u^{\alpha^i} - r(u^{\alpha^i})$. The function $\overline{f}_i$ are uniquely determined. Otherwise $\operatorname{supp}(\overline{f}_i - \overline{f}'_i) \subset \Gamma_0$. Then $\exp(\overline{f}_i - \overline{f}'_i) \in \exp_{\overline{T}}(\mathcal{I}) \cap \Gamma_0 = \Delta \cap \Gamma_0 = \emptyset$. □

Proposition 4.0.4. *Let $f_1, \dots, f_k$ be the standard basis of the ideal $\mathcal{I} \subset K\{u\}$ (with respect to a total normalized order $\overline{T}$). Then*

(1) *Any function $f \in \mathcal{I}$ can be written uniquely as $f = \sum_j h^i_j f_j$ with* $\operatorname{supd}(h_i) \subseteq \Gamma_i$.
(2) *Any function $f \in K\{u\}$ can be written uniquely as $f = \sum_j h^i_j f_j + h_0$ with* $\operatorname{supd}(h_i) \subseteq \Delta_i$.
(3) *The initial forms* $\operatorname{in}(f_i)$ *generate* $\operatorname{in}(\mathcal{I}) = \{\operatorname{in}(f) \mid f \in \mathcal{I}\}$.

Any ideal $\mathcal{I} \subset K\{u\}$ defines the Hilbert-Samuel function $\mathcal{H}_{\mathcal{I}} : \mathbb{N} \to \mathbb{N}$, $\mathcal{H}_{\mathcal{I}}(s) := \dim_K \mathcal{O}_x/(\mathcal{I} + m^{s+1})$, where m is the maximal ideal.

One can also associate with any diagram of initial exponents Δ in $\mathbb{N}^n$ its Hilbert-Samuel function $\mathcal{H}(\Delta) : \mathbb{N} \to \mathbb{N}$, $\mathcal{H}(\Delta)(s) := \{\alpha \in \Gamma \mid |\alpha| \leq s\}$.

Corollary 4.0.5. *If $\overline{T}$ is total and normalized then there is a natural isomorphism of of K-modules*

$$r_{\overline{T}} : K\{u\}/\mathcal{I} \to K\{u\}^\Gamma, \quad f \mapsto r_{\overline{T}}(f),$$

where $K\{u\}^\Gamma := K\{u^\alpha \mid \alpha \in \Gamma\}$ is a K- submodule of $K\{u\}$, of all the formal power series f with $\operatorname{supp}(f) \subset \Gamma$*, and $r(f)$ is the remainder of the division by the standard basis of $\mathcal{I}$. In particular $\mathcal{H}_{\mathcal{I}} = \mathcal{H}(\Delta)$.*

In view of Lemmas 3.0.5, Lemma 3.0.3 one can find a positive linear form L such that $\exp_L(f_i) = \alpha_i$, and then a norm ρ such that $\exp_L(f_i) = \alpha_i$. Thus we can replace the total order with the norm ρ in the definition of the standard basis, relaxing the conditions.

Definition 4.0.6. Let Δ be a diagram of initial exponents in $\mathbb{N}^s$, and $\rho = \rho_1, \dots, \rho_s, \geq 0$. By the *standard basis with respect to the diagram Δ and the seminorm ρ* we mean the set of functions f_i such that

(1) $\exp_\rho(f_i) = \alpha_i \in \mathbb{N}^s$ are vertices of Δ, and $\operatorname{ord}(f_i) = |\alpha_i|$.

(2) $f_i := u^{\alpha^i} + r_i$ such that $\mathrm{supd}(r_i)_{i=0}^n$ contained in Γ.
(3) $H(\Delta \times \mathbb{N}^{m-s}) = H(\mathcal{I})$

Combining the above we get

Theorem 4.0.7. *For any ideal $\mathcal{I} \subset K\{u\}$ there exists a standard basis with respect to a certain norm ρ and a diagram Δ.*

Theorem 4.0.8. *Let $\mathcal{I}$ be an ideal in $K\{u\}$. Assume there exists a standard basis $\{f_i \in \mathcal{I}\}$ with respect to the seminorm $\rho = (\rho_1, \ldots, \rho_s, 0, \ldots, 0)$ with $\rho_i > 0$, and a diagram Δ. Then it is unique, and moreover:*

(1) $K[u]^\Gamma \cap \mathrm{in}(\mathcal{I}) = \{0\}$
(2) $K\{u\}^\Gamma \cap \mathcal{I} = \{0\}$.
(3) *Any function $f \in \mathcal{I}$ can be written uniquely as $f = \sum_j h^i_j f_j$ with* $\mathrm{supd}(h_i) \subseteq \Gamma_i$.

Proof. For any $g \in \mathcal{I}$, we get $\mathrm{in}_x(g) = \sum H_i \mathrm{in}_x(f_i) + H_0$, with $H_0 \in \mathrm{in}(\mathcal{I})$ and $\mathrm{supp}(H_0) \subset \Gamma$. This implies that

$$\mathrm{gr}(\phi) : K[u_1, \ldots, u_m]^\Gamma \to \mathrm{gr}(\mathcal{O}_x)_s/\mathrm{in}_x(\mathcal{I})_s = \mathrm{gr}(\mathcal{O}_x/\mathcal{I}_x) = \bigoplus_{s=0}^{\infty} (m^s_x)/((\mathcal{I} \cap m^s_x) + m^{s+1}_x).$$

is an epimorphism. Since $H(\Delta \times \mathbb{N}^{m-n}) = H(\mathcal{I})$ we get that $\dim(\mathrm{gr}(\mathcal{O}_x)_s/\mathrm{in}_x(\mathcal{I})_s) = \dim(K[u_1, \ldots, u_k]^\Gamma)_s$, which implies that $\mathrm{gr}(\phi)$ is an isomorphism, and its kernel $\ker \mathrm{gr}(\phi) = K[u]^\Gamma \cap \mathrm{in}(\mathcal{I}) = \{0\}$. Consequently $K\{u\}^\Gamma \cap \mathcal{I} = \{0\}$.

Now, for any $g \in \mathcal{I}$ we write $g = \sum h_i f_i + h_0$, where $h_0 \in \mathcal{I}$ and $\mathrm{supp}(h_0) \subset \Gamma$, $\mathrm{ord}(h_i f_i), \mathrm{ord}(h_0) \geq \mathrm{ord}(g)$. Since $\mathrm{in}_x(h_0) \in \mathrm{in}(\mathcal{I}) \cap K[u_1, \ldots, u_k]^\Gamma \subseteq \ker(\mathrm{gr}(\phi)) = \{0\}$, hence $h_0 = 0$ and for any $g \in \mathcal{I}$ we get a unique presentation $g = \sum h_i f_i$, with $\mathrm{supd}(h_i) \subseteq \Gamma_i$. In particular $\mathcal{I} \cap K\{u\}^\Gamma = \{0\}$.

If $f'_i = u^{\alpha_i} + r'_i$ are another functions then with the above properties the $f_i - f'_i \in \mathcal{I} \cap K\{u\}^\Gamma = \{0\}$.

□

5. Standard basis along Samuel stratum

From now let K be a complete non-discretely normed field. Let $\mathcal{I} \subset \mathcal{O}(X)$ be an ideal sheaf of finite type on a K-analytic manifold X, and $x \in X$ be a point. Recall that the *Samuel stratum* S_x through a closed point $x \in X$ on an analytic manifold X is a locally closed subset $S \subset X$ consisting of all the closed points $y \in X$ with the same Hilbert-Samuel function $H_{x,\mathcal{I}} = H_{y,\mathcal{I}}$.

Definition 5.0.1. Let $\mathcal{I} \subset \mathcal{O}(X)$ be an ideal sheaf of finite type on a K-analytic manifold X of dimension n, and $x \in X$ be a point. Let $\Delta \subset \mathbb{N}^s$, where $s \leq n$ be a diagram of with vertices $\alpha_1, \ldots, \alpha_k$. A set of analytic functions $f_1, \ldots, f_k \in \mathcal{I}$ will be called a *standard basis* of $\mathcal{I}$ with respect to Δ on X along Samuel stratum S_x if there exists an open neighborhood U of x, a coordinate system $u_1, \ldots, u_s, u_{s+1}, \ldots, u_n$ on U, and a seminorm $\rho = (\rho_1, \ldots, \rho_s, 0, \ldots, 0) \in \mathbb{R}^n_{\geq 0}$ with all $\rho_i > 0$, such that athe following conditions are staisfied:

(1) $f_i := u^{\alpha_i} + r_i$ on U such that $\mathrm{supd}(r_i)$ contained in $\Gamma \times \mathbb{N}^{n-s}$.
(2) $S_x = \{y \in U \mid H_y(\mathcal{I}) = H_x(\mathcal{I}) = H(\Delta \times \mathbb{N}^{n-s})\} = \{y \in U \mid \mathrm{ord}_x(f_i) = |\alpha_i| \text{ for all } i = 1, \ldots, k\}$.
(3) For any $j = 1, \ldots, s$ there exists $i = 1, \ldots, k$, and $\beta_j = \alpha_{i,0} - e_j$ such that $u_j = D_{u^{\beta_j}}(f_i)$ on U ,and it vanishes along $S_x \cap U$.
(4) If $y \in S_x \cap U$ then $\exp_\rho(f_i) = \alpha^i$, $\mathrm{mon}_\rho(f_i) = u^{\alpha_i}$ with respect to the coordinate system $u_1, \ldots, u_s, u_{s+1} - u_{s+1}(y), \ldots, u_n - u_n(y)$.

Theorem 5.0.2. *(Existence of the standard basis along Samuel stratum)*

Let X be a K -analytic manifold, and $\mathcal{I}$ be a sheaf of ideals on X of finite type. Then for any $x \in X$ there exist an open neighborhood $U \subset X$ of x and functions $f_1, \ldots, f_k$ are regular on U which form a standard basis of $\mathcal{I}$ at a point $x \in X$ with respect to vertices $\alpha_1, \ldots, \alpha_k$ of Δ and a coordinate system $u_1, \ldots, u_n$ at any $y \in U$ with

(1) *The Samuel stratum through $x \in X$ can be described as*
$$S = S_x = \{y \in U \mid H_{y,\mathcal{I}} = H_{y,\mathcal{I}} = H(\Delta)\} = \{y \in U \mid \mathrm{ord}_y(f_i) = |\alpha_i|\}.$$
(2) *$f_1, \ldots, f_k$ is a standard basis with respect to Δ and $u_1, \ldots, u_n$ at any point $y \in S$.*
(3) *For any $y \in U$ we have $H_{y,\mathcal{I}} \leq H(\Delta)$.*

Proof. By Corollary 4.0.3 there exists a standard basis $f_1,\ldots,f_k$ of $\mathcal{I}_x \subset K\{u_1,\ldots,u_k\}$ (at the point x) with respect to any total normalized order $\overline{T'}$ and set $\Delta := \exp_{\overline{T}}(\mathcal{I}_x)$. We can assume that all vertices of Δ are in the smallest sublattice $\mathbb{N}^s of \mathbb{N}^n$. Let $\overline{T}$ be the order on $\mathbb{N}^n$ induced by the restriction of $\overline{T'}$ to $\mathbb{N}^s$

Write $f_j = u^{\alpha_i} + r_i$, with $\exp_{\overline{T'}}(f_i) = \exp_{\overline{T}}(f_i) = \alpha_i \in \mathbb{N}^s$. For any $j = 1,\ldots,s$ there exists $i = 1,\ldots,k$ such such in the monomial $u^{\alpha_i} = u_1^{\alpha_{i,1}} \cdot \ldots \cdot u_s^{\alpha_{i,s}}$, that j-coordinate is nonzero, that is $\alpha_{i,j} > 0$. Then for $\beta_j = \alpha_i - e_j$ we get that for the derivative $u'_j := D_{u^{\beta_j}}(f_i)$ has a form $u'_j = u_j + h_i$ with $\exp(u'_j) = e_j$. Replacing u'_j with u_j defines a coordinate change which preserves exp. By Lemmas 3.0.5,and 3.0.3 there is a positive linear form L and a norm $\rho = \rho_{L,d}$ such that $\exp_\rho(f_i) = \alpha_i$. By Lemma 2.0.2 we find a seminorm $\rho = (\rho_1,\ldots,\rho_s,0,\ldots,0)$ such that $\alpha_i = \exp_\rho(f_i)$ at a sufficiently small neighborhood U of x.

Set $S := \{y \in U \mid \mathrm{ord}_y(f_i) = |\alpha_i|\}$. We can assume that $U \subset B(x,\rho)$ is an open neighborhood of $x \in X$ where (f_i) generate $\mathcal{I}$. By the definition all the coordinates $u_1,\ldots,u_s$ vanish along S

The condition (1),(3),(4) are satisfied at any point $y \in S \cap B(x,\rho)$. it suffices tos how that $S = S_x$.

If $y \notin S$, there is $r := \mathrm{ord}_y(f_i) < |\alpha_i|$ and $\mathrm{ord}_y(f_j) = |\alpha_j|$ for all f_j such that $\mathrm{ord}_y(f_j) < r$. The support of $\mathrm{in}_y(f_i)$ is contained in Γ. On the other hand $\exp_{\rho_y}(f_j) = \exp_{\rho_y}\mathrm{in}_y(f_j) = \alpha_j$ if $|\alpha_j| \leq r$. It follows by uniqueness of Weierstrass-Hironaka division of the homogenous polynomials $\mathrm{in}_y(f_i)$ by $\mathrm{in}_y(f_j), |\alpha_j| \leq r$ (with that $\mathrm{in}_y(f_i)$ is linearly independent from $\{\mathrm{in}_y(u^\alpha f_j), \alpha + \alpha_j \in \Delta(r)\}$. This implies that $H_y(\mathcal{I}) < H(\Delta)$ with respect to lexicographic order. Thus $S_x \subset S$.

If $y \in S$ then $\mathrm{ord}_y(f_i) = |\alpha_i|$ and $\mathrm{in}_{\rho'}(\mathrm{in}(f_i)) = u^{\alpha_i}$. Thus there exists a unique division in $\mathcal{O}(U)$ such that any $g \in \mathcal{O}(U)$ can be written uniquely as $g = \sum h_i \cdot f_i + h_0$, where $\mathrm{supd}(h_i) \subset \Gamma_i$. In particular $\mathrm{supd}(h_0) \subset \Gamma$. But this implies that $\mathrm{in}_x(h) \equiv 0$, and consequently $h \equiv 0$ and $g = \sum h_i \cdot f_i$. It follows from uniqueness of the homogenous division by $\mathrm{in}_y(f_i)$ in $K[u_1,\ldots,u_k]$ then $\mathrm{in}_y(g) \in \mathrm{in}_y(\mathcal{I})$ can be written uniquely as $\mathrm{in}_y(g) = \sum H_i \cdot \mathrm{in}_y(f_i)$ which defines and isomorphism

$$\mathrm{gr}(\phi_y) : K[u_1,\ldots,u_k]^\Gamma \to \mathrm{gr}(\mathcal{O}_y/\mathcal{I}_y) = \bigoplus_{s=0}^{\infty} (m_y^s)/(\mathcal{I} + m_y^{i+1}),$$

showing that that $H_{y,\mathrm{in}_y(\mathcal{I})} = H_{y,\mathcal{I}} = H(\Delta)$ and $S = S_x$ on U.

□

Note that the Samuel stratum S an analytic subspace of U defined by the equations

$$S = \{x \in U \mid D_{u^\alpha}(f_i(x)) = 0, \quad i = 1,\ldots,k, \quad |\alpha| < |\alpha_i|\}.$$

Definition 5.0.3. Let $\mathcal{I}$ be any ideal sheaf of finite type on a manifold M. Consider the blow-up $\sigma : M' \to M$ at a smooth closed center $C \subset M$ contained in the Samuel stratum. By the *strict transform* of $\mathcal{I}$ we mean here the ideal generated locally by all the functions $f \in \mathcal{O}(U')$ for which $y^k f$, for some k, is in the ideal $\sigma^*(\mathcal{I})$ generated by $\sigma^*(g)$, where $g \in \mathcal{O}(U)$. where y is a local equation of the exceptional divisor, and $c(f)$ is the maximal exponent for which $y^{c(f)}$ divides $\sigma^*(f)$.

If $Y \subset M$ be a closed analytic subspace then by the *strict transform* we mean the closure of $Y \setminus C$ on M'.

Lemma 5.0.4. *Let C be a smooth center of the blow-up $\sigma : X \leftarrow X'$ and let D denote the exceptional divisor. Assume that $\mathrm{ord}_x(\mathcal{I}) \geq \mu$ for all $x \in C$. Let $\mathcal{I}_C$ denote the sheaf of ideals defined by C. Then*

(1) $\mathcal{I} \subset \mathcal{I}_C^\mu$.
(2) $\sigma^*(\mathcal{I}) \subset (\mathcal{I}_D)^\mu$.

Proof. (1) Let $u_1,\ldots,u_k$ be local parameters generating $\mathcal{I}_C$ Suppose $f \in \mathcal{I} \setminus \mathcal{I}_C^\mu$. Then we can write $f = \sum_\alpha c_\alpha u^\alpha$, where either $|\alpha| \geq \mu$ or $|\alpha| < \mu$ and $c_\alpha \notin \mathcal{I}_C$. By the assumption there is α with $|\alpha| < \mu$ such that $c_\alpha \notin \mathcal{I}_C$. Take α with the smallest $|\alpha|$. There is a point $x \in C$ for which $c_\alpha(x) \neq 0$ and in the Taylor expansion of f at x there is a term $c_\alpha(x)u^\alpha$. Thus $\mathrm{ord}_x(\mathcal{I}) < \mu$. This contradicts to the assumption $C \subset \mathrm{cosupp}(\mathcal{I},\mu)$.

(2) $\sigma^*(\mathcal{I}) \subset \sigma^*(\mathcal{I}_C)^\mu = (\mathcal{I}_D)^\mu$. □

It follows form the proof that

Lemma 5.0.5. *If $f \notin \mathcal{I}^c\mu$ and y is the equation of D then y^μ does not divide $\sigma^* f$.*

Theorem 5.0.6 (Stability of standard basis under blow-ups)**.** *Let $\mathcal{I}$ be any ideal sheaf of finite type on an analytic manifold X, and let $C \subset X$ be a smooth submanifold. Consider the blow-up $\sigma : X' \to X$ at a smooth closed center $C \subset X$ contained in the Samuel stratum, and let $\mathcal{I}'$ be its strict transform. Let $U' \subset \sigma^{-1}(U)$ be an open subset where a coordinate y on U describes the exceptional divisor of σ. Then:*

(1) *If $f_1, \ldots, f_k$ is a standard basis of $\mathcal{I}$ on an open neighborhood U of X with respect to a diagram Δ and norm ρ then*
$$f'_1 := \sigma^*(f_1)/y^{\mu_1}, \ldots, f'_k := \sigma^*(f_k)/y^{\mu_k}$$
is a standard basis of $\mathcal{I}'$ on U' with respect to Δ and the induced coordinate systems.

(2) (Bennett) $H_{x,\mathcal{I}} \geq H_{x',(\mathcal{I}')}$.

Proof. $u_1, \ldots, u_c, \ldots, u_n$ Consider local parameters compatible with the standard basis and the center C, where the coordinates $u_1, \ldots, u_c$ define C locally, and $u_1, \ldots, u_s$, with $s \leq c$ are essential unknowns. This can be done by a suitable coordinate change $u_{s+1}, \ldots, u_n$ which does not affect the conditions of the standard basis.

Let μ_i denote the multiplicity of the function f_i. It follows from Lemma 5.0.4 that $f_i \in \mathcal{I}^{\mu_i} \setminus \mathcal{I}_C^{\mu_i+1}$ and correspondingly $\sigma^*(f) \in \mathcal{I}_D^{\mu_i} \setminus \mathcal{I}_D^{\mu_i+1}$, where $D = \sigma^{-1}(C)$ is the exceptional divisor of σ. Then $f'_i := (1/y^{\mu_i})\sigma^*(f_i)$. Consider the effect of the blow-up of C near a point $x \in X$.

The points x' in $\sigma^{-1}(x)$ are defined by the lines in the normal space N_C, where $N_C^* = \mathcal{I}_C/\mathcal{I}_C^2$. For the line $t[a_1, \ldots, a_k]$, where $a_i \in K$, we consider the dual hyperplane in N_C^* and the corresponding linear system. More precisely, let $r = \max\{i : a_i \neq 0\}$ and consider the change of coordinates

(1) $$\overline{u}_1 = u_1 - (a_1/a_r)u_r, \ldots, \overline{u}_{r-1} = u_{r-1} - (a_{r-1}/a_r)u_r, \overline{u}_r = u_r, \overline{u}_{r+1} = u_{r+1}, \ldots, \overline{u}_n = u_n.$$

The effect of the blow-up at x' (in new coordinates) is described by
$$u'_1 = \overline{u}_1/y, \ldots, u'_1/y = \overline{u}_1, u'_r = \overline{u}_r = y, u'_{r+1} = \overline{u}_{r+1}/y, \ldots, u'_n = \overline{u}_n/y$$
with the exceptional divisor $y = \overline{u}_r$. Since $f_i \in \mathcal{I}_C^{\mu_i} \setminus \mathcal{I}_C^{\mu_i+1}$, we can write $f_i = \sum_{|\alpha| \geq \mu_i} c_{i\alpha}(v)\overline{u}^\alpha$, where $\overline{u} := (\overline{u}_1, \ldots, \overline{u}_c)$ and $v := (\overline{u}_{c+1}, \ldots, \overline{u}_n)$.

Let us define
$$\alpha' := (\alpha_1, \ldots, 0_r, \ldots, \alpha_c)$$
with 0 as the r-th coordinate. The transformed function will have the form
$$f'_i := (1/y^{\mu_i})\sigma^*(f_i) = (1/y^{\mu_i}) = \sum \sigma^*(c_i(v))(u')^{\beta'} \cdot y^{|\beta| - \mu_i}.$$

If $r > s$ then we consider the seminorm $\rho = \rho_1, \ldots, \rho_s, 0 \ldots, 0$. For $(u^{\alpha_i})' := (1/y^{\mu_i})\sigma^*((u^\alpha)$ we have
$$||u^{\alpha_i}|| = ||((u')^{\alpha_i})'|| \quad ||\sigma^*(c_i(v))|| = ||c_i(v))|| = ||c_i(v))(x')||$$
and if $\beta \in \operatorname{supd}(f_i)$ then
$$||u^\beta|| \leq ||(u')^{\beta'} \cdot y^{|\beta|-\mu_i}||.$$
This gives $||f'_i - (u')^{\alpha_i}||_\rho \leq ||f_i - u^{\alpha_i}||_\rho < ||u^{\alpha_{i0}}||_\rho = ||(u')^{\alpha_{i0}}||_\rho$. If $r \leq s$ then it follows $\operatorname{ord}_y(f'_i) \leq \operatorname{ord}_y(u^{\alpha_{i0}})' < \mu_i$ for some i. We can assume that $\operatorname{ord}_y(f'_j) = \operatorname{ord}_x(f_j)$, for all j with $\mu_j < \mu_i$. Then as before $\exp_\rho(f'_j) = \alpha_{0,j}$.

Also $\operatorname{supd}((f'_i) \subset \Gamma_0$, and thus $\operatorname{in}_y((f'_i)$ is independent of $\operatorname{in}(u^\alpha(f'_j)$, which shows that $H_{y,\mathcal{I}'} < H_{x,\mathcal{I}}$.

The other conditions of the standard basis are satisfied:

If $g \in \mathcal{I}$, and $y^\mu|\sigma^*(g)$ then and $g \in \mathcal{I}_C^\mu$, which means that for any $x \in C \cap U$, we have $\operatorname{ord}_x(g) \geq \mu$. Then by WHDT (Theorem 1.0.3) for the presentation $g = \sum h_i f_i + r(g)$, where $\operatorname{ord}_x(h_i) \geq \mu - \mu_i$, $\operatorname{ord}_x(h_i) \geq \mu$ where $\mu_i = |\alpha_i| = \operatorname{ord}_x(f_i)$. Thus $f_i \in \mathcal{I}_C^{\mu-\mu_i}$, and
$$y^{\mu-\mu_i}|\sigma^*(h_i), \quad y^\mu|\sigma^*(r(g)), \quad y^{\mu_i}|\sigma^*(f_i),$$
and we can write $\sigma^*(h_i) = y^{\mu-\mu_i}(h_i)'$,
$$\sigma^*(r(g)) = y^\mu(r(g))', \quad \sigma^*(h_i) = y^{\mu-\mu_i}(h_i)', \quad \sigma^*(f_i) = y^{\mu_i}(f_i)'.$$
We get $\sigma^*(g)/y^\mu = \sum h'_i f'_i + r(g)' \in (f'_1, \ldots, f'_k)$.

Suppose now that $g' \in (f'_1, \ldots, f'_k)$ is in the ideal generated by f'_i, and that y divides $g' = yg''$. Thus we can write $g' = \sum h'_i f'_i$, and $g'' = \sum h''_i f'_i + r(g)''$. The latter gives $g' = \sum yh''_i f'_i + yr(g)''$. Since multiplication by $y = u'_r$, with $r > s$ preserves elements from $\Delta_i \times \mathbb{N}^{n-s}$, we see that $\operatorname{supd}(yh''_i) \in \Gamma_i$, $\operatorname{supd}(yr(g)'') \in \Gamma$ we get by uniqueness of presentations that $yr(g)'' \equiv 0$. This implies that $r(g)'' \equiv 0$ and $g'' \in (f'_1, \ldots, f'_k)$.

Thus f_i' generate the ideal of the strict transform $\mathcal{I}'$.

If $u_j = D_{u^{\beta_j}}(f_i)$ on U then we see directly then $u_j' = D_{(u')^{\beta_j}}(f_i')$.

□

6. Homogenous Rees Algebra and essential variables

Let $F \in K[x_1, \dots, x_n]$ be a form of degree d. For any nonnegative integer $k \le d$ denote by $\overline{D}^d(F)$ the vector space spanned by the derivatives of order d. This definition does not depend upon a linear change of coordinates. Using this operation one can define a homogenous counterpart of Rees algebra and Rees ideal.

Definition 6.0.1. By *the homogenous Rees Algebra* generated by the homogenous polynomials $F_i \in K[x_1, \dots, x_n]$ of degree d_i we mean the smallest graded subalgebra

$$R^{\cdot} = R^{\cdot}(F_1, \dots, F_r) = \bigoplus_{d \in \mathbb{N}} R^d$$

containing $F_i \in R^{d_i}$ and which is $\overline{D}^{\cdot}$-stable, that is

$$\overline{D}^a(\mathcal{R}^d) \subset R^{d-a}$$

if $a \in \mathbb{Z}_{>0}$, $d \ge a$.

The graded ideal

$$I^{\cdot} = I^{\cdot}(F_1, \dots, F_r) \subset R^{\cdot} = R^{\cdot}(F_1, \dots, F_r)$$

generated over $R^{\cdot}$ by (F_i) will be called the *Rees ideal* generated by (F_i).

Definition 6.0.2. Let $\mathcal{I} = \bigoplus \mathcal{I}_{a \in \mathbb{N}} \subset K[x_1, \dots, x_n]$ be a homogenous ideal. By the *essential set of coordinates* we mean a set of lineally independent linear forms $u_1, \dots, u_k$ such that

(1) There exists a set of homogenous generators $F_1, \dots, F_r \in \mathcal{I}$, such that $F_i = F_i(u_1, \dots, u_k)$.
(2) The vector space $V := \operatorname{span}(u_1, \dots, u_k) \subset \operatorname{span}(x_1, \dots, x_n)$ is minimal for all sets $u_1, \dots, u'_{k'}$ satisfying the condition (1). The vector space V will be called the *essential space* of $\mathcal{I}$.

The notion of essential coordinates makes sense for homogenous polynomials or their sets.

Lemma 6.0.3. *([7], Lemma 6.2a) In characteristic 0 the vector space $V := \overline{D}^{d-1}(F)$ is essential for F.*

Proof. If $u_1, \dots, u_k$ is a basis of V then after extending the set to a complete coordinate system $u_1, \dots, u_n$ and F does not depend upon $u_{k+1}, \dots, u_n$ thus $F = F(u_1, \dots, u_k)$. On the other hand if $u_1', \dots, u_l'$ are essential unknowns then $F = F(u_1', \dots, u_l')$ and $\operatorname{span}(u_1', \dots, u_l') \supseteq V = \overline{D}^{d-1}(F)$. □

Proposition 6.0.4. *Let K be a field of the characteristic 0 and $F_1, \dots, F_r \in K[x_1, \dots, x_n]$ denote homogenous polynomials. The homogenous Rees algebra $R^{\cdot}(F_1, \dots, F_r)$ is generated by the essential unknowns $u_1, \dots u_s$ for $F_1, \dots, F_r$. That is*

$$R^{\cdot} := R^{\cdot}(F_1, \dots, F_r) = K[u_1, \dots, u_k]$$

Proof. The result follows from the fact that $u_1, \dots, u_k \in R^1$ and any form $G \in R^d$ can be expressed as function in $u_1, \dots, u_k$. This is true since the generators have this form and the property is preserved by the derivations, sums, and products.

□

Proposition 6.0.5. *Let K be a field and $I \subset K[x_1, \dots, x_n]$ be a homogenous ideal, and let $F_1, \dots, F_r \in I$ denote a standard basis of $\mathcal{I}$ (at 0). Then*

(1) *$F_1, \dots, F_r$ are homogenous and moreover $F_i = F_i(u_1, \dots, u_k)$, where $u_1, \dots, u_k$ is essential set of coordinates for I.*
(2) *$R^{\cdot}(I) := R^{\cdot}(F_1, \dots, F_r)$ is independent of the choice of the standard basis of I*
(3) *$R^{\cdot}(I) = K[u_1, \dots, u_k]$ and $J^{\cdot}(I) = I \cap K[u_1, \dots, u_k]$.*

Proof. Let $F_1 = F_1(u_1, \dots, u_k), \dots, F_r = F_r(u_1, \dots, u_k)$ be a basis of I which depends on essential unknowns. Then $D_{u_j}(F_i) = 0$ where $j \geq k$ and for any $G = \sum H_i F_i \in I$, we have $D_{u_j}(G) \in I$. On the other hand for the standard basis $G_i = v^{\alpha_i} + r_i$ with $\mathrm{supd}(r_i) \subset \Gamma$. More precisely, by Condition (3) of Definition **??**, $D_{v^\alpha}(G_i) = 0$ for $\alpha \in \Delta \setminus \{\alpha_i\}$, and $D_{v^{\alpha_i}}(G_i) = 1$. In both cases, that is, if $\alpha \in \Delta$ we have $D_{v^\alpha}(D_{\overline{u}_j})(G_i)) = D_{\overline{u}_j^{p^t}})(D_{v^\alpha}(G_i) \equiv 0$ which means that $\mathrm{supd}(D_{\overline{u}_j})(G_i) \subset \Gamma$.

Since additionally $(D_{\overline{u}_j})(G_i) \in I$ we conclude that $\mathrm{supd}(D_{\overline{u}_j})(G_i) = 0$. The latter implies that $G_i \in R^{\cdot}(F_1, \dots, F_r)$, and $R^{\cdot}(G_1, \dots, G_s) \subseteq R^{\cdot}(F_1, \dots, F_r)$. This implies that if (F_i) is another standard basis then by symmetry we get the equality $R^{\cdot}(F_1, \dots, F_r) = R^{\cdot}(G_1, \dots, G_s)$. □

7. Resolution of marked ideals

Definition 7.0.1. Let M be an analytic space over K and $Z \subset M$ be a compact subset. By *a germ* M_Z of M at Z we mean a pair (U, Z) where $U \subset M$ is any open subset of M containing Z. We say that any two open subsets U, U' of M containing Z define the same germ M_Z. We write $M_Z = (U, Z)$ and call U a *neighborhood* of a germ M_Z. If M is a manifold then M_Z will be called a smooth germ By a *morphism* $f : M_Z \to M'_{Z'}$ we mean a morphism $f_U : U \to U'$ between some neighborhoods of M_Z and $M'_{Z'}$ such that $f(Z) \subset Z'$. The morphism f is *proper*, *projective*, is an *open* or *closed inclusion* if f_U has this property for the corresponding neighborhoods U, U'.

We introduce the operation of union and intersection of germs : If $U, U' \subset M$ then
$$(U, Z) \cup (U', Z') := (U \cup U', Z \cup Z'), \quad (U, Z) \cap (U', Z') := (U \cap U', Z \cap Z')$$
Then $(U, Z) \to (U, Z) \cup (U', Z')$ and $(U, Z) \cap (U', Z') \to (U, Z)$ are open inclusions.

We shall consider ideal sheaves and divisors on smooth germs M_Z. An ideal sheaf of finite type on M_Z is an ideal sheaf $\mathcal{I}$ on some open neighborhood U of M_Z. For any ideal sheaf $\mathcal{I}$ on $M_Z = (U, Z)$ and any point $x \in U$ we denote by
$$\mathrm{ord}_x(\mathcal{I}) := \max\{i \mid \mathcal{I}_x \subset m_x^i\}$$
the *order* of $\mathcal{I}$ at x. (Here m_x denotes the maximal ideal of x.)

Definition 7.0.2. (Hironaka [14], [16], Bierstone-Milman [7], Villamayor [**?**]) A *marked ideal* is a collection $(M_Z, \mathcal{I}, E, \mu)$, where M_Z is a smooth germ, $\mathcal{I}$ is an ideal sheaf of finite type on M_Z, μ is a nonnegative integer and E is a totally ordered collection of divisors on M_Z whose irreducible components are pairwise disjoint and all have multiplicity one. Moreover the irreducible components of divisors in E have simultaneously simple normal crossings.

Let $(M_Z, \mathcal{I}, E, \mu)$ be a marked ideal such that the ideal sheaf $\mathcal{I}$ is defined on an open neighborhood U of M_Z. One can show that the set
$$\mathrm{cosupp}_Z(M_Z, \mathcal{I}, E, \mu) := \{x \in Z \mid \mathrm{ord}_x(\mathcal{I}) \geq \mu\}$$
is compact. On the other hand the set
$$\mathrm{cosupp}_U(M_Z, \mathcal{I}, E, \mu) := \{x \in U \mid \mathrm{ord}_x(\mathcal{I}) \geq \mu\}$$
defines a closed analytic subspace of U. (see Lemma 8.0.2).

Definition 7.0.3. (Hironaka [14], [16], Bierstone-Milman [7], Villamayor [**?**]) By the *cosupport* of $(M_Z, \mathcal{I}, E, \mu)$ we mean the germ of analytic space
$$\mathrm{cosupp}(M_Z, \mathcal{I}, E, \mu) := (\mathrm{cosupp}_U(M_Z, \mathcal{I}, E, \mu), \mathrm{cosupp}_Z(M_Z, \mathcal{I}, E, \mu)),$$

A collection of marked ideals $\{((M_Z, \mathcal{I}_i, E, \mu_i)\}$ will be called a *multiple marked ideal.* Marked functions (f, μ) are pairs of regular functions on X and $\mu \in \mathbb{N}$. Similarly
$$\mathrm{cosupp}\{((M_Z, \mathcal{I}_i, E, \mu_i)\} := \{x \in M \mid \mathrm{ord}_x(\mathcal{I}_i) \geq \mu_i\} = \bigcap_i \mathrm{cosupp}(M, \mathcal{I}_i, E, \mu_i).$$

Remarks. (1) For any sheaf of ideals $\mathcal{I}$ on M we have $\mathrm{cosupp}(\mathcal{I}, 1) = \mathrm{cosupp}(\mathcal{I})$.
(2) For any marked ideals $(\mathcal{I}, \mu)$ on X, $\mathrm{cosupp}(\mathcal{I}, \mu)$ is a closed subset of X (Lemma 8.0.2).

Definition 7.0.4. Let M_Z be a germ of an analytic manifold M. Let $C \subset U$ be a smooth closed subspace of a neighborhood $U \supset Z$. Let $\sigma_U : U' \to U$ denote the blow-up of a smooth center C. Set $Z' := \sigma_U^{-1}(Z)$, $M'_{Z'} := (U', Z')$. We shall call a bimeromorphic morphism $\sigma : M'_{Z'} \to M_Z$ a *blow-up* of M_Z at the center $C \subset M_Z$.

Definition 7.0.5. The blow-ups with the smooth centers $C \subset \operatorname{cosupp}(M_Z, \mathcal{I}, E, \mu)$ will be called *admissible for* $(M_Z, \mathcal{I}, E, \mu)$, if the centers are contained in the cosupport of marked ideals and have SNC with E. Likewise we called the centers *admissible* for the marked ideals.

Definition 7.0.6. Let $\sigma : {M_Z}'' \to M_Z$ be an admissible blow-up for $(M_Z, \mathcal{I}, E, \mu)$ with the exceptional divisor D then a marked ideal $(M'Z, \mathcal{I}', E', \mu) = \sigma^c(M_Z, \mathcal{I}, E, \mu)$ is called *the controlled transform* of $(X, \mathcal{I}, E, \mu)$ if

(1) $\mathcal{I}' = \mathcal{I}(D)^{-\mu}\sigma^*(\mathcal{I})$.
(2) $E' = \sigma^c(E) \cup \{D\}$, where $\sigma^c(E)$ is the set of strict transforms of divisors in E_{i-1}.
(3) The order on $\sigma^c(E)$ is defined by the order on E while D is the maximal element of E.

Similarly *the controlled transform* of $\{(M_Z, \mathcal{I}_i, E, \mu_i)\}$ is given as the collection of the controlled transforms of $(M_Z, \mathcal{I}_i, E, \mu_i)$.

It follows from Lemma 5.0.4 that the contolled transforms are well defined.

Definition 7.0.7. (Hironaka (see [14],), Bierstone-Milman (see [3]),Villamayor (see [**?**])) By a *admissible sequence of blow-ups* of $(M_Z, \mathcal{I}, E, \mu)$ we mean a sequence of blow-ups $\sigma_i : M_{i,Z_i} \to M_{i-1,Z_{i-1}}$ of of smooth centers $C_{i-1} \subset M_{Z,i-1}$,

$$M_{0,Z_0} \xleftarrow{\sigma_1} M_{1,Z_1} \xleftarrow{\sigma_2} M_{2,Z_2} \xleftarrow{\sigma_3} \ldots M_{i,Z_i} \longleftarrow \ldots \xleftarrow{\sigma_r} M_{r,Z_r},$$

which defines a sequence of marked ideals $(M_{i,Z_i}, \mathcal{I}_i, E_i, \mu)$ such that the centers C_{i-1} are admissible for $(M_{i-1,Z_{i-1}}, \mathcal{I}_{i-1}, E_{i-1}, \mu)$, are controlled transforms of $(M_{i,Z_i}, \mathcal{I}_i, E_i, \mu)$. If additionally

$$\operatorname{cosupp}(M_{i,Z_i}, \mathcal{I}_i, E_i, \mu) = \emptyset$$

then we call the sequence a *resolution* of $(M_Z, \mathcal{I}, E, \mu)$.

The definition of admissible sequence and a resolution sequence applies also to multiple marked ideals .

8. Ideals of derivatives

Ideals of derivatives were first introduced and studied in the resolution context by Giraud. Villamayor developed and applied this language to his *basic objects*.

Definition 8.0.1. (Giraud, Villamayor) Let $\mathcal{I}$ be a ideal sheaf of finite type on a germ of a manifold M_Z. By the *first derivative* (originally *extension*) $\mathcal{D}_{M_Z}(\mathcal{I})$ of $\mathcal{I}$ (or simply $\mathcal{D}(\mathcal{I})$) we mean the ideal sheaf of finite type generated by all functions $f \in \mathcal{I}$ with their first derivatives. Then the *i-th derivative* $\mathcal{D}^i(\mathcal{I})$ is defined to be $\mathcal{D}(\mathcal{D}^{i-1}(\mathcal{I}))$. If $(\mathcal{I}, \mu)$ is a marked ideal and $i \le \mu$ then we define

$$\mathcal{D}^i(\mathcal{I}, \mu) := (\mathcal{D}^i(\mathcal{I}), \mu - i).$$

Recall that on a manifold M there is a locally free sheaf of differentials $\Omega_{M/K}$ generated locally by $du_1, \ldots, du_n$ for a set of local parameters $u_1, \ldots, u_n$. The dual sheaf of derivations $\operatorname{Der}_K(\mathcal{O}_M)$ is locally generated by the derivations $\frac{\partial}{\partial u_i}$. Immediately from the definition we observe that $\mathcal{D}(\mathcal{I})$ is an ideal sheaf of finite type defined locally by generators f_j of $\mathcal{I}$ and all their partial derivatives $\frac{\partial f_j}{\partial u_i}$. We see by induction that $\mathcal{D}^i(\mathcal{I})$ is a ideal sheaf of finite type defined locally by the generators f_j of $\mathcal{I}$ and their derivatives $\frac{\partial^{|\alpha|} f_j}{\partial u^\alpha}$ for all multiindices $\alpha = (\alpha_1, \ldots, \alpha_n)$, where $|\alpha| := \alpha_1 + \ldots + \alpha_n \le i$.

Lemma 8.0.2. *(Giraud, Villamayor) For any* $i \le \mu - 1$,

$$\operatorname{cosupp}(\mathcal{I}, \mu) = \operatorname{cosupp}(\mathcal{D}^i(\mathcal{I}), \mu - i)).$$

In particular $\operatorname{cosupp}(\mathcal{I}, \mu) = \operatorname{cosupp}(\mathcal{D}^{\mu-1}(\mathcal{I}), 1) = V(\mathcal{D}^{\mu-1}(\mathcal{I}))$ *is a closed set (*$i = \mu - 1$*).*

Proof. It suffices to prove the lemma for $i = 1$. If $x \in \operatorname{cosupp}(\mathcal{I}, \mu)$ then for any $f \in \mathcal{I}$ we have $\operatorname{ord}_x(f) \ge \mu$. This implies $\operatorname{ord}_x(Df) \ge \mu - 1$ for any derivative D and consequently $x \in \operatorname{cosupp}(\mathcal{D}(\mathcal{I}), \mu - 1))$. Now, let $x \in \operatorname{cosupp}(\mathcal{D}(\mathcal{I}), \mu - 1))$. Then for any $f \in \mathcal{I}$ we have $\operatorname{ord}_x(f) \ge \mu - 1$. Suppose $\operatorname{ord}_x(f) = \mu - 1$ for some $f \in \mathcal{I}$. Then $f = \sum_{|\alpha| \ge \mu-1} c_\alpha x^\alpha$ and there is α such that $\alpha = \mu - 1$ and $c_\alpha \ne 0$. We find $\frac{\partial}{\partial x_i}$ for which $\operatorname{ord}_x(\frac{\partial x^\alpha}{\partial x_i}) = \mu - 2$ and thus $\operatorname{ord}_x(\frac{\partial f}{\partial x_i}) = \mu - 2$ and $x \notin \operatorname{cosupp}(\mathcal{D}(\mathcal{I}), \mu - 1))$. □

We write $(\mathcal{I}, \mu) \subset (\mathcal{J}, \mu)$ if $\mathcal{I} \subset \mathcal{J}$.

Lemma 8.0.3. *(Giraud, Villamayor) Let* $(\mathcal{I},\mu)$ *be a marked ideal and* $C\subset \mathrm{cosupp}(\mathcal{I},\mu)$ *be a smooth center and* $r\leq\mu$. *Let* $\sigma: M_Z\leftarrow M'_Z$ *be a blow-up at* C. *Then*

$$\sigma^{\mathrm{c}}(\mathcal{D}^r_{M_Z}(\mathcal{I},\mu))\subseteq \mathcal{D}^r_{M_{Z'}}(\sigma^{\mathrm{c}}(\mathcal{I},\mu)).$$

Proof. Follows easily from chain rule.

□

9. Canonical Rees algebra and standard basis

Definition 9.0.1. Let X be an analytic manifold.

By *the Rees Algebra*

$$\mathcal{R}^{\cdot}=\bigoplus_{\mu\in\mathbb{N}}(\mathcal{R}^{\mu},\mu)$$

we mean a graded algebra satisfying the conditions

(1) $\mathcal{R}^0=\mathcal{O}(U)$
(2) $\mathcal{R}^\mu\subset\mathcal{O}(U)$ is an ideal sheaf of finite type of $\mathcal{O}_X$.
(3) $\mathcal{R}^\mu\subset\mathcal{R}^{\mu'}$ if $\mu'\geq\mu$
(4) $\mathcal{R}^\mu\cdot\mathcal{R}^{\mu'}\subset\mathcal{R}^{\mu+\mu'}$
By *the Rees Algebra generated by* $\{\overline{\mathcal{I}}\}:=(\mathcal{I}_i,\mu_i)$ we mean the smallest graded algebra $\mathcal{R}^{\cdot}=\bigoplus_{\mu\in\mathbb{N}}(\mathcal{R}^\mu,\mu)$ such that $\mathcal{I}_i\subseteq\mathcal{R}^{\mu_i}$.
A Rees algebra will be called *a differential Rees Algebra* if it satisfies
(5) $\mathcal{D}^a(\mathcal{R}^\mu)\subset\mathcal{R}^{\mu-a}$ if $a\in\mathbb{Z}_{>0}$, $\mu\geq a$.

Similarly the differential Rees Algebra $\mathcal{R}^{\cdot}=\mathcal{R}^{\cdot}(\{\overline{\mathcal{I}}\})$ is *diff-generated by* $\overline{\mathcal{I}}:=\{(\mathcal{I}_i,\mu_i)\}$ if it is the smallest differential Rees algebra for which $\mathcal{I}_i\in\mathcal{R}^{\mu_i}$.

Remark. Different notions of Rees algebras defined by marked ideals were studied in the context of resolution by Giraud, Hironaka, Oda, and more recently Kawanoue-Matsuki, and Villamayor. The above definition is essentially equivalent to the one used Villamayor's papers. (See [**?**],[15],[21],[**?**],[**?**].)

The differential Rees algebras are natural extensions of marked ideals. They possess important properties generalizing the notion of coefficient ideals and homognization used in the simple proofs of the (weaker) desingularization in characterist zero [25], [**?**]

In this paper the notion will be used mainly to study more subtle properties related to the Hilbert-Samuel function, and strong resolution (see Definition 9.0.6).

It follows from the definition that

$$\mathrm{cosupp}(\mathcal{R}^{\cdot}(\overline{\mathcal{I}})=\bigcap_{\mu\in\mathbb{N}}\mathrm{cosupp}(\mathcal{R}^\mu,\mu)=\mathrm{cosupp}(\overline{\mathcal{I}}),$$

for any multiple marked ideal $\overline{\mathcal{I}}$.

Moreover, an immediate consequence of the definition is the foloowing:

Proposition 9.0.2. *Let* $\overline{\mathcal{I}}=\{(\mathcal{I}_i,\mu_i)\}$ *be a finite collection of marked ideals on an analytic manifold. Let* $u_1,\ldots,u_n$ *be a system of cooordinates on* X. *Denote by* $f_{i1},\ldots,f_{ij_i}$ *the finite sets of generators of* $\mathcal{I}_i$. *Then the differential Rees algebra* $R^{\cdot}(\overline{\mathcal{I}})$ *is (finitely) generated by marked functions* $(D_{u^\alpha}f_{ij},\mu_i-|\alpha|)$, *where* $\alpha\in\mathbb{N}^n$, $|\alpha|\leq\mu_i$.

We shall need the following result:

Lemma 9.0.3. *Let* (f_i,d_i) *be marked functions of maximal orders and consider the generated Rees algebra* $\mathcal{R}^{\cdot}=\mathcal{R}^{\cdot}(f_1,\ldots,f_r)$ *and the homogenous Rees algebra* $R(\mathrm{in}_x(f_1),\ldots,\mathrm{in}_x(f_1))$. *Then*

$$R^{\cdot}(\mathrm{in}_x(f_1),\ldots,\mathrm{in}_x(f_1))=\mathrm{in}_x(\mathcal{R}^{\cdot}(f_1,\ldots,f_r)$$

Proof. It follows by definition that $\mathrm{in}_x(Df)=D(\mathrm{in}_x(f))$, for any $D\in\overline{D}^a$, and $f\in\mathcal{R}$. Moreover in_x preserves products and sums of the marked functions in Rees algebra $\mathcal{R}$.

□

Theorem 9.0.4. *Assume X is an analytic manifold. Let $\mathcal{I}$ be a ideal sheaf of finite type on X, and (f_i, d_i) be a standard basis of $\mathcal{I}$ along a Samuel Stratum S. Then the Rees algebras $\mathcal{R}^{\cdot}(\mathcal{I}) = \mathcal{R}^{\cdot}(f_1, \ldots, f_k)$ is independent of choice of standard basis of $\mathcal{I}$ in a neighborhood of the Samuel stratum S.*

Proof. Let $f_1, \ldots, f_r$, and $g_1, \ldots, g_m$ be two standard bases of $\mathcal{I}$ along S on an open neighborhood of x . Denote by $u_1, \ldots, u_n$, and $v_1, \ldots, v_n$ the corresponding compatible coordinates. We can assume here that $u_1, \ldots, u_s$ are distinguished , and $u_1, \ldots, u_k$ are essential, with $s \leq k \leq n$. By symmetry it suffices to show that $f_i \in \mathcal{J}^{\mu_i}(g_1, \ldots, g_s)$. We will show that f_i is in the completion of $\widehat{\mathcal{J}^{\mu_i}}(g_1, \ldots, g_s)_x \subset \widehat{\mathcal{O}_{X',x}} = \widehat{\mathcal{O}_{X,x}}$ in a neighborhood of any $x \in S = \operatorname{cosupp}(\mathcal{J}^{\cdot})$.

First observe that the initial forms $\operatorname{in}_x(f_1), \ldots, \operatorname{in}_x(f_r)$, and $\operatorname{in}_x(g_1), \ldots, \operatorname{in}_x(g_s)$ form two different bases of the initial ideal $\operatorname{in}_x(\mathcal{I})$. Then we can find the essential linear forms

$$\overline{u}_1 = \operatorname{in}_x(u_1) = \operatorname{in}_x(\widetilde{u}_1), \ldots, \overline{u}_s = \operatorname{in}_x(\widetilde{u}_s)$$

in the grading

$$R^1(\operatorname{in}_x(I)) = R^1(\operatorname{in}_x(g_1), \ldots, \operatorname{in}_x(g_s)),$$

for a certain coordinates $\widetilde{u}_1, \ldots, \widetilde{u}_k \in \mathcal{R}^1(g_1, \ldots, g_s)$, and $\widetilde{f}_i \in \mathcal{R}^{d_i}(g_1, \ldots, g_s)$, such that

$$(\operatorname{in}_x(\widetilde{f}_i))(\overline{u}_1, \ldots, \overline{u}_s, 0, \ldots, 0) = \operatorname{in}_x(f_i)(u_1, \ldots, u_s, 0, \ldots, 0).$$

This implies that $(\widetilde{f}_i)$ satisfies (in particular) the condition (4) of the standard basis. Using it one can perform in the completion ring $\widehat{\mathcal{O}_{X,x}}$ the following Euclidean divison algorithm. Consider the function $\widetilde{f}_j$. Its initial form coincides with that of f_j. We shall modify $\widetilde{f}_j$ to get f_j with all intermediate steps performed in $\widehat{\mathcal{J}^{\mu_i}}(g_1, \ldots, g_s)_x$. We just need to eliminate all the higher degree monomials in $\widetilde{f}_j$ which are not in $\Delta \times \mathbb{N}^{n-s}$. Set $h_0 := \widetilde{f}_j$

For any natural s let $K[u_1, \ldots, u_n]_s$ be the vector space of the forms of degree s in $u_1, \ldots, u_n$ and $V_s \subset K[u_1, \ldots, u_n]_s$ be the vector space spanned by the ordered set of forms

$$V_s := \operatorname{span}\{u^\alpha \mid \alpha \in \Delta = \bigcup \Delta_i, |\alpha| = s\}.$$

Consider the natural projection

$$\pi_s : K[[u_1, \ldots, u_n]] \to V_s.$$

Let

$$T^s := [t_{\alpha,\beta}] = [\ldots, \pi_s(\widetilde{f}_\alpha), \ldots]$$

be the square matrix whose subscripts are labeled by the ordered set

$$\Delta^s =: \{\alpha \in \Delta, |\alpha| = s\}$$

and containing as $\alpha = \alpha' + \alpha_i$- column the vector $\pi_s(\widetilde{f}_\alpha)$, where $\widetilde{f}_\alpha := \widetilde{u}^{\alpha'} \widetilde{f}_i$, for $\alpha \in \Delta^s, \alpha = \alpha_i + \alpha' \in \Delta_i^s \subset \Delta^s$, with α_i the vertex of Δ_i. Then we show the Lemma

Lemma 9.0.5. *T^s are invertible*

Proof. The map between sets $u^\alpha \mapsto u^\alpha$ for $\alpha \in \Gamma^s$ and $u^\alpha \mapsto \pi_s(\widetilde{f}_\alpha)$, for $\alpha \in \Delta^s$ extends to a linear map

$$\overline{T}^s : K[u_1, \ldots, u_n]_s \to K[u_1, \ldots, u_n]_s$$

Moreover as in the proof of Theorem 1.0.3, $||\overline{T}^s - id||_\rho < 1$ and is invertible which implies that its matrix $\overline{T}^s = \begin{bmatrix} id_{\Gamma^s} & * \\ 0 & T^s \end{bmatrix}$ is invertible. The latter yields invertibility of the submatrix T^s. □

Let $(T^s)^{-1} := [r^s_{\alpha,\beta}]$.

For any $h = \sum_{(\alpha,\gamma) \in \mathbb{N}^s \times \mathbb{N}^{n-s}} c_{\alpha,\gamma} u^{\alpha,\gamma} \in K[[u_1, \ldots, u_n]] = \widehat{\mathcal{O}_{X,x}}$ put

$$\mu(h) := \inf\{|\alpha| + |\gamma|, |\gamma|, \gamma) \mid \quad \alpha \in \Delta, \quad c_{\alpha,\gamma} \neq 0\} = (\beta, \gamma)$$

(with lexicographic order) and let $s := |\beta|$, and $t = |\gamma|$. Then for

$$\overline{h}_1 := h_0 - c_{\beta,\gamma} u^{0,\gamma} \sum_{\alpha \in \Delta^s} r^s_{\beta,\alpha} \; \widetilde{f}_\alpha$$

we have $\mu(\overline{h_1}) > \mu(h)$.

Likewise since $\mathrm{in}_x(D_{u_\beta}h_0) = c_{\beta,\gamma}u^{0,\gamma} + \sum_{\gamma'>\gamma} a_{\gamma'}u^{0,\gamma}$ the same is true for the function

$$h_1 := h_0 - D_{u_\beta}h_0 \sum_{\alpha\in\Delta^s} r^s_{\alpha,\beta}\ \widetilde{f}_\alpha.$$

The latter function remains to be in $\widehat{\mathcal{J}^{\mu_i}}(g_1,\ldots,g_s)_x$ since $D_{u_\beta}h_0 \in \mathcal{J}^{\mu_i-s}$ and $\widetilde{f}_\alpha := \widetilde{u}^{\alpha'}\widetilde{f}_i \in \mathcal{J}^s(g_1,\ldots,g_s)_x$ for any $\alpha\in\Delta^s$.

This defines a convergent sequence $(h_n)\to h_\infty \in \widehat{\mathcal{J}^{\mu_i}}(g_1,\ldots,g_s)_x$. Moreover the function $h_\infty \in \widehat{\mathcal{I}_x}$ has a form

$$h_\infty = u^{\alpha_j} + r_\infty$$

, with $\mathrm{supp}(r_\infty)\in\Gamma\times\mathbb{N}^s$. Then $\overline{h} := h_\infty - f_j = r_\infty - r_j \in \widehat{\mathcal{I}_x}$, and $\mathrm{supd}(\overline{h})\subset\Gamma$. Likewise $\mathrm{in}(\overline{h})\in\mathrm{in}(\mathcal{I})$ has a support in Γ. Thus , by Corollary 4.0.8 we deduce that $\overline{h}=0$, and

$$f_j = u^{\alpha_j} + r_\infty = h_\infty \in \widehat{\mathcal{J}^{d_j}}(g_1,\ldots,g_s)_x,$$

Then $\widehat{\mathcal{J}^{d_j}}(f_1,\ldots,f_r)_x \subseteq \widehat{\mathcal{J}^{d_j}}(g_1,\ldots,g_m)_x$, and by symmetry

$$\widehat{\mathcal{J}^{d_j}}(f_1,\ldots,f_r)_x = \widehat{\mathcal{J}^{d_j}}(g_1,\ldots,g_m)_x.$$

The latter implies, by flatness that

$$\mathcal{J}^{d_j}(f_1,\ldots,f_r)_x = \mathcal{J}^{d_j}(g_1,\ldots,g_m)_x.$$

□

Definition 9.0.6. Let $\mathcal{I}$ be an ideal sheaf of finite type on a manifold M over a field K. Consider a Samuel stratum S of $\mathcal{I}$ on X, and a standard basis (f'_i, d'_i) of $\mathcal{I}$.

We shall call the (multiple) marked ideal $\mathcal{R}^{\cdot}(\mathcal{I}) = \mathcal{R}^{\cdot}(f'_1,\ldots,f'_k)$ (respectively $\mathcal{J}^{\cdot}(\mathcal{I})$) *the canonical Rees algebra algebra along Samuel stratum* (respectively *canonical Rees ideal*) of $\mathcal{I}$ along a Samuel stratum S.

Theorem 9.0.7 (Descending chain condition of the Hilbert-Samuel function)**.** *The set of values of the Hilbert-Samuel function (ordered lexicographically)*

$$\mathcal{H}(n) := \{H_{\mathcal{I}}(k) = \dim(K[x_1,\ldots,x_n]/(\mathcal{I}+m^{k+1})) \mid \mathcal{I}\subset K[x_1,\ldots,x_n])\}$$

is d.c.c. (satisfies the descending chain condition). In other words, any decreasing sequence of functions

$$H_1 \geq \ldots \geq H_n \geq \ldots \tag{2}$$

stabilizes: $H_s = H_{s+1} = \ldots$ for sufficiently large s.

Corollary 9.0.8 (Bennett [6])**.** *Let M be a manifold over a field K (or a compact analytic or differentiable manifold). Let $\mathcal{I}$ be a sheaf of ideals of finite type on M. Then the Hilbert-Samuel function $H_{x,\mathcal{I}}(k)$ of $\mathcal{I}$ on M is upper semicontinuous and attains only finitely many values on any compact subset Z. Consequently, there is a locally finite* Samuel decomposition *into locally closed strata such that two closed points are in the same stratum if they have the same Hilbert-Samuel function $H_{\mathcal{I},x}(s) = \dim_{K_x}\mathcal{O}_X/(m_x^{s+1}+\mathcal{I}_x)$.*

Corollary 9.0.9. *There exist only finitely many diagrams of initial exponents Δ in $\mathbb{N}^k$ having the same Hilbert-Samuel function.*

Corollary 9.0.10. *Let a be the maximal multiplicity of all possible vertices of all possible diagrams $\Delta\in\mathbb{N}^n$. It is finite by Corollary 9.0.9. Then its canonical capacitor is defined as $\overline{R} = R^{a!}$.*

10. Strong resolution of singularities

10.1. **Resolution of marked ideals.** The thorems can be derived as in the weaker case of Hironaka desingularization from the following result

Theorem 10.1.1. *For any marked ideal $(M_Z,\mathcal{I},E,\mu)$ such that $\mathcal{I}\neq 0$ there is an associated resolution $(M_{iZ_i}))_{0\leq i\leq m_M}$, called* <u>*canonical*</u>*, satisfying the following conditions:*

1. *For any surjective locally analytic isomophism $\phi : M'_{Z'} \to M_Z$ the induced sequence $(M'_{iZ'_i}) = \phi^*(M_{iZ_i})$ is the canonical resolution of $(M'_{Z'},\mathcal{I}',E',\mu) := \phi^*(M_Z,\mathcal{I},E,\mu)$.*
2. *For any locally analytic isomophism $\phi : M' \to M$ the induced sequence $(M'_{iZ_i}) = \phi^*(M_{iZ_i})$ is an extension of the canonical resolution of $(M'_{Z'},\mathcal{I}',E',\mu) := \phi^*(M_Z,\mathcal{I},E,\mu)$.*

Proof. The proof uses induction on two invariants: the order of the (nonmonomial part) of the ideal, and the dimension of the ambient manifold,. We refer the reader to [25] for details briefly sketch the resolution strategy.

The algorithm relies on modifications of marked ideals. There are two reasons for such modification. First, we reduce resolution of a marked ideal to a simpler one. Second, we modify the marked ideal in such a way that its resolution is equivalent to the resolution of the original ideal but it has better properties. Recall tha two multiple marked ideals are equivalent

$$(M_Z, \overline{\mathcal{I}}, E_{\overline{\mathcal{I}}}) \simeq (M_Z, \overline{\mathcal{J}}, E_{\overline{\mathcal{J}}},)$$

if $E_{\overline{\mathcal{I}}} = E_{\overline{\mathcal{J}}}$ and their cosupport remain equal for any sequences of admissible blow-ups (including trivial ones). If

$$\operatorname{cosupp}(\overline{\mathcal{I}}) \subseteq \operatorname{cosupp}(\overline{\mathcal{J}})$$

after any sequences of admissible blow-ups then we write

$$\overline{\mathcal{I}} \subseteq \overline{\mathcal{J}}$$

. The sum of marked ideals $(\mathcal{I}_i, \mu_i)$ is the marked ideal

$$\sum (\mathcal{I}_i, \mu_i) := ((\sum (\mathcal{I}_i^{a_i}), \mu)$$

, where $\mu := \operatorname{lcm}(\mu_i)$, $a_i := \mu/\mu_i$. The product of $(\mathcal{I}, \mu_{\mathcal{I}})$ and $(\mathcal{J}, \mu_{\mathcal{J}})$ is defined as

$$(\mathcal{I}, \mu_{\mathcal{I}}) \cdot (\mathcal{J}, \mu_{\mathcal{J}}) := (\mathcal{I} \cdot J, \mu_{\mathcal{I}} + \mu_{\mathcal{J}}).$$

Then

$$\sum (\mathcal{I}_i, \mu_i) \equiv \{(\mathcal{I}_i, \mu_i)\}, \quad \{(\mathcal{I}, \mu_{\mathcal{I}}), (\mathcal{J}, \mu_{\mathcal{J}})\} \subseteq (\mathcal{I}, \mu_{\mathcal{I}}) \cdot (\mathcal{J}, \mu_{\mathcal{J}}).$$

Also, by Lemma,

$$\mathcal{D}^a(\mathcal{I}) \subseteq \mathcal{I}$$

. The resolution runs on the simple inductive scheme. We assume that there exists a resolution of marked ideals on the manifolds of smaller dimension. The resolution algorithm consists of two steps and can be represented by the following scheme. The algorithm is presented here in the reverse order with respect to the proof.

Step 2. In this final step we resolve $(\mathcal{I}, \mu)$. The process is obtained by two steps:

Step 2a. We reduce $(\mathcal{I}, \mu)$ to a principal monomial marked ideal. In order to do this we write $\mathcal{I}$ as the product $\mathcal{I}$ of maximal principal monomial ideal $M(\mathcal{I})$ and decrease the maximal order $\mu_{N(I)}$ of the nonmonomial part $\mathcal{N}(\mathcal{I})$. This can be done by resolving the companion ideal:

$$O(\mathcal{I}, \mu) = \begin{cases} (\mathcal{N}(\mathcal{I}), \operatorname{ord}_{\mathcal{N}(\mathcal{I})}) + (\mathcal{M}(\mathcal{I}), \mu - \operatorname{ord}_{\mathcal{N}(\mathcal{I})}) & \text{if } \operatorname{ord}_{\mathcal{N}(\mathcal{I})} < \mu, \\ (\mathcal{N}(\mathcal{I}), \operatorname{ord}_{\mathcal{N}(\mathcal{I})}) & \text{if } \operatorname{ord}_{\mathcal{N}(\mathcal{I})} \geq \mu. \end{cases}$$

Note that $\operatorname{cosupp}(O(\mathcal{I}, \mu)) = \{x \in \operatorname{cosupp}(\mathcal{I}, \mu) \mid \operatorname{ord}_x(N(\mathcal{I}) = \mu_{N(I)}$. Thus resolving companion ideal will eliminate all the points with maximal order $\operatorname{ord}_x(N(\mathcal{I}) \geq = \mu_{N(I)}$. on $\operatorname{cosupp}(\mathcal{I}, \mu)$, and thus $\operatorname{ord}_x(N(\mathcal{I})$ will drop.

Step 1. In this step we resolve the companion ideal $(\mathcal{J}, \mu_{\mathcal{J}}) := O(\mathcal{I}, \mu)$.

Note that this ideal is of maximal order $(\mathcal{J}, \mu_{\mathcal{J}})$, and thus it satisfies the condition $\operatorname{ord}_x(\mathcal{J}) \leq \mu_{\mathcal{J}})$. Thus by Lemma the cosuport $(\mathcal{J}, \mu_{\mathcal{J}})$ is contained locally in the smooth submainfold (hypersurface of maximal contact) $V(u)$, where $u \in D^{\mu_{\mathcal{J}}-1}(\mathcal{J}), 1)$, is a function of multiplicity 1. This allows to reduce resolution of $(\mathcal{J}, \mu_{\mathcal{J}})$ to a submanifold of codimension one $V(u)$ and use the inductive assumption. There are 3 problems to solve.

Problem 1. The choice of the hypersurface if maximal contact is not unique.

We replace $(\mathcal{J}, \mu_{\mathcal{J}})$ with its homogenization Set $T(\mathcal{I}) := \mathcal{D}^{\mu_{\mathcal{J}}-1}\mathcal{I}$. By the *homogenized ideal* we mean

$$\mathcal{H}(\mathcal{J}, \mu) := (\mathcal{H}(\mathcal{J}), \mu) = (\mathcal{J} + \mathcal{D}\mathcal{J} \cdot T(\mathcal{J}) + \ldots + \mathcal{D}^i \mathcal{J} \cdot T(\mathcal{J})^i + \ldots + \mathcal{D}^{\mu-1} \mathcal{J} \cdot T(\mathcal{J})^{\mu-1}, \mu).$$

Observe that, by the above properties $\mathcal{J} \simeq \mathcal{H}(\mathcal{J}, \mu)$.

Lemma 10.1.2. *(Gluing lemma). $u, v \in T(\mathcal{I}, \mu)_x = \mathcal{D}^{\mu-1}(\mathcal{I}, \mu)$ at $x \in \operatorname{supp}(\mathcal{I}, \mu)$. Then there exists an open neighborhood V of x such that $\overline{V}$ is compact and an automorphism ϕ_{uv} of M_S where $S := Z \cap \overline{V} \cap \operatorname{cosupp}(\mathcal{I}, \mu)$ such that $\phi_{uv}^*(\mathcal{H}\mathcal{I})_{|M_S} = (\mathcal{H}\mathcal{I})_{M_S}$. and $\phi_{uv}^*(u) = v$. $\operatorname{supp}(\mathcal{I}, \mu) := V(T(\mathcal{I}, \mu))$ is contained in the fixed point set of ϕ.*

Find parameters $u_2, \ldots, u_n$ transversal to u and v such that $u = u_1, u_2, \ldots, u_n$ and $v, u_2, \ldots, u_n$ form two sets of parameters at x and divisors in E are described by some parameters u_i where $i \geq 2$. Set

$$\phi_{uv}(u_1) = v, \quad \phi_{uv}(u_i) = u_i \quad \text{for} \quad i > 1.$$

The morphism $\phi_{uv} : U \to U'$ defines an open embedding from some neighborhood U of x to another neighborhood U' of x.

Let $h := v - u \in T(\mathcal{I})$. For any $f \in \mathcal{I}$,

$$\phi_{uv}^*(f) = f(u_1 + h, u_2, \ldots, u_n) = f(u_1, \ldots, u_n) + \frac{\partial f}{\partial u_1} \cdot h + \frac{1}{2!}\frac{\partial^2 f}{\partial u_1^2} \cdot h^2 + \ldots + \frac{1}{i!}\frac{\partial^i f}{\partial u_1^i} \cdot h^i + \ldots$$

The latter element belongs to

$$\mathcal{I} + \mathcal{D}\mathcal{I} \cdot T(\mathcal{I}) + \ldots + \mathcal{D}^i\mathcal{I} \cdot T(\mathcal{I})^i + \ldots + \mathcal{D}^{\mu-1}\mathcal{I} \cdot T(\mathcal{I})^{\mu-1} = \mathcal{H}\mathcal{I}.$$

By the construction the map preserves the ideal $T(\mathcal{I})$ describing the cosupport. Thus its restriction to the support is identical.

Also, for any $y \in \mathrm{cosupp}(\mathcal{I})$, $\phi_{uv}^*(\mathcal{I})_y \subset \mathcal{H}\mathcal{I}_y$. Analogously $\phi_{uv}^*(\mathcal{D}^i\mathcal{I})_y \subset \mathcal{D}^i\mathcal{I}_y + (\mathcal{D}^{i+1}\mathcal{I} \cdot T(\mathcal{I})_y + \ldots + (\mathcal{D}^{\mu-1}\mathcal{I} \cdot T(\mathcal{I})^{\mu-i-1}) = (\mathcal{H}\mathcal{D}^i I)_y$. Thus

$$\phi_{uv}^*(\mathcal{H}\mathcal{I}_y \subset \mathcal{H}\mathcal{I}_y.$$

Since the stalks of the ideals are noetherian this implies the equality of stalks $\phi_{uv}^*(\mathcal{H}\mathcal{I}_y = \mathcal{H}\mathcal{I}_y$, and sheaves in the neighborhood of S.

Problem 2. In order to restrict the algorithm of resolving $\mathcal{J}$ to the hypersurface of maximal contact $V(u)$ (or any other smooth subvariety) one needs to find the ideal $\mathcal{J}'$ on $V(u)$ such that the resolution of $\mathcal{J}'$ will determine the resolution of $\mathcal{J}$.

In order to resolve the problem we introduce the *coefficient ideal* of any ideal $\mathcal{I}, \mu$ of maximal order:

$$\mathcal{C}(\mathcal{I}, \mu) = (\mathcal{I}, \mu) + (\mathcal{D}\mathcal{I}, \mu - 1) + \ldots + (\mathcal{D}^{\mu-1}\mathcal{I}, 1).$$

Lemma 10.1.3. (1) *$\mathcal{C}(\mathcal{I})$ is equivalent to $\mathcal{I}$.*

(2) *The intersection of the support of $(\mathcal{I}, \mu)$ with any submanifold S is the support of the restriction of $\mathcal{C}(\mathcal{I})$ to S:*

$$\mathrm{supp}(\mathcal{I}) \cap S = \mathrm{supp}(\mathcal{C}(\mathcal{I})_{|S}).$$

Moreover this condition is persistent under sequences of admissible blow-ups.

(1) follows from the above porperties of marked ideals

(2) Let $x_1, \ldots, x_k, y_1, \ldots, y_{n-k}$ be local parameters at p such that $\{x_1 = 0, \ldots, x_k = 0\}$ describes S. Then write a function $f \in \mathcal{I}$ can be written as

$$f = \sum c_{\alpha f}(y) x^\alpha.$$

Now $x \in \mathrm{supp}(\mathcal{I}, \mu) \cap S$ iff $\mathrm{ord}_x(c_{\alpha,f}) \geq \mu - |\alpha|$ for all $f \in \mathcal{I}$ and $0 \leq |\alpha| < \mu$. Note that

$$c_{\alpha f|S} = \left(\frac{1}{\alpha!}\frac{\partial^{|\alpha|}(f)}{\partial x^\alpha}\right)_{|S} \in \mathcal{D}^{|\alpha|}(\mathcal{I})_{|S}$$

and consequently $\mathrm{supp}(\mathcal{I}, \mu) \cap S = \bigcap_{f \in \mathcal{I}, |\alpha| \leq \mu} \mathrm{supp}(c_{\alpha f|S}, \mu - |\alpha|) \supseteq \bigcap_{0 \leq i < \mu} \mathrm{supp}((\mathcal{D}^i\mathcal{I})_{|S}) = \mathrm{supp}(\mathcal{C}(\mathcal{I}, \mu)_{|S})$.

Replace $\mathcal{J}$ with $\overline{\mathcal{J}} := \mathcal{C}(\mathcal{H}(\mathcal{J})) \simeq \mathcal{J}$. (*)

Problem 3. The restriction of existing exceptional divisors to hypersurface of maximal contact will create analytic subspaces which are not smooth. The existing, so called, "old" divisors need to be separated from the cosupport of marked ideal. The new exceptional divisors are created from the smooth center on the maximal constact $V(u)$. This ensures that they are transversal to $V(u)$, and their restrictions to $V(u)$ are SNC divisors.

In Step 1 we replace $\mathcal{H}(\mathcal{J})$ with $\mathcal{C}(\mathcal{H}(\mathcal{J}))$. This addresses the problems 1, and 2.

Step 1a. Move apart all strict transforms of $E = \bigcup E_i$ and $\mathrm{supp}(\overline{\mathcal{J}}, \mu)$ by resolving $(\overline{\mathcal{J}}_{|E_i}, \mu)$ one by one (according the order).

Step 1b If the strict transforms of E do not intersect $\mathrm{supp}(\overline{\mathcal{J}}, \mu)$, resolve $(\overline{\mathcal{J}}, \mu)$. For simplicity we drop the compact supports in the considerations below.

We can now restrict $(\overline{\mathcal{J}}, \mu)$ locally to the hypersufacaces of maximal contact $V(u)$. The created new exceptional divisors are now transversal to the maximal contacts $H_\alpha := V(u_\alpha)$ on open neighborhoods U_α. Consider the analytic manifold $\widetilde{M} := \coprod U_\alpha$ and its closed submanifold $\widetilde{H} = \coprod U_\alpha \subset \widetilde{M}$, and the induced sheaf $\overline{\mathcal{J}}_{\widetilde{H}} := \coprod_\alpha \overline{\overline{\mathcal{J}}}_{|H_\alpha}$.

By the inductive assumption there exists a canonical resolution $\widetilde{H}_i$ of $\overline{\mathcal{J}}_{\widetilde{H}}$ with centers $C_i = \coprod C_{i\alpha}$ on $\widetilde{H}_i$. Let $\pi : \widetilde{M} \to M$ be the natural projection. We need to show that the resolution $(\widetilde{H}_i)$ and (induced resolution $(\widetilde{M}_i)$ descends to (M_i). In other words we shall prove that the centers $C_i := \coprod C_{i\alpha}$, descend to the smooth centers $\overline{C}_i$ on M_i. Here $C_{i\alpha} \subset U_{i\alpha}$ are smooth closed (possibly empty) centers on the induced local resolutions $(U_{i\alpha}) \subset (\widetilde{M}_i)$. It suffice to show that $\pi(C_i) \cap U_{i\alpha} = C_{i\alpha}$. If $x \in \pi(C_i) \cap U_{i\alpha}$ then $x \in \pi(C_{i\beta}) \cap U_{i\alpha} \subset U_{i\alpha} \cap U_{i\beta} := U_{i\alpha} \cap U_{i\beta}$. Let $H_{i\alpha\beta} := H_{i\alpha} \cap H_{i\beta}$.

The canonical resolution $(\widetilde{H}_i)$ restricts to the canonical resolution on its open subsets $H_{i\alpha\beta}$ and $H_{i\beta\alpha}$. The latter are related by the lifting isomorhism $\phi_{i\alpha\beta}$ of $\phi_{\alpha\beta}$. Thus by canonicity of resolutions $H_{i\alpha\beta}$ and $H_{i\beta\alpha}$ (with respect to surjective local isomorphsisms) we get

$$C_{i\beta} \cap U_{i\alpha\beta} = \phi_{i\alpha\beta}(C_{i\alpha} \cap U_{i\beta\alpha}) = C_{i\alpha} \cap U_{i\beta\alpha}$$

, since $\phi_{i\alpha\beta}$ is identical on the cossuport of $\overline{J}_{|H_{i\alpha\beta}}$. This shows that

$$\pi(C_{i\beta}) \cap U_{i\alpha} = C_{i\beta} \cap U_{i\alpha\beta} = C_{i\alpha} \cap U_{i\alpha\beta} \subset C_{i\alpha}$$

Consequently $\pi(C_i) \cap U_{i\alpha} = C_{i\alpha}$ is smooth and the centers C_i on $(\widetilde{H}_i)$ descend to the smooth centers $\pi(C_i)$ on M_i.

The induced canonical resolution (M_i) of $\overline{\mathcal{J}}$ completes the Step 1b.

Step 2b. Resolve the monomial marked ideal $\mathcal{I} = \mathcal{M}(\mathcal{I})$. The marked ideal $\mathcal{I}$ is locally descibed by the equation (x^α, μ). Its cosupport is a union of maximal components

$$C_I := \{x_i = 0 \mid i \in I\}$$

, with $I \subset \{1, \ldots, n\}$ such that $\sum_{i\in I} \alpha_i \geq \mu$ and for any proper subset $I' \subset I$ we have that $\sum_{i\in I'} \alpha_i < \mu$. The blow-ups of maximal components creates a new marked ideal of the smaller order ($|\alpha|$ drops) and leads to the simple combinatorial resolution algorithm.

□

10.2. **Canonical embedded desingularization of germs of analytic spaces.**

Theorem 10.2.1. *Let M_Z be a germ of an analytic manifold and Y_T be a germ of analytic subspace $Y \subset M$, where $T = Z \cap Y$. There exists an embedded desingularization of $Y_Z \subset M_Z$ that is, a finite sequence*

$$M_Z = M_{0,Z_0} \xleftarrow{\sigma_1} M_{1Z_1} \xleftarrow{\sigma_2} M_{2,Z_2} \longleftarrow \ldots \longleftarrow M_{i,Z_i} \longleftarrow \ldots \longleftarrow M_{r,Z_r} = \widetilde{M}_{\widetilde{Z}}$$

of blow-ups with smooth centers $C_{i-1} \subset M_{i-1,Z_{i-1}}$ such that

1. *The exceptional divisor E_i of the induced morphism $\sigma^i = \sigma_1 \circ \ldots \circ \sigma_i : U_i \to U$ has only simple normal crossings and C_i has simple normal crossings with E_i.*
2. *The strict transform $\widetilde{Y}_{\widetilde{Z}} := Y_{r,Z_r}$ of Y_Z is smooth and has only simple normal crossings with the exceptional divisor E_r.*
3. *The morphism $(M_Z, Y_Z) \leftarrow (\widetilde{M}_{\widetilde{Z}}, \widetilde{Y}_{\widetilde{Z}})$ defined by the embedded desingularization commutes with locally analytic isomorphisms, embeddings of ambient manifolds.*

Proof. Since the Hilbert-Samuel function has a dcc property, it admits a finite decomposition into locally closed Samuel strata along a compact set. Consider the maximum value (with respect to lexicographic order) of the H-S function. One can assign the canonical marked ideal (capacitor of Rees algebra along the stratum. The latter can be resolved (along the compact subset) in a canonical way. The resolution leads to a new ideal with a lower maximum value of HS-function. We run the algorithm until we reach the lowest possible value of HS function on one of the components. The value of HS function on that component will coincide with the value of the HS-function at smooth generic points of the components. The HS-function will determine that the ideal of the strict transform along that component is generated by the (independent) local parameters, and thus the component will become smooth and (thus) disjoint from other copmponents. We will continue the resolution algorithm for remaining components until all all the components are smooth and disjoint. □

10.3. **Locally Finite Embedded Desingularization.**

Theorem 10.3.1. *Let Y be an analytic subspace of an analytic manifold M. There exists a manifold $\widetilde{M}$ a simple normal crossing divivisor $\widetilde{E}$ on $\widetilde{M}$ and a birational projective morphism $\mathrm{res}_{Y,M} : \widetilde{M} \to M$ such that the strict transform $\widetilde{Y} \subset \widetilde{M}$ is smooth and have simple normal crossings with the divisor E. The support $|E|$ of the divisor E is the inverse image of the exceptional locus of $\mathrm{res}_{Y,M}$. The morphism $\mathrm{res}_{Y,M}$ locally factors into a sequence of blow-ups. That is for any compact set $Z \subset Y$ there is an open subset $U \subset M$ and $\widetilde{U} = \mathrm{res}^{-1}_{Y,M}(U) \subset \widetilde{M}$. There exists a sequence*

$$U_0 = U \stackrel{\sigma_1}{\longleftarrow} U_1 \stackrel{\sigma_2}{\longleftarrow} U_2 \longleftarrow \ldots \longleftarrow U_i \longleftarrow \ldots \longleftarrow U_r = \widetilde{U}$$

of blow-ups $\sigma_i : U_{i-1} \longleftarrow U_i$ with smooth centers $C_{i-1} \subset U_{i-1}$ such that

1. *The exceptional divisor E_i of the induced morphism $\sigma^i = \sigma_1 \circ \ldots \circ \sigma_i : U_i \to U$ has only simple normal crossings and C_i has simple normal crossings with E_i.*
2. *Let $Y_i \subset U_i$ be the strict transform of Y. All centers C_i are disjoint from the set $\mathrm{Reg}(Y_i) \subset Y_i$ of points where Y is smooth.*
3. *The strict transform $\widetilde{Y} := Y_r$ of Y is smooth and has only simple normal crossings with the exceptional divisor E_r.*
4. *The morphism $(M,Y) \leftarrow (\widetilde{M},\widetilde{Y})$ defined by the embedded desingularization commutes with locally analytic isomorphisms, embeddings of ambient varieties and (separable) ground field extensions.*
5. *If $Z_1 \subset Z_2$ are compact sets and $U_1 \subset U_2$ are corresponding open neighborhoods such that $U_1 \subset U_2$ then the restriction of the factorization of blow-ups of $\mathrm{res}_{Y,M|\widetilde{U_2}} : \widetilde{U}_2 \to U_2$ to $\widetilde{U_1}$ determines the factorization of $\mathrm{res}_{Y,M|\widetilde{U_1}} : \widetilde{U}_1 \to U_1$.*

Proof. Consider any locally finite cover of M by compact neighborhoods. By the previous theorem the canonical resolutions over compact sets glue to give the lcoally finite embedded resolution $(\widetilde{M},\widetilde{Y})$ of (M,Y). □

10.4. **Canonical desingularization of germs analytic spaces.**

Theorem 10.4.1. *Let M_Z be a germ of an analytic manifold and Y_T be a germ of analytic subspace $Y \subset M$, where $T = Z \cap Y$. There exists an embedded desingularization of $Y_Z \subset M_Z$ that is, a finite sequence*

$$M_Z = M_{0,Z_0} \stackrel{\sigma_1}{\longleftarrow} M_{1Z_1} \stackrel{\sigma_2}{\longleftarrow} M_{2,Z_2} \longleftarrow \ldots \longleftarrow M_{i,Z_i} \longleftarrow \ldots \longleftarrow M_{r,Z_r} = \widetilde{M}_{\widetilde{Z}}$$

of blow-ups with smooth centers $C_{i-1} \subset M_{i-1,Z_{i-1}}$ such that

1. *The exceptional divisor E_i of the induced morphism $\sigma^i = \sigma_1 \circ \ldots \circ \sigma_i : U_i \to U$ has only simple normal crossings and C_i has simple normal crossings with E_i.*
2. *The strict transform $\widetilde{Y}_{\widetilde{Z}} := Y_{r,Z_r}$ of Y_Z is smooth and has only simple normal crossings with the exceptional divisor E_r.*
3. *The morphism $(M_Z,Y_Z) \leftarrow (\widetilde{M}_{\widetilde{Z}},\widetilde{Y}_{\widetilde{Z}})$ defined by the embedded desingularization commutes with locally analytic isomorphisms, embeddings of ambient manifolds.*

Proof. Let Y_Z be a germ of an analytic space. Let S be the Samuel stratum of Y in a neighborhood of Z corresponding to the maximum value $\mathcal{H}_{Y,y}$ on Z.

Every point of $y \in Y$ has a neighborhood V which is locally isomorphic to a closed analytic subset of an open ball $U \subset \mathbb{A}^n$. The parameters $u_1, \ldots u_n$ on Y define a minimal embedding $Y \supset V \to U$ into an open subset U of $\mathbb{A}^n$. Let $Z \subset V = Y \cap U$ be a compact set. Then Y_Z can be identified with V_Z. Observe that the dimension of minimal embedding is deteremined by the Hilbert-Samuel functions and is constant along the staratum.

Consider the canonical resolution of marked ideals of the capacitor of Reee algebra $(\widetilde{U_Z},\widetilde{Y_Z}) \to (U_Z,Y_Z)$. Two minimal embeddings $\phi_1 : Z \subset V_1 \to U_1 \supset Z_1 = \phi_1(Z)$ and $\phi_2 : Z \subset V_2 \to U_2 \supset Z_2 = \phi_2(Z)$ of two different open subsets V_1, V_2 containing Z are defined by two different sets of parameters $u_1, \ldots u_n$ and $u'_1, \ldots u'_n$ differ by an isomorphism

$$\psi := \phi_2^{-1}\phi_1 : (U_{1Z_1}, (\phi_1(V_1)_{Z_1}) \to (U_{2Z_2}, (\phi_2(V_2)_{Z_2})$$

mapping coordinates $x_1, \ldots, x_n$ to $x'_1, \ldots, x'_n$. Note that both $\phi_1(V_1)_{Z_1}$ and $\phi_2(V_1)_{Z_2}$ can be identified with $\widetilde{Y}_{\widetilde{Z}}$. The isomorphism ψ, by canonicity, lifts to the isomorphisms between resolutions of marked ideals. It thus defines a unique sequence of blow-ups of smooth centers contained in the Samuel stratum on the disjoint union $\Pi V_{i,Z}$ which descends to a unique sequnce on the $Y_{j,Z}$ resulting in dropping the maximum value of the HS-function. We repeat those steps until we reach the minimum value of the H-S function on one of the (finitely many components) $\widetilde{Y}^1$. Then the H-S function on the component indicates that that Y^1 is smooth in a neighborhood of the compact set, and it is disjoint from other support. We continue the process for the remaining set of components until all become smooth and disjoint.

Let Y_Z denote the analytic germ of Y at Z. Consider an open cover of Z with the open subsets $V_i \subset W_i \subset U_i$ of Y, such that $\overline{V_i} \subset W_i$ and $\overline{V_i} \subset U_i$ are compact and U_i is isomorphic to an open balls in $\mathbb{A}^n$ as above. Set $S_i := \overline{V_i}$, $Z_i := \overline{W_i} \cap Z$.

It follows from the definition that it commutes with locally analytic isomorphisms and open embeddings. □

10.5. **Canonical Resolution of Singularities.**

Theorem 10.5.1. *Let Y be an analytic manifold. There exists a canonical desingularization of Y that is a manifold $\widetilde{Y}$ together with a sequence of blow-ups of smooth normally flat centers $\mathrm{res}_Y : \widetilde{Y} \to Y$ which is functorial with respect to locally analytic isomorphisms. For any local analytic isomorphism $\phi : Y' \to Y$ there is a natural lifting $\widetilde{\phi} : \widetilde{Y'} \to \widetilde{Y}$ which is a local analytic isomorphism.*

In particular $\mathrm{res}_Y : \widetilde{Y} \to Y$ is an isomorphism over the nonsingular part of Y.

Proof. Let Y be an analytic space. Consider an open cover $\{U_i\}_{i\in I}$ of Y, such that $Z_i := \overline{U_i}$ are compact. For every i let $\mathrm{res}_i : \widetilde{Y}_{Z_i} \to Y_{Z_i}$ be the canonical desingularization of the germ Y_{Z_i}. Let $\widetilde{U}_i := \mathrm{res}_i^{-1}(U_i) \to U_i$ be its restriction. As before we define $\widetilde{Y}$ to be a manifold obtained by gluing $\widetilde{U}_i$ along $\widetilde{U}_{ij}$. Then $\mathrm{res}_Y : \widetilde{Y} \to Y$ is a proper bimeromorphic morphism. Moreover for any compact $Z \subset Y$, the morphism $\widetilde{Y}_{\widetilde{Z}} \to Y_Z$ is the canonical desingularization of germ Y_Z. □

11. Weierstrass-Hironaka for smooth convergent functions

The technique of Weierstass-Hironaka division with respect to the norm can be applied to some classes of smooth functions. Recallt that Weierstass division for smooth functions (Malgrange-Mather divsion) is not unique and the proofs are not easy. In this paper we show existence of much more general Weierstass-Hironaka division defined for special class of smooth convergent functions. It is more convenient to use Hasse dervations in the considerations below. Set:

$$D_{u^\alpha}(f) := \frac{1}{\alpha!}\frac{\partial^\alpha f}{\partial u^\alpha}$$

It follows from the definition that

$$D_{u^\alpha} D_{u^\beta} = \binom{\alpha+\beta}{\alpha} D_{u^{\alpha+\beta}}.$$

Definition 11.0.1. Let $D \subset \mathbb{R}^n$ be a compact set. Denote by $C^0(D)$ the Banach algebra with the supremum norm

$$||f||_D := \sup\{f(x) \mid x \in D\}.$$

A smooth function $f \in C^\infty(D)$ is called *convergent* on D if there exists $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$, where $\epsilon_i > 0$ such that

$$\sum ||D_{u^\alpha}(f)||_D \cdot \epsilon^\alpha$$

converges.

A smooth function $f \in C^\infty_x(\mathbb{R}^n)$ is *convergent at a point* $x \in \mathbb{R}^n$ if it is convergent on a certain compact neighborhood D of x. We shall denote the ring of germs of convergent smooth functions at x by $C^\infty_x\{\mathbb{R}^n\}$, and the ring of smooth functions on D by $(C^\infty(D))$.

It follows from Lemma 2.0.2(2) that $\mathbb{R}$-analytic functions are smooth convergent.

Let $(C^\infty(D))^\infty_\epsilon$ denote the ring of smooth convergent functions on D with respect to the ϵ- norm

$$||f||^\infty_{D,\epsilon} := \sum ||D_{u^\alpha}(f)||_D \cdot \epsilon^\alpha.$$

Then $(C^\infty(D))^\infty_\epsilon$ is a Banach space.

Consider the Banach algebra

$$C^0(D)\{X_1,\ldots,X_n\}_\epsilon$$

of convergent power series over the Banach algebra $C^0(D)$ with respect to the norm defined for $\rho=\epsilon$. (We use here notation from Section 1.1.) Immediately from the definition we conclude:

Lemma 11.0.2. *Let $T:(C^\infty(D))^\infty_\epsilon\to C^0(D)\{X_1,\ldots,X_n\}_\epsilon$,*

$$T(f):=\sum D_{u^\alpha}(f)X^\alpha$$

denote the Taylor homomorfism of Banach algebras. Then T defines a closed immersion of Banach algebras, preserving the norm. In particular $(C^\infty(D))^\infty_\epsilon$ can be identified with Banach subalgebra of $C^0(D)\{X_1,\ldots,X_n\}_\epsilon$

□

Lemma 11.0.3. *If $f\in(C^\infty(D))^\infty_\epsilon$ then $T(f)\in(C^\infty(D))^\infty_{\epsilon'}[[X_1,\ldots,X_n]]_{\epsilon'}$, where $\epsilon'\le\epsilon/2$, and has a finite norm*

$$||T(f)||_{D,\epsilon'}=||f||^\infty_{D,2\epsilon'}$$

Proof.

$$||T(f)||_{D,\epsilon'}=\sum_\alpha||D_{u^\alpha}(f)||^\infty_{D,\epsilon'}\cdot(\epsilon')^{|\alpha|}=$$

$$\sum_\alpha\sum_\beta||D_{u^\alpha}D_{u^\beta}(f)||_D\cdot(\epsilon')^{|\alpha|+|\beta|}=\sum_{\alpha,\beta}\binom{\alpha+\beta}{\alpha}\sum_{\alpha+\beta}||D_{u^{\alpha+\beta}}(f)||_D\cdot(\epsilon')^{|\alpha|+|\beta|}$$

Substituting $\gamma:=\alpha+\beta$, $\sum_{\alpha+\beta=\gamma}\binom{\alpha+\beta}{\alpha}=2^{|\gamma|}$ yields

$$||T(f)||_{D,\epsilon'}=2^{|\gamma|}\sum_\gamma||D_{u^\beta}(f)||_D\cdot(\epsilon')^{|\gamma|}=\sum_\gamma||D_{u^\beta}(f)||_D\cdot(\epsilon')^{|\gamma|}\cdot(2\epsilon')^{|\gamma|}=||f||^\infty_{D,2\epsilon'}.$$

□

Lemma 11.0.4. *Let $T:C^\infty(D)\to C^\infty(D)[[X_1,\ldots,X_n]]$ be a Taylor homomorphism. If $g\in C^\infty(D)[[X_1,\ldots,X_n]]$ then $g=T(f)$ for some $f\in C^\infty(D)$ iff $(D_{u_i}-D_{X_i})(g)=0$ for $i=1,\ldots,n$.*

Proof. Denote by $e_i=(0,\ldots,1,\ldots0$, with 1 as the i-th component, the vector of the standard basis of $\mathbb{N}^n$. Let $g=\sum c_\alpha X^\alpha$. If $(D_{u_i}-D_{X_i})(g)=0$ then

$$D_{u_i}(c_\alpha)=(\alpha_i+1)(c_{\alpha+e_i}),$$

where $\alpha+e_i=(\alpha_1,\ldots,\alpha_i+1,\ldots,\alpha_n)$. Thus

$$\frac{1}{\alpha_i+1}D_{u_i}(c_\alpha)=c_{\alpha+e_i}.$$

This implies, by induction, that

$$\frac{1}{\alpha!}\frac{\partial^\alpha c_0}{\partial u^\alpha}=c_\alpha.$$

That is $g=T(f)$ for $f=c_0$. Conversely if $g=T(f)$ then $c_\alpha=D_{u^\alpha}(f)$. On the other hand the coefficients of $D_{X_i}(g)$ are equal $c'_\alpha=(\alpha_i+1)c_{\alpha+e_i}$, and are equal to the coefficients

$$c''_\alpha=D_uD_{u^\alpha}(f)=(\alpha_i+1)D_{u^{\alpha+e_i}}(f)=(\alpha_i+1)c_{\alpha+e_i}$$

of $D_{u_i}(g)$. □

Corollary 11.0.5. *The Taylor homomorphism*

$$T:(C^\infty(D))^\infty_\epsilon\to(C^\infty(D))^\infty_{\epsilon/2}\{X_1,\ldots,X_n\}_{\epsilon/2},$$

determines an is isomorphism of the Banach space $(C^\infty(D))^\infty_\epsilon$ with a closed Banach subspace of $(C^\infty(D))^\infty_{\epsilon/2}\{X_1,\ldots,X_n\}_{\epsilon/2}$, defined by the equations $(D_{u_i}-D_{X_i})(g)=0$ for $i=1,\ldots,n$.

We can extend the definition of the dominating weights and the initial exponents in the case of smooth functions.

Definition 11.0.6. Let $u = (u_1, \dots, u_n)$ be a coordinate system at $x \in \mathbb{R}^n$.

Let $f \in C^\infty(D)^\infty_\epsilon$ be a smooth convergent function at $x \in D$, and be $T_x(f) \in R\{u_1, \dots, u_n\}$ the the Taylor series of f at x. We say that α dominates f respect to ϵ at the point x , and write $\exp_{x,\epsilon}(f) = \alpha$ if $\exp_\epsilon(T_x(f)) = \alpha$, and $\mathrm{ord}_x(f) = |\alpha|$, where

Lemma 11.0.7. *Let $u = (u_1, \dots, u_n)$ be a coordinate system at $x \in \mathbb{R}^n$. Assume $\exp_{x,\epsilon}(f) = \alpha$ for $f \in C^\infty_x(D)_\epsilon$, where D is a compact neighborhood of x.*

Then there is a sufficiently small compact neighborhood $D' \subset D$ of x such that

$$T(f) \in (C^\infty(D'))^\infty_{\epsilon/2}\{X_1, \dots, X_n\})_{\epsilon/2,D'}$$

and the weight

$$\alpha = \exp_{\epsilon/2,D'}(T(f))$$

is dominating for $T(f)$ with respect to the norm defined for $\rho = \epsilon/2$.

Proof. For any $G \in (C^\infty(D))^\infty_\epsilon\{X_1, \dots, X_n\})_\epsilon$ let $||G||_{D,\epsilon}$ denote the induced norm. Then

$$T(u^\alpha) = X^\alpha + \sum_{\beta<\alpha} ||D_{u^\beta}(u^\alpha)||_D \cdot (X)^\beta$$

Since $D_{u^\beta}(u^\alpha)(x) = 0$ its norm $||D_{u^\beta}(u^\alpha)||$ can be arbitrarily small when D gets small we have that

$$\lim_{D\to 0} \frac{||T(u^\alpha) - X^\alpha||_{D,\epsilon}}{||X^\alpha||_{D,\epsilon}} = 0$$

Similarly

$$\lim_{D\to 0} \{||D_{u^\alpha}(f) - D_{u^\alpha}(f)(x)||_{D,\epsilon} = 0$$

Assume $\exp_{x,\epsilon,}(f) = \alpha$. Thus using Lemma 11.0.3 we get

$$\lim_{D\to 0} \frac{||T(f) - D_{u^\alpha}(f) \cdot X^\alpha||_{D,\epsilon/2}}{||D_{u^\alpha}(f) \cdot X^\alpha||_{D,\epsilon/2}} = \lim_{D\to 0} \frac{||T(f) - D_{u^\alpha}(f)(x) \cdot X^\alpha||_{D,\epsilon/2}}{||D_{u^\alpha}(f)(x) \cdot X^\alpha||_{D,\epsilon/2}} =$$

$$\lim_{D\to 0} \frac{||T(f) - T(D_{u^\alpha}(f)(x) \cdot u^\alpha)||_{D,\epsilon/2}}{||D_{u^\alpha}(f)(x)T(u)^\alpha||_{D,\epsilon/2}} = \lim_{D\to 0} \frac{||f - D_{u^\alpha}(f)(x)u^\alpha||_{D,\epsilon}}{||D_{u^\alpha}(f)(x) \cdot u^\alpha||_{D,\epsilon}} =$$

$$= \frac{||T_x(f) - D_{u^\alpha}(f)(x)u^\alpha||_\epsilon}{||D_{u^\alpha}(f)(x) \cdot u^\alpha||_\epsilon} < 1.$$

□

Definition 11.0.8. Let $f \in C^\infty(D)$ be a smooth function on $D \ni x$, and let $\overline{T} = (T_1, \dots, T_r)$ be an r-tuple of nonnegative functionals defined on $\mathbb{N}^n$ then we set

$$\exp_{x,\overline{T}}(f) := \exp_{\overline{T}}(T_x(f)),$$

where $T_x(f) = T(f)(x)$ is a Taylor expansion of f at x.

Corollary 11.0.9. *For sufficiently small $\epsilon > 0$, and a compact neighborhhod D we have*

$$\exp_{x,\overline{T}}(f) = \exp_{x,\epsilon}(f) = \exp_{\epsilon/2,D}(T(f))$$

□

Theorem 11.0.10. *Let $f_1, \dots, f_k \in C^\infty_x\{\mathbb{R}^{n+m}\}$ be smooth convergent functions, $\epsilon > 0$. Consider the diagram of initial exponents for*

$$\alpha_1 := \exp_{x,\epsilon}(f_1(u, 0)), \dots, \alpha_k := \exp_{x,\epsilon}(f_r(u, 0)) \in \mathbb{N}^n.$$

(1) *Then for every $g \in C^\infty_x\{\mathbb{R}^{n+m}\}$, there exist unique $h_i \in C^\infty_x\{\mathbb{R}^{n+m}\}$ and $r(g) \in C^\infty_x\{\mathbb{R}^{n+m}\}$ such that $\mathrm{supp}(h_i) \subset \Gamma_i$, $\mathrm{supp}(r) \subset \Gamma_0$, and*

$$g = \sum h_i f_i + r(g)$$

(2) *Moreover, if* $\mathrm{ord}_x(f_1(u,0)) = |\alpha_i|$ *for any* i *(respectively* $\mathrm{ord}_x(f_i) = |\alpha_i|$*) we have that*

$$\mathrm{ord}_x(r(g)(u,0)) \geq \mathrm{ord}_x(g(u,0)),$$

$$\mathrm{ord}_x(h_i(u,0)) \geq \mathrm{ord}_x(g(u,0)) - |\alpha_i|$$

or respectively

$$\mathrm{ord}_x(r(g)) \geq \mathrm{ord}(g)), \quad \mathrm{ord}_x(h_i) \geq \mathrm{ord}_x(g) - |\alpha_i|$$

Proof. By definition, we can assume that all the functions $f_i \in (C^\infty(D))^\infty_{2\epsilon}$ are convergent on a certain compact neighborhood D for a certain $\epsilon > 0$. Then by decreasing ϵ, if necessary, and using Corollary 11.0.3 we get that $T(f_i) \in (C^\infty(D))^\infty_\epsilon\{X_1,\ldots,X_n\}_\epsilon$ for all i, and sufficiently small ϵ.

By Lemmas 11.0.9, and 2.0.2(2), there exists a a sufficiently small ϵ-norm such that $\exp_\epsilon T(f_i) = \alpha_i$, for $T(f_i) \in (C^\infty(D))^\infty_\epsilon\{X_1,\ldots,X_n\}_\epsilon$ for all k. Then by Theorem 1.0.3, for any function $T(g)$ there exists a unique decomposition

$$T(g) = \sum \overline{h_i}T(f_i) + \overline{h_0},$$

where $h_i \in (C^\infty(D))^\infty_\epsilon\{X_1,\ldots,X_n\}_\epsilon$, $\quad \mathrm{supp}(\overline{h_i}) \subset \Gamma_i$. Consequently, by uniquenes $\overline{h_i} \in C^\infty(D)[[X_1,\ldots,X_n]]$. By Lemma 11.0.4, $(D_{u_i} - D_{X_i})(T(g)) = (D_{u_i} - D_{X_i})(T(f_i)) = 0$. Applying the diffential operator to the above presenation we get

$$0 = \sum (D_{u_i} - D_{X_i})(\overline{h_i})T(f_i) + (D_{u_i} - D_{X_i})(\overline{h_0}),$$

Observe that $\mathrm{supp}((D_{u_i} - D_{X_i})(h_i) \subset \Gamma_0$. By uniqueness of Weierstrass-Hironka division in $C^0(D)\{X_1,\ldots,X_n\}_\epsilon$ we conclude that $(D_{u_i} - D_{X_i})(\overline{h_i}) = 0$, which implies that $\overline{h_i} = T(h_i)$. Then

$$T(g) = \sum T(h_i)T(f_i) + T(h_0) = T(h_if_i + h_0),$$

Since T is a monomorphism we get a unique decomposition

$$g = \sum h_if_i + h_0,$$

with

$$\mathrm{supp}(h_i) = \mathrm{supp}(T(h_i)) \subset \Gamma_i.$$

Since $T(h_i) \in C^0(D)\{X_1,\ldots,X_n\}_\epsilon$ are smooth convergent the functions $h_i \in (C^\infty(D))^\infty_\epsilon$ are convergent by Lemma 11.0.2. ☐

Corollary 11.0.11. *(Preparation Theorem for smooth convergent functions)*

Let $\{f_i(u,v)\}$ *be a finite set of the convergent functions in* $C^\infty_x\{\mathbb{R}^{n+m}\}$. *Let* Δ *be a diagram of the initial exponents generated by its vertices of* $\exp_{\overline{T}}((f_i(u,0)) = \alpha^i$. *Then there is a set of generators* $\overline{f_i} := u^{\alpha^i} + r_i$ *of the ideal* $(f_1,\ldots,f_k)$ *such that*

$$\overline{f_i} := u^{\alpha^i} + r_i, \tag{3}$$

where $\mathrm{supp}(r_i)^n_{i=0}$ *is contained in* $\Gamma \times \mathbb{N}^m$, $\exp_{\overline{T}}(f_i(u,0)) = \alpha^i$. *Moreover if*

$$\{f \in (f_1,\ldots,f_k) \mid \mathrm{supp}(f(u,0)) \subset \Gamma_0\} = \{0\}$$

then such a basis is unique.

Proof. We construct $\overline{f_i}$ and show by induction that the set $\overline{f_1},\ldots,\overline{f_{i-1}},\overline{f_i},f_{i+1},\ldots,f_k$ generate $\mathcal{I}$. This is true for $i=0$. Suppose it is valid for $i-1$. Consider the division with remainder:

$$u^{\alpha^i} = \sum_{j=1}^{i-1} h_j\overline{f_j} + \sum_{j=i}^{k} h_jf_j + r(u^{\alpha^i})$$

Where $\mathrm{supp}(r(u^{\alpha^i})) \subset \Gamma_0 \times \mathbb{N}^m$. Then put

$$\overline{f_i} := u^{\alpha^i} - r(u^{\alpha^i}) = \sum_{j=1}^{i-1} h_j\overline{f_j} + \sum_{j=i}^{k} h_jf_j.$$

Note that $\exp(h_jf_j)(u,0) = \exp(h_jf_j)(u,0) \in \Delta_j \times \mathbb{N}^m$, and thus all distict for distinct j.

$$\exp(\overline{f_i}(u,0)) = \min(\exp(h_jf_j)(u,0)) = \exp((h_if_1)(u,0)) = \exp((h_i)(u,0)) + \exp((f_i)(u,0))$$

$\overline{\text{Consequently}}$ $\exp((h_i)(u,0)) = 0$, and $h_i(u,0)$ and h_i are invertible, and thus the ideals generated by $\overline{f_1}, \ldots, \overline{f_{i-1}}, \overline{f_i}, f_{i+1}, \ldots, f_k$ and $\overline{f_1}, \ldots, \overline{f_{i-1}}, \overline{f_{i-1}}, f_i, \ldots, f_k$ are the same.

We need to show uniqueness of $\overline{f_i}$. If there exist two bases $(\overline{f_i^1})$ and $(\overline{f_i^2})$ then for functions $g_i := \overline{f_i^1} - \overline{f_i^2} \in \mathcal{I}$ the support $\mathrm{supp}(g) \in \Gamma \times \mathbb{N}^m$, which, by the assumption implies $g_i \equiv 0$.

□

Corollary 11.0.12. *(***Weierstrass preparation for smooth convergent function***) If $f = f(u,v) = f(u, v_1, \ldots, v_m) \in C_x^{m+1}\{\mathbb{R}^{m+1}\}$ is smooth convergent and d-regular at x in u then*

1. *f can be expressed uniquely in a neighborhood of x as $f = \alpha \cdot P$, where $P = u^d + c_1(v)u^{d-1} + \ldots + c_d(v)$ is a Weierstrass polynomial and a smooth convergent function at x, and α is invertible at x*
2. *For any smooth convergent function g on at x there exist unique smooth convergent h, r such that $f = gh + r$ in a neighborhood of x, where $r = r_1(v)u^{d-1} + \ldots + r_d(v)$ is a polynomial in u of degree $d-1$.*

Proof. (1) Follows from Theorem 11.0.11. In this case the diagram Δ is generated by $d \in \mathbb{N}$. (2) Follows from Theorems 11.0.10, and Lemma 11.0.9. Again we use the diagram Δ is generated by $d \in \mathbb{N}$. □

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