

RELATIVE DESINGULARIZATION AND PRINCIPALIZATION OF IDEALS

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ABSTRACT. In characteristic zero, we construct relative principalization of ideals for logarithmically regular morphisms of log schemes, and use it to construct logarithmically regular desingularization of morphisms. These constructions are canonical and even functorial with respect to log regular morphisms and arbitrary base changes. As a consequence we deduce the semistable reduction theorem over arbitrary valuation rings.

1. INTRODUCTION

¹ If not said to the contrary, all schemes considered in this paper are assumed to be noetherian and of characteristic zero. ←1

1.1. Motivation. Comparing to desingularization of schemes and varieties, the theory of resolution of morphisms or families has a much shorter history. Excluding stable reduction of curves, which is a low-dimensional exception, there were two main achievements in the characteristic zero case, and both were based on breakthroughs in the absolute desingularization theory.

First, Mumford et al. observed in [KKMSD73] that Hironaka's theorem implies that if R is a DVR and X is flat, of finite type and with a smooth generic fiber X_η over $S = \text{Spec}(R)$, then there exists a modification $X' \rightarrow X$ such that $X'_\eta = X_\eta$ and $X' \rightarrow S$ is log smooth (or toroidal). The implication is in fact obvious and the main goal of the book is to prove that X' can be made semistable by combinatorial methods. By later works Hironaka's method can be made canonical, and then the modification $X' \rightarrow X$ becomes canonical, but it heavily depends on R and changes completely after any ramified extension of R .

Second, Abramovich and Karu used in [AK00] de Jong's method of alterations to prove that any dominant morphism of finite type between integral schemes $Z \rightarrow B$ can be made log smooth after modifying Z and B . This was the only known result applying to any dimension of the base and of the fibers, but the method is non-canonical and even a smooth generic fiber Z_η can be modified by it.

In particular, the following questions were open until now: Can one resolve Z/B canonically? Can one at least achieve that the smooth locus of Z/B is kept

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¹(Michael) This is a preliminary version. Sections 2, 3 and 4.2.6 will be revised. Some foundational material from §2 will be outsourced to [MT19], and the generality will be improved at a couple of places. Also, will describe applications in more detail, especially addressing wider audience and generalist journals.

unchanged? And, finally, an even more specific but famous question extending the semistable reduction to non-discrete valuation rings R : given a smooth proper variety X_η over $\text{Frac}(R)$, can one extend it to a proper log smooth R -scheme X ?

In this paper we answer all these questions affirmatively (in characteristic zero). Moreover, we construct a canonical resolution of morphisms compatible with arbitrary base changes $B' \rightarrow B$ and log smooth morphisms $Z' \rightarrow Z$. Our construction is strongly based on ideas and methods of the classical desingularization, but involves completely new ingredients, such as extensive use of logarithmic geometry and non-representable modifications. To the best of our knowledge, this is the first relative desingularization algorithm.

mainres

1.2. Statement of main results.

1.2.1. *Overview.* In this paper we accomplish our program on functorial desingularization (or semistable reduction) of morphisms in characteristic zero. Oversimplifying, the main idea is to take the classical desingularization method of Hironaka, Giraud, Bierstone and Milman, Villamayor, Włodarczyk, Kollar, and others and adjust all its ingredients to logarithmic and relative settings. In particular, one should first obtain a logarithmic relative principalization of ideals on log smooth morphisms, and deduce desingularization from it. Half of this work was done in [ATW17a], where we constructed logarithmically functorial desingularization of log varieties. It turned out that the stronger functoriality forces one to use non-representable stack theoretic modifications, that we called Kummer blow ups, but the algorithm's structure became simpler than in the classical case.

In this paper, we show that the algorithm constructed in [ATW17a] applies to morphisms $Z \rightarrow B$ once one adjusts all definitions to the relative setting. The main new issue we have to solve is that one might need to modify the base B before (or in the process of) running the desingularization algorithm. Technically this happens because relative differential saturation does not have to produce a monomial ideal, and we prove a (surprisingly hard) monomialization theorem 3.4.8, which reconciles this by a log modification of the base.

An additional challenge we took on in this paper is to deal with morphisms not necessarily of finite type. In particular, we study principalization for log regular morphisms (which we also have to define) rather than for log smooth ones, and this has the advantage of applications to other categories, such as formal schemes and analytic spaces. This direction is novel even in the classical desingularization, since [Tem18] does not construct principalization for qc schemes.

1.2.2. *Functorial relative principalization.* Throughout this paper, we say that a morphism $f: X \rightarrow B$ of fs logarithmic DM stacks is a (relative) log orbifold if $X \rightarrow B$ is logarithmically regular and has enough derivations, see §2.4–2.6. A Kummer ideal $\mathcal{J} \subseteq \mathcal{O}_{X_{\text{két}}}$ is called *submonomial* if it is the sum of a suborbifold ideal and a Kummer monomial ideal, and the submonomial Kummer blow up along \mathcal{J} is the universal morphism of log DM stacks $h: X' \rightarrow X$ such that $h^{-1}\mathcal{J}$ is an invertible ideal, see §4.2.6. By a *principalization method* we mean a rule \mathcal{F} which obtains a relative log orbifold $f: X \rightarrow B$ and an ideal $\mathcal{I} \subseteq \mathcal{O}_X$ and outputs either $\mathcal{F}(f, \mathcal{I}) = \emptyset$, which means “ \mathcal{F} fails over the given B , blow it up first”, or a sequence of submonomial Kummer blowings up $\mathcal{F}(f, \mathcal{I}): X' = X_n \rightarrow \cdots \rightarrow X_0 = X$ such

that each $X_i \rightarrow B$ is a relative log orbifold, the center of each $X_{i+1} \rightarrow X_i$ is supported on $V(\mathcal{I}\mathcal{O}_{X_i})$, and $\mathcal{I}\mathcal{O}_{X'}$ is a monomial ideal.

Our first main result states that there exists a principalization method that does not fail after a large enough modification of the base and satisfies three functoriality conditions as follows:

Proj:principalization

Theorem 1.2.3 (Principalization). *There exists a principalization method \mathcal{F} satisfying the following properties:*

(i) *Existence: if $f: X \rightarrow B$ is a relative log orbifold with abundance of derivations and B is universally resolvable, then there exists a modification $g: B' \rightarrow B$ such that g is an isomorphism over the complement to the closure of the image of $V(\mathcal{I})$ in B and $\mathcal{F}(f', \mathcal{I}') \neq \emptyset$, where $f': X' \rightarrow B'$ is the saturated pullback of f and $\mathcal{I}' = \mathcal{I}\mathcal{O}_{X'}$.*

(ii) *Base change functoriality: if $\mathcal{F}(f, \mathcal{I}) \neq \emptyset$ and $B' \rightarrow B$ is any morphism of logarithmic stacks with saturated pullback $f': X' \rightarrow B'$ and $\mathcal{I}' = \mathcal{I}\mathcal{O}_{X'}$, then the sequence $\mathcal{F}(f', \mathcal{I}')$ is obtained from the saturated pullback sequence $\mathcal{F}(f, \mathcal{I}) \times_B B'$ by removing Kummer blowings up with empty centers.*

(iii) *Log regular functoriality: if $\mathcal{F}(f, \mathcal{I}) \neq \emptyset$ and $\mathcal{I}' = \mathcal{I}\mathcal{O}_{X'}$ for a logarithmically regular morphism $X' \rightarrow X$ such that $X' \rightarrow B$ is a relative log orbifold, then $\mathcal{F}(f', \mathcal{I}')$ is obtained from the saturated pullback sequence $\mathcal{F}(f, \mathcal{I}) \times_X X'$ by removing Kummer blowings up with empty centers.*

(iv) *Compatibility with closed embeddings: if $\bar{f}: \bar{X} \rightarrow B$ is another relative log orbifold and $i: X \hookrightarrow \bar{X}$ is a B -suborbifold embedding of pure codimension, then the pushforward sequence $i_*(\mathcal{F}(f, \mathcal{I}))$ coincides with $\mathcal{F}(\bar{f}, i_*\mathcal{I})$, where $i_*\mathcal{I}$ is the preimage of \mathcal{I} in $\mathcal{O}_{\bar{X}}$.*

Remark 1.2.4. (i) In brief, the main result is that there exists a relative principalization algorithm \mathcal{F} , which works after a fine enough modification of B assuming B is reasonable and X/B has plenty of derivations, for example, $X \rightarrow B$ is log smooth.

(ii) Once B is fine enough, our method is constructive and applies to different geometric spaces. However, the result on existence of the modification of B is purely existential and can vary in different contexts, see §1.2.16. This is the main reason why we prefer the formulation which separates construction of $X' \rightarrow X$ over the variant where $\mathcal{F}(f, \mathcal{I})$ outputs a modification $B' \rightarrow B$ (non-canonically) and a blow up sequence $X' \rightarrow X \times_B B'$, which is canonical once B' is chosen.

Remark 1.2.5. The functoriality property (iii) is the analogue of the functoriality in [ATW17a]. Property (iv) strengthens the re-embedding principle in [ATW17a]. Finally, property (ii) only makes sense in the relative situation, and it is very important since the algorithm often fails without base changes.

1.2.6. *Order reduction.* As in other cases, the principalization theorem is a particular case of an order reduction theorem for marked ideals on X . The formulation is similar and will be given in 7.1.1. In fact, this is the main theorem of the paper, since principalization and desingularization are its corollaries.

1.2.7. *Functorial relative desingularization.* Similarly to the absolute case, the main application of relative principalization is the following relative desingularization theorem that was the main motivation for our project started at [ATW17a]. By a

relative desingularization method we mean a rule that given a morphism $g: Z \rightarrow B$ of log DM stacks, which is locally embeddable into qe log orbifolds, see §8.3, outputs either the empty value or a modification $Z_{\text{res}} \rightarrow Z$ such that $g_{\text{res}}: Z' \rightarrow B$ is log regular. The following theorem will be deduced from Theorem 1.2.3 in §8.

Proj:toroidalization

Theorem 1.2.8 (Relative desingularization). *There exists a relative desingularization method \mathcal{R} such that*

- 2→ (i) *Existence: if $g: Z \rightarrow B$ is locally embeddable into a log orbifold with abundance of derivations and B is nice², then there exists a stack theoretic modification $B' \rightarrow B$ with saturated base change $g': Z' \rightarrow B'$ such that $\mathcal{R}(g')$ does not fail.*
- (ii) *Base change functoriality: if $\mathcal{R}(g)$ does not fail and $B' \rightarrow B$ is a morphism of log schemes with saturated base change $g': Z' \rightarrow B'$, then $\mathcal{R}(g')$ is the saturated pullback of $\mathcal{R}(g)$.*
- (iii) *Log regular functoriality: if $\mathcal{R}(g)$ does not fail and $Z' \rightarrow Z$ is a log regular morphism such that $g': Z' \rightarrow B$ is locally embeddable into a log orbifold, then $\mathcal{R}(g')$ is the saturated pullback of $\mathcal{R}(g)$.*

1.2.9. *Destackification.* It is still a natural question whether there exists a *scheme-theoretic relative desingularization method*, which assigns to a locally embeddable morphism of log schemes $Z \rightarrow B$ a projective modification $Z_{\text{res}} \rightarrow Z$ such that $g_{\text{res}}: Z_{\text{res}} \rightarrow B$ is log regular. As in [ATW17a], it is constructed by composing the sequence $\mathcal{R}(g): Z_1 \rightarrow Z$ with the destackification morphism $\mathcal{D}(Z_1): Z_2 \rightarrow Z_1$ and passing to the coarse space $Z_{\text{res}} = (Z_2)_{\text{cs}}$.³ A careful examination of the construction shows that it is compatible with base changes, and the same argument as in the proof of [ATW17a, Theorem 8.3.4] shows that if $Z' \rightarrow Z$ is log regular and quasi-saturated, then $Z'_{\text{res}} \rightarrow Z_{\text{res}}$ is representable and hence is compatible with destackification. Therefore we obtain the following

3→

Proj:schemedesing

Theorem 1.2.10. *Composing \mathcal{R} from Theorem 1.2.8 with destackification and taking coarse moduli space, one obtains a relative desingularization method \mathcal{R}' which satisfies properties (i) and (ii) of Theorem 1.2.8, while (iii) is only satisfied for log regular quasi-saturated morphisms $Z' \rightarrow Z$.*

Remark 1.2.11. After a finite covering of B , one can further improve the log regular morphism $g_{\text{res}}: Z_{\text{res}} \rightarrow B$ by methods of toroidal geometry. In particular, one can make it weakly semistable applying the methods of [AK00] with care given to functoriality. Furthermore, in [ALT18] it will be proved that by toroidal methods g_{res} can be even made semistable.

1.2.12. *Morphisms of finite presentation.* It is easy to see that the assumptions on abundance of derivations are always satisfied for morphisms of finite type. In particular, all assertions of Theorems 1.2.3, 1.2.8 and 1.2.10 hold for this class of morphisms. Moreover, using standard approximation theory we will remove in §9.1 the noetherianity assumption on B obtaining the following result:

finpres

Theorem 1.2.13. (i) *All statements of Theorem 1.2.3 hold for log smooth morphisms $X \rightarrow B$, where B is a qcqs integral scheme provided with a quasi-coherent log structure.⁴*

4→

²(Michael) Will sum up all minor assumptions on B later.

³(Michael) Need to prove that $(Z_2)_{\text{cs}} \rightarrow B$ is log regular.

⁴(Michael) Maybe solid or quasi-solid log scheme?

(ii) All statements of Theorems 1.2.8 and 1.2.10 hold for morphisms $Z \rightarrow B$ of locally finite presentation such that the underlying stack of B is qcqs and integral.

In particular, this covers the case when B is the spectrum of a valuation ring R and $M_B = R \setminus \{0\}$.

1.2.14. *Log smooth reduction theorem.* Combining Nagata compactification theorem (e.g. see [Con07]) with Theorem 1.2.13(ii) we obtain the following result:

Theorem 1.2.15. *Assume that B is a qcqs integral scheme provided with a quasi-coherent log structure and $f: X \rightarrow B$ is a separable log smooth morphism. Then there exists a proper log smooth morphism $\overline{X} \rightarrow B$ and an open immersion $X \hookrightarrow \overline{X}$ of log schemes over B .*

Again, in [ALT18] this will be strengthened by combinatorial methods to a semistable reduction theorem.

othersec

1.2.16. *Extension to other geometries.* Part of our motivation stems from profound questions arising in non-archimedean geometry, see [BLR95, Page 364]. Being functorial for regular morphisms, our relative principalization and desingularization methods extend to these categories immediately, and existence would be equivalent to extending the monomialization theorem to these contexts. Remark 1.3.9 below shows that at the very least, one should either restrict the class of objects one resolves or allow non-proper base changes. This is a separate question not related much to the content of this paper, so we will not study it here.

methodssec

1.3. **Methods.** Now, let us describe our methods and discuss certain choices we make and possible alternatives.

1.3.1. *Ambient spaces.* We resolve morphisms $Z \rightarrow B$ by embedding them into log regular morphisms $X \rightarrow B$ and principalizing \mathcal{I}_Z on X/B . Similarly to [ATW17a], the latter is done by a sequence of non-representable modifications $\sigma: X' \rightarrow X$ with finite diagonalizable inertia $I_{X'/X}$, which are called *submonomial Kummer blow ups* in this paper. This forces us to work with stacks. Similarly to [ATW17a] one could consider only stacks with finite diagonalizable inertia, but we decided to extend the generality to log DM stacks because this does not change a single argument.

1.3.2. *Derivations.* As in [ATW17a], all constructions of our method are expressed in terms of sheaves of logarithmic derivations and differential operators of order at most a , but this time we use the relative sheaves $\mathcal{D}_{X/B}^{(\leq a)}$. In particular, the *log order* a of \mathcal{I} is the minimal number such that $\mathcal{D}_{X/B}^{(\leq a)}(\mathcal{I}) = \mathcal{O}_X$, the ideal \mathcal{I} is *clean* if $a < \infty$, and the (*hypersurface of*) *maximal contact* to \mathcal{I} is $H = V(t)$, where t is an element of log order one in $\mathcal{D}_{X/B}^{(\leq a-1)}(\mathcal{I})$.

1.3.3. *Marked ideals.* As usually, order reduction operates with marked (or weighted) ideals denoted $\underline{\mathcal{I}} = (\mathcal{I}, a)$ throughout the paper. We define products, equivalence and domination as usually, but replace sums with the homogenized ones, see §5.1 and Remark 5.1.8. Also, we use the *homogenized coefficient ideals* $\underline{\mathcal{C}} = \sum_{i=0}^{a-1} \mathcal{D}_{X/B}^{(\leq i)} \underline{\mathcal{I}}$ defined by use of homogenized sums. In fact, this is precisely the analogue of Kollar's tuning ideal $W_{a!}(\underline{\mathcal{I}})$, see [Kol07].

1.3.4. *Modules of derivations.* Certain submodules of \mathcal{D}_X suffice to run various parts of the principalization algorithms, for example, see [BM08, Exercise 4.4]. This was essentially used in [BMT11] to show that the algorithm of Bierstone-Milman depends only on the (huge) module of absolute derivations and hence resolutions of varieties over different fields are compatible. Submodules of \mathcal{D}_X were also used in [ATW17a].

For an arbitrary log regular morphism the module $\mathcal{D}_{X/B}$ can be very large or very small, even 0. Also it may be not quasi-coherent and does not satisfy good functoriality properties, see §2.5. Therefore, we prefer to work with concrete \mathcal{O}_X -submodules $\mathcal{F} \subseteq \mathcal{D}_{X/B}$ and their transforms. In particular, in Proposition 4.3.7 and Theorem 5.3.6 we obtain equalities for transforms of derivations and coefficient ideals, while in the classical situation and in [ATW17a] one only proved inclusions. This equality, which only holds for homogenized coefficient ideals, allows us to simplify the formalism of [ATW17a] – we do not have to consider normal closures of marked ideals anymore, see [ATW17a, Proposition 6.1.3] for comparison. Though, we still have to consider normal closures of powers of admissible centers, see Remark 4.2.4 and Lemma 5.1.13(ii).

1.3.5. *Relative log orbifolds.* At each step of the algorithm we precisely describe the conditions modules of derivations \mathcal{F} should satisfy. We say that \mathcal{F} is *separating* (resp. *log separating*) if it separates all parameters (resp. and log parameters) on X/B . It turns out that almost all stages of the algorithm only require that \mathcal{F} is separating, for example, this provides enough derivations to compute the log order and define the maximal contact. However, only the property of being log separating is stable under blow ups, so we call a log regular $X \rightarrow B$ a *log orbifold* if $\mathcal{D}_{X/B}$ is log separating, and we construct relative principalization for arbitrary log orbifolds.

1.3.6. *Base change.* Loosely speaking everything described in §1.3 up to now is obtained from the method of [ATW17a] by passing to the relative and log regular setting and making some (relatively minor) improvements. The really new issue is that one has to take base changes into account. First, they provide a new form of functoriality which one should establish. This is done straightforwardly and causes no problems. Second, it might happen that in order to succeed the algorithm requires to modify the base first, and here is the simplest example.

stuckexample

Example 1.3.7. (i) We start with a “non-trivial” example. Take $B = \mathbb{A}_k^2 = \text{Spec}(k[x, y])$ and $X = \mathbb{A}_k^3 = \text{Spec}(k[x, y, t])$ with the trivial log structures, and let $\mathcal{I} = (x, y, t)$. A minimal submonomial center \mathcal{J} through $P = V(\mathcal{I})$ is two-dimensional, for example (t) , hence there exists no \mathcal{J} with support at P , and \mathcal{I} cannot be principalized. The situation can be solved by modifying B . In this case, the most natural way is to blow up (x, y) and enrich the log structure with the exceptional divisor, but one can even simply enlarge the log structure to $x^{\mathbb{N}} \times y^{\mathbb{N}}$ so that $\mathcal{I} = (t) + (x, y)$ becomes submonomial.

(ii) Precisely the same example works when $B = \text{Spec}(k[x])$, but it is less illuminating since blowing up (x) and increasing the log structure to $x^{\mathbb{N}}$ have the same effect.

(iii) In fact, even in the “most trivial” case of $X = B = \mathbb{A}_k^2$ or $X = B = \mathbb{A}_k^1$, the only way to principalize $\mathcal{I} = (x, y)$ or $\mathcal{I} = (x)$ is by base change.

intronomomsec

1.3.8. *Monomialization.* Similarly to the absolute case in [ATW17a], our relative order reduction of $\underline{\mathcal{I}} = (\mathcal{I}, a)$ must start with blowing up the weighted differential saturation $\mathcal{W}(\underline{\mathcal{I}}) := (\mathcal{D}_{X/B}^{(\leq \infty)}(\mathcal{I}))^{1/a}$. The main new issue one has to deal with is that $\mathcal{D}_{X/B}^{(\leq \infty)}(\mathcal{I})$ can be strictly smaller than the monomial saturation $\mathcal{M}(\mathcal{I})$. In this case the center is not submonomial and the algorithm fails. For example this is what happens in Example 1.3.7(iii). Our main result in this direction is the monomialization theorem 3.4.8, which says that $\mathcal{D}_{X/B}^{(\leq \infty)}(\mathcal{I})$ can be monomialized by a modification of B if X/B has abundance of derivations. The latter notion is defined in §2.5.10, in particular, any log smooth morphism has abundance of derivations. The proof is surprisingly difficult. Another indication of subtlety of this result is that its naive generalizations fail.

intronomomrem

Remark 1.3.9. (i) Monomialization does not hold for complex analytic spaces. For example, take $X \hookrightarrow B$ an open immersion and $\mathcal{I} \subseteq \mathcal{O}_X$ an ideal which cannot be extended to \mathcal{O}_B . Then \mathcal{I} cannot be monomialized by modifications of B . Similar “non-trivial” examples exist when $X \rightarrow B$ factors through an open immersion. This indicates that monomialization requires some restrictions: either restrict the class of objects one resolves, for example, by imposing a properness assumption, or allow non-proper base changes, for example, open covers of B . We plan to study this question elsewhere.

(ii) The above remark also suggests to try to extend Theorem 3.4.8 to more general relative log orbifolds by enlarging the class of base changes.

1.3.10. *The relative order reduction algorithm.* As usually, the principalization algorithm of \mathcal{I} makes an order reduction of $(\mathcal{I}, 1)$. Our order reduction algorithm for a marked ideal $\underline{\mathcal{I}} = (\mathcal{I}, a)$ is a copy of the algorithm from [ATW17a]. Here is an outline and see §7 and §7.3 for details. The algorithm runs by induction on the relative dimension of X/B and consists of three steps.

Step 1. *Initial cleaning.* If $\mathcal{D}_{X/B}^{(\leq \infty)}(\mathcal{I})$ is not monomial output the fail value, otherwise blow up $(\mathcal{D}_{X/B}^{(\leq \infty)}(\mathcal{I}))^{1/a}$ making the controlled transform clean.

Step 2. *Reducing the log order of the clean part below a .* Throughout this step the ideal is balanced, that is, $\mathcal{I} = \mathcal{M} \cdot \mathcal{I}^{\text{cln}}$ with an invertible monomial $\mathcal{M} = \mathcal{M}(\mathcal{I})$ and a clean \mathcal{I}^{cln} of order b . By induction, one simply performs the order reduction of the maximal order marked ideal $\underline{\mathcal{I}}^{\text{cln}} = (\mathcal{I}^{\text{cln}}, b)$, which automatically results in reducing the order of the clean part of the transform. The order reduction of $\underline{\mathcal{I}}^{\text{cln}}$ is constructed by finding étale-locally a maximal contact H to $\underline{\mathcal{I}}^{\text{cln}}$ and descending the order reduction of the equivalent marked ideal $(\mathcal{C}(\mathcal{I}^{\text{cln}})|_H, b!)$ on H to X .

Step 3. *Final cleaning.* At this stage the clean part is resolved, so one simply blows up $\mathcal{M}(\mathcal{I})^{1/a}$.

1.3.11. *Equivalence classes.* As usually, the most subtle thing is to show that the process is independent of the choice of H and hence descends to a Kummer blow up sequence of X . As in [ATW17a], one possible way would be to prove that $\mathcal{C}(\mathcal{I}^{\text{cln}})|_H$ is unique up to a formal isomorphism, where we use that our coefficient ideal is homogenized. However, we decided to use equivalence classes instead. This is a slightly more elaborate way which also proves the stronger claim that the order reduction only depends on the functorial equivalence class of $\underline{\mathcal{I}}$ as defined in §5.2

and §6.2.4. We explain in Remark 5.2.2 why it is more natural to consider functorial logarithmic equivalence, and how it sheds a new light on the classical definitions of Hironaka and Bierstone-Milman. In addition, we prove in Theorem 7.2.11 that the functorial equivalence class determines the main invariants of $\underline{\mathcal{L}}$ used in the order reduction – the weighted log order and the weighted monomial part. This is a logarithmic version of Hironaka’s trick, and the proof is more straightforward than in the classical case, see §7.2.1.

1.3.12. *Induction on length.* All proofs in this paper are designed so that claims about blow up sequences are proved by simple and formal induction on the length. In this way we achieve that any coordinate-dependent check is done for a single Kummer blow up, and there is no need to consider charts of sequences, which might be unpleasant due to non-representability of Kummer blow ups. This becomes possible because we use transforms of derivations in Theorem 5.3.6. In particular, we manage to use explicit Taylor series (or another form of a Weierstrass preparation) only in the proof of Proposition 6.3.6, which deals with lift of admissibility of a single Kummer center and not a whole blow up sequence.

1.3.13. *The relative desingularization method.* The relative desingularization is obtained by embedding into relative log orbifolds and principalizing ideals there, and the main issue is to prove that there is a unique minimal embedding. In [ATW17a] we did this in étale topology. In this paper we use formal topology instead. This is possible because principalization was constructed for arbitrary log regular morphisms, and this even leads to simpler proofs because uniqueness of minimal formal embedding is less technically demanding.

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2. RELATIVE LOG ORBIFOLDS

basicsec

Some notions introduced in this section are of interest in any characteristic, so we deviate for some time from the characteristic-zero assumption, and it will be restored when needed.

assumpsec

2.1. **A brief summary.** All log schemes in this paper are assumed to be fs.⁵ We will develop the principalization algorithm in the case when B is a *solid* and *universally resolvable* logarithmic scheme, and $f: X \rightarrow B$ is a *relative logarithmic orbifold*, that is, X is a logarithmic DM stack, f is *logarithmically regular* with a *separating* sheaf of logarithmic derivations $\mathcal{D}_{X/B}$. Section 2 is devoted to introducing these notions and studying their basic properties. ←5

2.2. Logarithmic schemes.

2.2.1. *Formally split logarithmic schemes.* We say that a logarithmic scheme Y is *formally split* at a point $y \in Y$ if there is an isomorphism of complete local logarithmic rings $A[[P]] \xrightarrow{\sim} \widehat{\mathcal{O}}_{Y,y}$, where $P = \overline{M}_y$ and the logarithmic structure on the left is given by P . We say that Y is *formally split* if there exists a strict étale covering $Y' \rightarrow Y$ such that Y' is formally split everywhere. This happens if and only if Y is formally split at its geometric points.

splitexam

Example 2.2.2. If Y is a log regular logarithmic scheme of equal characteristic (the latter means that Y admits a morphism to a field), then Y is formally split by [Kat94, Theorem 3.2(1)].

2.2.3. *Solid log schemes.*⁶ Given a log scheme Y , consider the open embedding let $i: U = Y_{\text{tr}} \hookrightarrow Y$ denote the triviality locus of the logarithmic structure. We say that Y is *solid* if the underlying scheme is integral and $M_Y = \mathcal{O}_{Y_{\text{ét}}} \cap i_*(\mathcal{O}_{U_{\text{ét}}}^\times)$. Unless stated oppositely, we will only work with solid log schemes, and we will sometimes emphasize the logarithmic structure by writing (Y, D_Y) , where $D_Y = Y \setminus Y_{\text{tr}}$. ←6

splitlem

Lemma 2.2.4. *Assume that Y is a formally split noetherian logarithmic scheme whose underlying scheme is integral. Then Y is solid.*

Proof. Clearly, the map $M_Y \rightarrow \mathcal{O}_{Y_{\text{ét}}}$ factors through $\mathcal{O}_{Y_{\text{ét}}} \cap i_*(\mathcal{O}_{U_{\text{ét}}}^\times)$. To check that it is an isomorphism we can work étale locally. In particular, we can assume that Y is Zariski and formally split at all its points. We should prove that $\phi_y: M_y \rightarrow \mathcal{O}_y \cap i_*(\mathcal{O}_U^\times)_y$ is an isomorphism for any $y \in Y$, and this reduces to showing that $M_y \hookrightarrow \mathcal{O}_y$ and any element $f \in \mathcal{O}_y$ dividing a monomial $m \in M_y$ is a monomial. Injectivity of ϕ_y follows from the injectivity of the composition $M_y \rightarrow \mathcal{O}_y \rightarrow \widehat{\mathcal{O}}_y \xrightarrow{\sim}$

⁵(Michael) Check in the end that fine non-fs logarithmic schemes are not used.
⁶(Michael) Ogus uses the notion "solid" in a more general way.

$A[\overline{M}_y]$. Since $f|m$ in the logarithmic ring $\widehat{\mathcal{O}}_y = A[\overline{M}_y]$, it is easy to see that f is a monomial in $\widehat{\mathcal{O}}_y$, and hence also in \mathcal{O}_y . ♣

2.2.5. *Monomial ideals.* We say that an ideal \mathcal{I} on a logarithmic scheme Y is *monomial* at $y \in Y$ if it is locally generated by monomials $u^q \in M_y$. At least on formally split logarithmic schemes this property is Kummer-local.

monomcoverlem

Lemma 2.2.6. *Assume that Z is a formally split logarithmic scheme, $f: Y \rightarrow Z$ is a Kummer logarithmically étale cover, and $\mathcal{I} \subseteq \mathcal{O}_Z$ is an ideal. Then $\mathcal{I}\mathcal{O}_Y$ is monomial if and only if \mathcal{I} is monomial.*

Proof. The claim is local, so fix a point $y \in Y$ with $z = f(y)$. Being monomial is an étale-local condition, hence we can assume that the logarithmic structures are Zariski and $k(z) = k(y)$. Then $\mathcal{J} = \mathcal{I}\mathcal{O}_Y$ is monomial at y if and only if its completion $\widehat{\mathcal{J}}$ is a monomial ideal of $\widehat{\mathcal{O}}_y$, and similarly for \mathcal{I} and z . Therefore the claim follows from the following lemma. ♣

Lemma 2.2.7. *Assume that $P \hookrightarrow Q$ is a Kummer embedding of sharp fs monoids, O is a ring, $A = O[[P]]$ and $C = O[[Q]]$. Then an ideal $I \subset A$ is monomial if and only if $J = IC$ is monomial.*

Proof. Only the inverse implication needs a proof, so assume that J is monomial. If $f = \sum_{p \in P} f_p u^p$ is an element of I , then each u^p lies in J , say, $u^p = \sum c_i g_i$ with $g_i \in I$ and $c_i \in C$. Note that C is $Q^{\text{gp}}/P^{\text{gp}}$ -graded and the component of weight zero is A . Taking the weight-zero component of the above equality we obtain that $u^p = \sum (c_i)_0 g_i$ with $(c_i)_0 \in A$, and hence $u^p \in I$. So I is monomial, as claimed. ♣

2.2.8. *Integral closure and saturation.* We refer to [ATW17a, §4.3.1] for the definition of the integral closure I^{nor} of an ideal. Clearly, integral closure of monomial ideals contains their saturation, and on good enough log schemes they always coincide. This was proved for log regular log schemes in [AT17, Corollary 5.3.6]. Here is another case, which is a vast generalization when the characteristic is not mixed.

norsatlem

7→

Lemma 2.2.9. *Assume that X is a formally split log scheme and \mathcal{I} is a monomial ideal. Assume that X has reduced formal fibers.⁷ Then $\mathcal{I}^{\text{nor}} = \mathcal{I}^{\text{sat}} + \mathcal{N}$, where \mathcal{N} is the nilradical of \mathcal{O}_X .*

Proof. The claim is étale-local at a point x , hence we can assume that $X = \text{Spec}(A)$ for $A = \mathcal{O}_x$, the log structure is induced by $P \rightarrow \mathcal{O}_x$ for $P = \overline{M}_x$, and $\widehat{A} = O[[P]]$. We should prove that $I = \mathcal{I}_x$ satisfies $I^{\text{nor}} = I^{\text{sat}} + N$, where $N = \text{Rad}(A)$. Only the inclusion $I^{\text{nor}} \subseteq I^{\text{sat}} + N$ needs a proof. Furthermore, it suffices to prove that $\widehat{I}^{\text{nor}} = \widehat{I}^{\text{sat}} + \text{Rad}(\widehat{A})$ because $\text{Rad}(\widehat{A}) = N\widehat{A}$ by our assumption and hence $I^{\text{nor}} \subseteq \widehat{I}^{\text{nor}} \cap A = (I^{\text{sat}} + N)\widehat{A} \cap A = I^{\text{sat}} + N$. Finally, the claim easily reduces further to the ring $O[P]$: it suffices to prove that the monomial ideal $J = \widehat{I} \cap O[P]$ satisfies $J^{\text{nor}} \subseteq J^{\text{sat}} + \text{Rad}(O)[P]$.

For any prime ideal $p \subset O$ and $B(p) = \text{Frac}(A/p)[P]$ we have that $J^{\text{nor}}B(p) \subseteq (JB(p))^{\text{nor}}$. Since $\text{Spec}(B(p))$ is log regular, by [AT17, Corollary 5.3.6] we obtain that $(JB(p))^{\text{nor}} = (JB(p))^{\text{sat}}$. Thus $J^{\text{nor}}B(p) = J^{\text{sat}}B(p)$ for any p , and hence $J^{\text{nor}} \subseteq J^{\text{sat}} + \text{Rad}(O)[P]$. ♣

⁷(Michael) This is an annoying minor technical assumption. May consider later how to get rid of it – reducedness of X does not help.

2.2.10. *Blow ups.* By a *modification* of B we mean any morphism $f: B' \rightarrow B$ of solid log schemes which is proper and birational on the level of schemes. In fact, it will suffice to work with a special type of modifications. We say that f is a *blow up along \mathcal{I}* if the underlying morphism of schemes is the blow up along \mathcal{I} and $B'_{\text{tr}} = f^{-1}(B_{\text{tr}} \setminus V(\mathcal{I}))$. In other words, the solid logarithmic structure of B' is obtained by pulling back the logarithmic structure of B and enriching it by the Cartier divisor $\mathcal{I}\mathcal{O}_{B'}$. In this case, we will use the notation $B' = \text{Bl}_{\mathcal{I}}(B)$.

2.2.11. *Supports.* The following terminology will be used to control the supports. Assume given a morphism $g: B \rightarrow S$ and a closed subset $T \subseteq S$. Then, a modification $B' \rightarrow B$ is called a *T -modification* if it induces an isomorphism over $g^{-1}(S \setminus T)$. The important special case is obtained when f is a blow up along \mathcal{I} such that $V(\mathcal{I}) \subseteq g^{-1}(T)$. In this case we say that f is a *T -supported* blow up.

2.2.12. *Basic properties of blow ups.* The following properties immediately follow from the analogous well-known properties of usual blow ups: T -supported blow ups of logarithmic schemes are preserved by compositions, compatible with flat morphisms, cofinal in the family of all T -modifications, and can be extended from open subschemes. We will also need the following stronger cofinality property.

2.2.13. *Universal resolvability.* We say that B is *universally resolvable* if it is noetherian and any T -modification $B' \rightarrow B$ with $T \subseteq B$ is dominated by a T -supported blow up $B'' \rightarrow B$ with a formally split source. In the following lemma the assumption on universal resolvability is only needed to simplify the proof, but in few other results in the paper it will be essential.

cofinalth

Lemma 2.2.14. *Assume that $Y \rightarrow Z$ is a Kummer étale covering of solid logarithmic schemes, Z is universally resolvable, and $T \subseteq Z$ is closed. Then for any T -modification $Y' \rightarrow Y$ there exists a T -supported blow up $Z' \rightarrow Z$ such that $(Y \times_Z Z')^{\text{sat}} \rightarrow Y$ factors through Y' .*

Proof. We will denote the logarithmic schemes (Y, D_Y) , $(Y', D_{Y'})$, etc. In particular, $D_{Y'}$ is covered by the preimages of D_Y and T . It is an easy consequence of the flattening theorem of Raynaud-Gruson that there exists a T -supported blow up $Z' = \text{Bl}_{\mathcal{I}}(Z) \rightarrow Z$ such that $Y \times_Z Z' \rightarrow Y$ factors through Y' on the level of schemes. In addition, multiplying \mathcal{I} by the ideal of T we refine the blow up and also achieve that $T = V(\mathcal{I})$. Finally, replacing $Z' \rightarrow Z$ by an even larger blow up, we can also achieve that Z' is logarithmically regular.

Since $Y'' = (Y \times_Z Z')^{\text{sat}}$ is logarithmically étale over Z' , it is logarithmically regular and hence solid. By the construction, the morphism of schemes $Y'' \rightarrow Y$ factors through Y' , and $D_{Y''}$ contains the preimages of T and D_Y and hence also the preimage of $D_{Y'}$. Since Y'' is solid this implies that $Y'' \rightarrow Y$ factors through Y' as a morphism of logarithmic schemes. ♣

chafrtssec

2.3. **Charts and log fibers.** In Section 2.3 we will generalize the log stratification of log schemes to the case of morphisms $f: Y \rightarrow Z$. For an integral f , this will be just the log stratification of its fibers, but the general case requires more care. A conceptual definition of log fibers is that they are the fibers of the associated morphism $Y \rightarrow \mathbf{Log}(Z)$, whose target is Olsson's stack parameterizing log structures on Z -schemes, see [Ols03]. The latter work requires a strong stack-theoretic background, so to encourage the reader our exposition will be as follows: in this

section we provide a more involved but less technical definition of log fibers, which only uses independence of charts. Using this circle of ideas we will also provide an interpretation of the stack $\mathbf{Log}(Z)$ and the morphism $Y \rightarrow \mathbf{Log}(Z)$ skipping all technical details. In fact, we will obtain an explicit description of étale-local charts of $\mathbf{Log}(Z)$ which will be useful in sequel sections (mainly to circumvent technical points related to log flatness).

2.3.1. *Base change of monoids.* By \mathbf{A} we denote the natural functor from monoids to affine schemes: $\mathbf{A}_P = \mathrm{Spec}(\mathbb{Z}[P])$ and a homomorphism of monoids $\phi: P \rightarrow Q$ induces the morphism $\mathbf{A}_\phi: \mathbf{A}_Q \rightarrow \mathbf{A}_P$.

In the sequel, given a chart $u: P \rightarrow A$ of a logarithmic ring and a homomorphism of monoids $P \rightarrow Q$ we will use the notation $A[u^{Q/P}] = A \otimes_{\mathbb{Z}[P]} \mathbb{Z}[Q]$. If (A, m) is local and Q is sharp, then the completion of $A[u^{Q/P}]$ with respect to the maximal ideal generated by m and Q_+ will be denoted $\hat{A}[[u^{Q/P}]]$.

In the same fashion, if $Z \rightarrow \mathbf{A}_P$ is a global chart of a log scheme Z and $P \rightarrow Q$ is a homomorphism of monoids, we will use the notation $Z[u^{Q/P}] = Z \times_{\mathbf{A}_P} \mathbf{A}_Q$. In particular, any chart $Z \rightarrow \mathbf{A}_P, Y \rightarrow \mathbf{A}_Q, \phi: P \rightarrow Q$ of a morphism $f: Y \rightarrow Z$ factors f into a composition of a strict morphism $Y \rightarrow Z[u^{Q/P}]$ and a pullback $Z[u^{Q/P}] \rightarrow Z$ of the toric morphism \mathbf{A}_ϕ .

refchartsec

2.3.2. *Refinements of charts.* We say that a global chart $p': Z \rightarrow \mathbf{A}_{P'}$ of a log scheme Z *refines* a global chart $p: Z \rightarrow \mathbf{A}_P$ if there exists a homomorphism $\phi: P \rightarrow P'$ such that $p = \mathbf{A}_\phi \circ p'$. Similarly, refinement of a global chart $Z \rightarrow \mathbf{A}_P, Y \rightarrow \mathbf{A}_Q, P \rightarrow Q$ of a morphism $f: Y \rightarrow Z$ is a chart $Z \rightarrow \mathbf{A}_{P'}, Y \rightarrow \mathbf{A}_{Q'}, P \rightarrow Q'$ such that there exists compatible homomorphisms $P \rightarrow P', Q \rightarrow Q'$ as above.

refchartlem

Lemma 2.3.3. *Any two global charts of a log scheme or a morphism of log schemes possess a joint refinement.*

Proof. Let $M = M_Z(Z)$. Then global charts $P \rightarrow M$ and $P' \rightarrow M$ are refined by $P \times_M P' \rightarrow M$. The proof for morphisms is similar. ♣

indepchartsec

2.3.4. *Independence of charts.* Refinements of charts provides a natural tool to establish chart-independence of various constructions. It is even more natural to work directly with the following chart-independent quotient, which is however a stack:

indepchartcor

Corollary 2.3.5. *Assume that a morphism $f: Y \rightarrow Z$ of log schemes possesses a global chart $Z \rightarrow \mathbf{A}_P, Y \rightarrow \mathbf{A}_Q$. Consider the torus $T_{Q^{\mathrm{gp}}/P^{\mathrm{gp}}} = \mathbf{A}_{Q^{\mathrm{gp}}/P^{\mathrm{gp}}}$. Then the stacky quotient $\mathbf{Log}(Z)_Y = [Z[u^{Q/P}]/T_{Q^{\mathrm{gp}}/P^{\mathrm{gp}}}]$ and the factoring of the morphism of schemes $Y \rightarrow Z$ through $\mathbf{Log}(Z)_Y$ depend only on f and are independent of the choice of the chart.*

Note that the stack $\mathbf{Log}(Z)_Y$ is not provided with any log structure.

Proof. By the previous lemma it suffices to compare this chart and its refinement $Z \rightarrow \mathbf{A}_{P'}, Y \rightarrow \mathbf{A}_{Q'}$. In the latter case, the claim reduces to a simple diagram chase and the observations that $\mathbf{A}_{P'} \rightarrow \mathbf{A}_P$ is a torus fibration with fiber $T_{P'^{\mathrm{gp}}/P^{\mathrm{gp}}}$ and similarly for Q , and hence $\mathbf{A}_{Q'} \rightarrow \mathbf{A}_Q \times_{\mathbf{A}_P} \mathbf{A}_{P'}$ is a torus fibration with fiber $T_{Q'^{\mathrm{gp}}/(Q^{\mathrm{gp}}+P'^{\mathrm{gp}})}$. ♣

In fact, $\mathbf{Log}(Z)_Y$ is a part of a more general theory of Olsson’s stacks $\mathbf{Log}(Z)$ introduced in [Ols03].

Remark 2.3.6. Recall that the morphism of schemes $Y \rightarrow Z$ factors through $\mathbf{Log}(Z)$. In fact, $\mathbf{Log}(Z)_Y$ is an open substack in $\mathbf{Log}(Z)$ and the latter is actually glued from open substacks of this form with varying Y . In particular, the factoring $Y \rightarrow \mathbf{Log}(Z)$ exists without any assumption on global charts. In fact, charts and, hence, quotients $\mathbf{Log}(Z)_Y$ exist étale-locally on Z and the stack $\mathbf{Log}(Z)$ is glued from them by étale descent.

logfibsec

2.3.7. *Log fibers.* If f possesses a global chart $Y \rightarrow \mathbf{A}_Q, Z \rightarrow \mathbf{A}_P$, then consider the morphism $h: Y \rightarrow Z[u^{Q/P}]$ and define the stratum $S_y = S_{f,y}$ through a point $y \in Y$ to be the pullback of the orbit O_x of $T_{Q^{\text{gp}}/P^{\text{gp}}}$ through $x = h(y)$.

stabrem

Remark 2.3.8. We notice for future referencing that O_x is the $k(x)$ -torus with group of characters $R_y = Q_y^\times / (P^{\text{gp}} \cap Q_y^\times)$, where $Q_y^\times = Q^{\text{gp}} \cap \mathcal{O}_y^\times$.

It is easy to see that S_y is preserved under refinements of charts, hence it is independent of the chart by Lemma 2.3.3. This defines a covering of Y by locally closed subschemes, that we call a foliation (it can be a stratification or a covering by fibers of a morphism). Clearly, this foliation of Y is compatible with arbitrary strict morphisms $Y' \rightarrow Y$. In general, an étale neighborhood Y' of y possesses a global chart. The projections $Y'' = Y' \times_Y Y' \rightarrow Y'$ are strict, hence the strata of Y' and Y'' are compatible and via étale descent give rise to stratification of the fibers of f , whose strata will be called the *log fibers* of f .

Example 2.3.9. If $\phi: P \rightarrow Q$ is a homomorphism of monoids with $P^{\text{gp}} = Q^{\text{gp}}$, then the log smooth morphism \mathbf{A}_ϕ has trivial log fibers – the points of \mathbf{A}_Q . On the other hand, the fibers of \mathbf{A}_ϕ (and their log strata) may be large.

Remark 2.3.10. Log fibers do not have to be reduced. For example, it can freely happen when R_y contains torsion non-invertible in $k(y)$ and hence O_x is non-reduced.

logfibrem1

Remark 2.3.11. If a global chart exists, then S_y is the fiber of $h: Y \rightarrow \mathbf{Log}(Z)_Y$ over the “stacky point” $h(y) = O_x/T_{Q^{\text{gp}}/P^{\text{gp}}}$. (By Remark 2.3.8 we can also describe $h(y)$ as x/H_x , where $H_x = T_{Q^{\text{gp}}/(P^{\text{gp}}+Q_y^\times)}$ acts trivially on x .) It follows that in general the log fibers are nothing else but the fibers of the morphism $Y \rightarrow \mathbf{Log}(Z)$ (over the stacky points of $\mathbf{Log}(Z)$).

2.3.12. *Base change.* Log fibers are preserved by any base change:

fibchangelem

Lemma 2.3.13. *Assume that $f: Y \rightarrow Z$ and $g: Z' \rightarrow Z$ are morphism of log schemes and $f': Y' \rightarrow Z', g': Y' \rightarrow Y$ are the base changes. Then for any log fiber $C \hookrightarrow Y$ of f , the pullback $C \times_Y Y'$ is a log fiber of f' .*

Proof. ⁸

←8



logregsec

2.4. Log regular morphisms. ⁹

←9

⁸(Michael) Will be added in [MT19].

⁹(Michael) The topic will be developed in detail in [MT19].

2.4.1. *Log flatness.* Recall that a morphism of log schemes $f: Y \rightarrow Z$ is called *log flat* if étale-locally it admits charts $Y \rightarrow \mathbf{A}_Q$, $Z \rightarrow \mathbf{A}_P$ such that $P \rightarrow Q$ is injective and the morphism $Y \rightarrow Z[u^{Q/P}]$ is flat (see [Niz08, Definition 2.1]). This happens if and only if the morphism $h: Y \rightarrow \mathbf{Log}(Z)$ is flat, see [Ols03, Theorem 4.6(iv)].

2.4.2. *Logarithmically regular morphisms.* Regular morphisms are recalled in appendix §A.1. Unfortunately, the logarithmic analogue was not developed in the literature, but the basic theory is similar.

Definition 2.4.3. A morphism of log schemes $f: Y \rightarrow Z$ is called *log regular* at y if it is log flat at y and the log fiber S_y is geometrically regular at y . Furthermore, f is *log regular* if it is log regular at all points of Y .

For example, log smooth morphisms are log regular. It is also easy to see that log regular morphisms are preserved by base changes and compositions, but we postpone this to §2.7.

logregrem

Remark 2.4.4. (i) Since log fibers and log flatness of f correspond to fibers and flatness of the induced morphism $h: Y \rightarrow \mathbf{Log}(Z)$, we obtain that f is log regular (at y) if and only if h is regular (at y).

(ii) For morphisms of finite type regularity and smoothness are the same. Therefore, (i) and [Ols03, Lemm 4.8] imply that if f is of finite type, then it is smooth if and only if it is regular.

(iii) We will not prove this, but Kato's description of log smoothness generalizes directly: f is log regular at $y \in Y$ if and only if étale-locally at y there exists a chart such that the morphism $Y \rightarrow Z[u^{Q/P}]$ is regular at y , $\text{Ker}(P^{\text{gp}} \rightarrow Q^{\text{gp}})$ is a torsion invertible in $k(y)$, and the torsion of $\text{Coker}(P^{\text{gp}} \rightarrow Q^{\text{gp}})$ is invertible in $k(y)$.

2.4.5. *Regular parameters.* The notion of (relative) regular parameters applies only to simple regular points, see appendix A.1. To avoid this issue, we assume from now on that the characteristic is zero. Assume that $Y \rightarrow Z$ is log regular and $y \in Y$ is a point, and $S = S_y$ is its log fiber. By a *family of regular parameters* at y we mean any family $t_1, \dots, t_d \in \mathcal{O}_y$ whose image is a family of regular parameters of the regular ring $\mathcal{O}_{S,y}$ (that is, its image is a basis of the cotangent space $T'_{S,y} = m_{S,y}/m_{S,y}^2$ to S at y). Any element (resp. subset) of such a family will be called a *regular parameter at y* (resp. a *partial family* of regular parameters at y).

dimsec

2.4.6. *Relative log dimension.* Let $Y \rightarrow Z$ be a log regular morphism and $y \in Y$ a point with $z = f(y)$. By a *relative log dimension* of Y over Z at y we mean the number

$$\text{logdim}_y(Y/Z) = \dim_y(S_y) + \text{rk}(\text{Coker}(\overline{M}_z \rightarrow \overline{M}_y)).$$

In addition, $\text{logdim}(Y/Z) = \max_{y \in Y} \text{logdim}_y(Y/Z)$.

2.4.7. *Submanifolds.* An ideal $I \subseteq \mathcal{O}_Y$ will be called a *Z-submanifold ideal* if for any point $y \in Y$ there is a subset of a regular family of parameters at y that generates I locally at y . By a *log Z-submanifold of Y* we mean any strict closed subscheme given by a *Z-submanifold ideal*.

submanlem

Lemma 2.4.8. *If $Y \rightarrow Z$ is a log regular morphism and $T \hookrightarrow Y$ is a log Z-submanifold, then the morphism $T \rightarrow Y$ is log regular.*

Proof. Since $T \hookrightarrow Y$ is strict, the morphism $T \rightarrow \mathbf{Log}(Z)$ is obtained by restricting $Y \rightarrow \mathbf{Log}(Z)$ to T . Therefore the claim follows from Lemma A.1.3(ii). ♣

2.4.9. *Increasing and decreasing the log structure.* A standard tool in the theory of log regular schemes is to increase or decrease the log structure. For example, it is used in [AT17, Sections 3.4, 3.5] to study toroidal actions. We will use a relative version of this tool, which is also rather straightforward.

incrloglem

Lemma 2.4.10. *Assume that $f: Y \rightarrow Z$ is a log regular morphism and t_1, \dots, t_l a partial family of regular parameters at a point $y \in Y$. Let W be a neighborhood of y where t_i are defined and consider the log structure M'_W obtained from M_W by adding t_1, \dots, t_l , that is, M'_W is associated with the prelog structure $M_W \oplus \mathbb{N}^l \rightarrow \mathcal{O}_W$, where the basis elements of \mathbb{N}^l are sent to t_i . Then the morphism $(W, M'_W) \rightarrow Z$ is log regular at y and $\overline{M}'_y = \overline{M}_y \oplus \mathbb{N}^l$.*

Proof. Working étale-locally at y we can assume that $t_i \in \mathcal{O}_Y(Y)$ and f possesses a global chart $Y \rightarrow \mathbf{A}_Q, Z \rightarrow \mathbf{A}_P$. Let $Y' = (Y, M'_Y)$ denote the log scheme with enlarged log structure. Sending the basis of the second summand of $Q' = Q \oplus \mathbb{N}^l$ to t_i extends $Z \rightarrow \mathbf{A}_P$ to a chart $Y' \rightarrow \mathbf{A}_{Q'}$ of $f': Y' \rightarrow Z$. By our assumption $h: Y \rightarrow \mathbf{Log}(Z)_Y$ is regular at y . Since $\mathbf{A}_{Q'} = \mathbf{A}_Q \times \mathbb{A}^l$, we obtain that $\mathbf{Log}(Z)_{Y'} = [(Z_Y^{\log} \times \mathbb{A}^l)/\mathbb{G}_m^l]$ and the morphism $h': Y \rightarrow \mathbf{Log}(Z)_{Y'}$ is induced by h and t_1, \dots, t_l . By Lemma A.1.3(i), the morphism $Y \rightarrow (\mathbf{Log}(Z)_Y \times \mathbb{A}^l)$ is regular at y . Since $\mathbf{Log}(Z)_Y \times \mathbb{A}^l \rightarrow \mathbf{Log}(Z)_{Y'}$ is smooth, h' is regular at y too. ♣

The opposite direction is easier.

decrloglem

Lemma 2.4.11. *Let $f: Y \rightarrow Z$ be a log regular morphism and $y \in Y$ a point. Set $z = f(y)$, $P = \overline{M}_z$, $Q = \overline{M}_y$, and assume that Q splits as $Q' \oplus \mathbb{N}^l$ so that $P \rightarrow Q$ factors through Q' . Choose an étale neighborhood W of y such that generators of Q' extend to global sections of M_W , and let M'_W be the log structure induced by Q' . Then the morphism $f': (W, M'_W) \rightarrow Z$ is log regular at any preimage of y .*

Proof. The claim is étale-local, hence we can assume that f and f' possess global charts (not necessarily with Q and Q'). This time we have that

$$\mathbf{Log}(Z)_Y = [(\mathbf{Log}(Z)_{Y'} \times \mathbb{A}^l)/\mathbb{G}_m^l] = \mathbf{Log}(Z)_{Y'} \times [\mathbb{A}^l/\mathbb{G}_m^l],$$

and since the second factor is smooth and $h: Y \rightarrow \mathbf{Log}(Z)_Y$ is regular at y , we also have that $h': Y \rightarrow \mathbf{Log}(Z)_{Y'}$ is regular at y . ♣

logdifsec

2.5. Logarithmic differential operators.

2.5.1. *Logarithmic derivations and operators.* Let $f: Y \rightarrow Z$ be a morphism of solid logarithmic schemes. Recall that a *logarithmic \mathcal{O}_Z -derivations* on \mathcal{O}_Y with values in an \mathcal{O}_Y -module L consists of an \mathcal{O}_Z -derivation $\partial: \mathcal{O}_Y \rightarrow L$ and an (additive) homomorphism $\delta: M_Y \rightarrow L$ such that $\partial(u^m) = m\delta(m)$ for any monomial m , see [Ogu16, Definition IV.1.2.1]. Since Y is a locally integral scheme, giving (∂, δ) is equivalent to giving an \mathcal{O}_Z -derivations $\partial: \mathcal{O}_Y \rightarrow M$ preserving monomial ideals: $\partial(u^m) \in u^m \mathcal{O}_Y$. In particular, the sheaf of logarithmic derivations with values in \mathcal{O}_Y , that will be denoted $\mathcal{D}_{Y/Z}$, is a subsheaf of the usual sheaf of relative derivations $\mathcal{D}_{Y/Z}$. Note also that there always exists a universal derivation $d_{Y/Z}: \mathcal{O}_Y \rightarrow$

$\Omega_{Y/Z}^{\log}$, whose target is the *logarithmic module of differentials*. It is always quasi-coherent, unlike $\mathcal{D}_{Y/Z}$, and possesses better functoriality properties. Here is the most important example:

derfreelem

Lemma 2.5.2. *If f is logarithmically smooth, then $\Omega_{Y/Z}^{\log}$ and $\mathcal{D}_{Y/Z}$ are locally free, and the rank at a point $y \in Y$ equals $\dim_y(Y)$.*

Proof. Note that $\mathcal{D}_{Y/Z}^1$ is dual to the sheaf of logarithmic differentials, and by the work of Kato the latter is locally free of expected rank whenever f is logarithmically smooth. ♣

2.5.3. *The first fundamental sequence.* For a morphism $g: X \rightarrow Y$ we will denote by $\mathcal{D}_{Y/Z}(X)$ the \mathcal{O}_X -module of logarithmic $g^{-1}\mathcal{O}_Z$ -derivations $\partial: g^{-1}\mathcal{O}_Y \rightarrow \mathcal{O}_X$ with values in \mathcal{O}_X . (It is easily seen to coincide with the pullback of the module of \mathcal{O}_Z -derivations $\mathcal{O}_Y \rightarrow g_*\mathcal{O}_X$.)

firstseqlem

Lemma 2.5.4. *Given morphisms of log schemes $g: X \rightarrow Y$ and $f: Y \rightarrow Z$, there exist natural exact sequences*

$$0 \rightarrow \mathcal{D}_{X/Y} \rightarrow \mathcal{D}_{X/Z} \xrightarrow{\phi} \mathcal{D}_{Y/Z}(X), \quad g^*(\Omega_{Y/Z}^{\log}) \xrightarrow{\psi} \Omega_{X/Z}^{\log} \rightarrow \Omega_{X/Y}^{\log} \rightarrow 0.$$

In addition, if g is log smooth, then ψ is injective and ϕ is surjective. In particular, if $X \rightarrow Y$ is log étale then $\mathcal{D}_{X/Z} = \mathcal{D}_{Y/Z}(X)$.

Proof. The first exact sequence is tautological: any log Y -derivation on X is also a log Z -derivation, and any log Z -derivation on X can be restricted to Y . As usually, as a corollary one obtains the first fundamental sequence of log differentials, see also [Ogu16, Proposition IV.2.3.1]. (One can also construct the latter directly.) Note also that the first sequence can be obtained from the second one by applying the functor $\mathcal{H}om_{\mathcal{O}_X}(\cdot, \mathcal{O}_X)$. If g is log smooth, then $\Omega_{X/Y}^{\log}$ is a locally free module of finite rank and ψ is injective. Therefore ϕ is surjective in this case. ♣

difpullbacksec

2.5.5. *Pullback.* The embedding $g^{-1}\mathcal{D}_{Y/Z} \hookrightarrow \mathcal{D}_{Y/Z}(X)$ induces a natural homomorphism of \mathcal{O}_X -modules $g^*\mathcal{D}_{Y/Z} \rightarrow \mathcal{D}_{Y/Z}(X)$. If f is log smooth then $\Omega_{Y/Z}$ is locally free of finite rank, and hence $g^*\mathcal{D}_{Y/Z} = \mathcal{D}_{Y/Z}(X)$. By a slight abuse of language, for any \mathcal{O}_Y -submodule $\mathcal{F} \subseteq \mathcal{D}_{Y/Z}$ we will denote by $g^*\mathcal{F}$ its image in $\mathcal{D}_{Y/Z}(X)$. In general, $g^*\mathcal{D}_{Y/Z}$ can be much smaller than $\mathcal{D}_{Y/Z}(X)$, similarly to an infinite direct sum embedded into direct product.

2.5.6. *Base change.* The relation with base changes is as follows:

basechangelem

Lemma 2.5.7. *Let $f: Y \rightarrow Z$ and $g: Z' \rightarrow Z$ be morphisms of log schemes with base changes $f': Y' \rightarrow Z'$ and $g': Y' \rightarrow Y$. Then there is a natural isomorphism $\mathcal{D}_{Y/Z}(Y') = \mathcal{D}_{Y'/Z'}$.*

Proof. The inverse isomorphisms are given the restriction map and, conversely, by extending log \mathcal{O}_Z -derivations $\mathcal{O}_Y \rightarrow \mathcal{O}_{Y'}$ to log $\mathcal{O}_{Z'}$ -derivations $\mathcal{O}_{Y'} \rightarrow \mathcal{O}_{Y'}$ by $\mathcal{O}_{Z'}$ -linearity, see also [Ogu16, Proposition IV.1.2.3(2a)]. ♣

2.5.8. *Differential operators.* By the algebra of *logarithmic \mathcal{O}_Z -differential operators* $\mathcal{D}_{Y/Z}^{(\leq \infty)}$ on \mathcal{O}_Y we mean the algebra of \mathcal{O}_Z -linear operators on \mathcal{O}_Y generated by $\mathcal{D}_{Y/Z}$. It has a natural filtration by algebras of operators $\mathcal{D}_{Y/Z}^{(\leq d)}$ of degree at most d . In particular, $\mathcal{D}_{Y/Z}^{(\leq 1)}$ is a direct sum of \mathcal{O}_Y and $\mathcal{D}_{Y/Z}$. For any \mathcal{O}_Y -submodule $\mathcal{F} \subseteq \mathcal{D}_{Y/Z}$ we denote by $\mathcal{F}^{(\leq \infty)}$ the \mathcal{O}_Z -subalgebra of $\mathcal{D}_{Y/Z}^{(\leq \infty)}$ generated by \mathcal{F} and set $\mathcal{F}^{(\leq i)} = \mathcal{F}^{(\leq \infty)} \cap \mathcal{D}_{Y/Z}^{(\leq i)}$.

Remark 2.5.9. We will not need this, but $\mathcal{D}_{Y/Z} = \mathcal{D}er_{Y/\mathbf{Log}(Z)}$ and similarly $\mathcal{D}_{Y/Z}^{(\leq \infty)}$ are sheaves of differential operators of $Y/\mathbf{Log}(Z)$.

abundsec

2.5.10. *Abundance of derivations.* Recall that $\mathcal{D}er_{Y/Z}$ is coherent for morphisms of finite type, but, unlike the sheaves of differentials, it does not have to be even quasi-coherent in general because quasi-coherence is not preserved by filtered limits. See [Tem11, Example 2.3.5(ii)] for a pathological example with a regular $Y \rightarrow Z$. Our algorithm requires that $\mathcal{D}_{Y/Z}$ is large enough, and we formalize this as follows. Assume that $f: Y \rightarrow Z$ is a log regular morphism with a point $y \in Y$ and let $z = f(y)$, $\mathcal{D}_y = \mathcal{D}_{Y/Z,y}$, $P = \overline{M}_z$, $Q = \overline{M}_{f(y)}$, and $S = S_y$ the log fiber through y . Then the following three homomorphisms arise:

- (1) $\phi_1: \mathcal{D}_y \rightarrow M_1 = \text{Hom}_{k(y)}(m_{S,y}/m_{S,y}^2, k(y))$, where $\phi_1(\partial)(a) = \partial(a)$.
- (2) $\phi_2: \mathcal{D}_y \rightarrow M_2 = \text{Hom}_Z(\text{Coker}(P^{\text{gp}} \rightarrow Q^{\text{gp}}), k(y))$, where $\phi_2(\partial)(m) = m^{-1}\partial(m)$.
- (3) $\phi_3: \mathcal{D}_0 \rightarrow M_3 = \mathcal{D}_{k(y)/k(z)}$, where $\mathcal{D}_0 = \text{Ker}(\phi_1)$ the restriction map.¹⁰

←10

An \mathcal{O}_Y -submodule $\mathcal{F} \subseteq \mathcal{D}_{Y/Z}$ is called *separating at y* if $\mathcal{F}_y \rightarrow M_1$ is onto, *log separating at y* if $\mathcal{F}_y \rightarrow M_1 \oplus M_2$ is onto, *abundant at y* if $\mathcal{F}_y \rightarrow M_1 \oplus M_2 \oplus M_3$ has dense image with respect to the discrete topologies on M_1 and M_2 and the weak topology on M_3 . We say that \mathcal{F} is *separating*, *log separating* or *abundant* if it is so at all points of Y .

Remark 2.5.11. Informally speaking, \mathcal{F} is separating if it distinguishes regular parameters, it is log separating if it also distinguishes monomials, and it is abundant if it also distinguishes finite subsets of $k(y)$ transcendently independent over $k(z)$. This will be formalized in Lemma 2.5.15.

logmansec

2.5.12. *Relative logarithmic manifolds.* Let $f: Y \rightarrow Z$ be a log regular morphism. If $\mathcal{D}_{Y/Z}$ is log separating, then we say that f is a *relative log manifold* or Y is a *log Z -manifold*. If $\mathcal{D}_{Y/Z}$ is abundant, then we say that f has *abundance of derivations*.

relmanrem

Remark 2.5.13. (i) We do not know if this definition is étale-local on Y , but this will not cause to troubles in the sequel.

(ii) If f is log smooth, then it follows easily from Lemma 2.5.2 that it is a relative log manifold with abundance of derivations.

(iii) Intuitively, tangent spaces on a manifold are glued into a vector bundle $\mathcal{D}_{Y/Z}$, and this is what happens in the log smooth case. In general, the sheaf $\mathcal{D}_{Y/Z}$ can be very large and even not quasi-coherent, and the condition of being log separating is the minimal assumption needed to guarantee that $\mathcal{D}_{Y/Z}$ is large enough to be useful. Note also that it would not be enough to work with derivations of the

¹⁰(Michael) Needed corrections: exact morphism for abundance to be defined! Maybe move this to Section 3. Fix a chart $u: Q \rightarrow \mathcal{O}_y$ for ϕ_2 to be defined.

localizations $\overline{\mathcal{D}}_{\mathcal{O}_y/\mathcal{O}_{f(y)}}$, since it can be larger than the stalk $\mathcal{D}_{Y/Z,y}$. So, certain coherence of the tangent spaces is built into the definition.

2.5.14. *Derivations and parameters.* Given a family S of elements of \mathcal{O}_y , by a *dual family of derivations* we mean a subset of derivations $\partial_a \in \mathcal{D}_{Y/Z,y}$ indexed by elements of S such that $\partial_a(b) = 0$ if $a \neq b$, $\partial_a(a) = 1$ if a is not a monomial, and $\partial_a(a) = a$ if a is a monomial.

standardderiv

Lemma 2.5.15. *Let $f: Y \rightarrow Z$ be a log relative morphism, $y \in Y$ and $z = f(y)$. Let $t = \{t_1, \dots, t_n\} \in \mathcal{O}_y$ be a family of regular parameters at y and $q = \{q_1, \dots, q_r\} \in \overline{M}_y$ a family of monomials whose image is a basis of $\text{Coker}(\overline{M}_z^{gp} \rightarrow \overline{M}_y^{gp})$. If $\mathcal{F} \subseteq \mathcal{D}_{Y/Z}$ is an \mathcal{O}_Y -submodule, then*

(i) \mathcal{F} is separating at y if and only if the family t possesses a dual family of derivations from \mathcal{F}_y .

(ii) \mathcal{F} is log separating at y if and only if the family $\{t, u^q\}$ possesses a dual family of derivations from \mathcal{F}_y .

(iii) \mathcal{F} is abundant at y if and only if for any finite family of units $v = \{v_1, \dots, v_s\}$ whose image in $k(y)$ is algebraically independent over $k(z)$ the family $\{t, u^q, v\}$ possesses a dual family of derivations from \mathcal{F}_y .

Proof. All three claims are proved by the same argument, so we stick to (i) for simplicity of notation. Since \mathcal{F} is separating, there exist derivations $\partial'_1, \dots, \partial'_n \in \mathcal{F}_y$ such that $\partial'_k t_j \in \delta_{kj} + m_y$. In particular, the square matrix D with entries $d_{kj} = \partial'_k t_j$ lies in $\text{GL}_n(\mathcal{O}_y)$. We claim that for each $i \in \{1, \dots, n\}$ there exist $f_1, \dots, f_n \in \mathcal{O}_y$ such that $\partial_i = \sum_{k=1}^n f_k \partial'_k$ is as required. Indeed, this is equivalent to solving the system $\sum_{k=1}^n f_k d_{kj} = \delta_{ij}$, which is possible since D is invertible. ♣

logorbsec

2.6. **Relative log orbifolds.** As in the absolute case, the relative principalization algorithm involves non-representable analogues of blow ups. So, we are going to extend the theory of relative log manifolds to DM stacks.

2.6.1. *The definitions.* A morphism $f: Y \rightarrow Z$ of log DM stacks is called *log regular* if it is so étale-locally on Y and Z . The latter means that there exist a compatible strict étale coverings of Y and Z by log schemes Y_0 and Z_0 such that the morphism $f_0: Y_0 \rightarrow Z_0$ is log regular. Moreover, this property is actually independent of the choice of a covering because log regularity is local with respect to strict étale morphisms. A strict closed substack $Y' \hookrightarrow Y$ is called a *log Z -submanifold* if its preimage $Y' \times_Y Y_0$ is a log Z_0 -submanifold. Finally, we say that f is a *relative log orbifold* or Y is a *log Z -orbifold* if in addition there exists an étale covering with Y_0 a log Z_0 -manifold. In the latter case we have to choose the covering fine enough, see Remark 2.5.13(i).

Remark 2.6.2. Similarly to the absolute case studied in [ATW17a] one could also impose the condition that the relative inertia $I_{Y/Z}$ is finite and diagonalizable, since this condition persists under Kummer blow ups we will define later. Nevertheless we decided to work in the larger generality since imposing this condition as an additional assumption would not help to simplify any argument.

2.6.3. *Sheaves of derivations.* Given a log Z -orbifold Y find a strict étale covering $p: Y_0 \rightarrow Y$, $Z_0 \rightarrow Z$, and set $Y_1 = Y_0 \times_Y Y_0$. Strict étale (and even log étale) morphisms are compatible with the sheaves of log derivations, hence $p_1^* \mathcal{D}_{Y_0/Z_0} =$

$\mathcal{D}_{Y_1/Z_0} = p_2^* \mathcal{D}_{Y_0/Z_0}$ ¹¹ and this isomorphism satisfies the cocycle condition. By étale descent we obtain a sheaf $\mathcal{D}_{Y/Z}$ of log derivations of Y/Z . In the same way the sheaves of operators $\mathcal{D}_{Y/Z}^{(\leq i)}$ are defined. A submodule $\mathcal{F} \subseteq \mathcal{D}_{Y/Z}$ is *separating* if so is the submodule $\mathcal{F}_0 = p^* \mathcal{F}$ of $\mathcal{D}_{Y_0/Z} = p^* \mathcal{D}_{Y_0/Z_0}$.

functorsec

2.7. Functoriality. It will be important throughout the text to check that all our constructions on log Z -orbifolds Y are functorial with respect to relative log orbifolds $Y' \rightarrow Y$ and arbitrary base changes $Z' \rightarrow Z$. So, let us establish functoriality of notions introduced so far.

2.7.1. Base changes. Base change functoriality is very robust and easy for proofs.

basechangeprop

Proposition 2.7.2. *Let $f: Y \rightarrow Z$ and $g: Z' \rightarrow Z$ be morphisms of log DM stacks with base changes $f': Y' \rightarrow Z'$ and $g': Y' \rightarrow Y$, and assume that f is log regular. Then*

- (i) *The morphism f' is log regular. Moreover, if f is a relative log manifold or a relative log orbifold, then f' satisfies the same property.*
- (ii) *If a submodule \mathcal{F} of $\mathcal{D}_{Y/Z}$ is separating (resp. log separating, resp. abundant), then $g'^*(\mathcal{F})$ satisfies the same property.*
- (iii) *Assume that X, Y, Z are log schemes. If t_1, \dots, t_n are regular parameters at a point $y \in Y$, then their pullbacks are regular parameters at the preimages of y in Y' .*
- (iv) *If $T \hookrightarrow Y$ is a log Z -submanifold then $T' = T \times_Y Y'$ is a log Z' -submanifold of Y' .*

Proof. Log flatness is preserved by base changes by [Niz08, Lemma 2.7(1)], and log fibers are compatible with base changes by Lemma 2.3.13. Therefore, f' is log regular. Claim (iii) follows from Lemma 2.3.13, and claim (iv) is its consequence. The separating property in (ii) is preserved since the log fibers are preserved by the base change. This implies the second part of (i). ♣

2.7.3. Compositions. The composition functoriality is a little bit more subtle.

coverprop

Proposition 2.7.4. *Assume that $g: X \rightarrow Y$ and $f: Y \rightarrow Z$ are log regular morphisms of DM log stacks and $h: X \rightarrow Z$ the composition. Then*

- (i) *The morphism h is log regular. Moreover, if both f and g are relative log manifolds or relative log orbifolds, then the same is true for h .*
- (ii) *If f and g are log regular, and $\mathcal{D}_{X/Y}$ and a submodule $\mathcal{F}_{Y/Z}$ of $\mathcal{D}_{Y/Z}$ are separating (resp. log separating, resp. abundant), then the preimage of $g^* \mathcal{F}$ under the homomorphism $\mathcal{D}_{X/Z} \rightarrow \mathcal{D}_{Y/Z}(X)$ satisfies the same property.*
- (iii) *Assume that X, Y, Z are log schemes, t_1, \dots, t_n are regular parameters of g at a point $x \in X$, and s_1, \dots, s_l are regular parameters of f at $g(x)$. Then $t_1, \dots, t_n, g^\#(s_1), \dots, g^\#(s_l)$ are regular parameters of h at x .*
- (iv) *If $T \hookrightarrow Y$ is a log Z -submanifold, then $T \times_Y X$ is a log Z -submanifold of X .*

*Proof.*¹²

¹¹(Michael) Should be more careful – $p_i^{-1} \mathcal{D}_{Y_0/Z_0}(Y_1)$ should be used instead.

¹²(Michael) Will be given in [MT19]. In particular, (i) follows from the theorem proven there that f is log regular iff $\mathbf{Log}(f)$ is regular.

←11


←12



2.7.5. *Finite type stability.* Finally, we show that largeness of $\mathcal{D}_{Y/Z}$ is preserved under morphisms of finite type:

manifoldth

Theorem 2.7.6. *Assume that $f: Y \rightarrow Z$ is a relative log orbifold and $g: Y' \rightarrow Y$ is a morphism of finite type such that $f': Y' \rightarrow Z$ is log regular. Then f' is a relative log orbifold too. In particular, a log Z -submanifold of a log Z -orbifold is a log Z -orbifold too.*

Proof. Find strict étale covers $Y_0 \rightarrow Y$ and $Y'_0 \rightarrow Y'$ such that Y' and Y'_0 are schemes and $Y'_0 \rightarrow Y$ factors through $g': Y'_0 \rightarrow Y_0$. Then replacing g by g' reduces the question to the case of log Z -manifolds. Now, we can work locally at a point $y' \in Y'$ with image $y \in Y$. It is easy to see that locally Y' can be embedded as a log Z -submanifold into a log Z -manifold W such that W is log smooth over Y . Therefore, it suffices to consider two cases: $Y' \rightarrow Y$ is log smooth, or $Y' \hookrightarrow Y$ is a log Z -submanifold. The first case is covered by Proposition 2.7.4(i). In the second case, choose parameters $t_1, \dots, t_n \in \mathcal{O}_y$ such that Y' is given by vanishing of t_1, \dots, t_l , and find derivations $\partial_1, \dots, \partial_n \in \mathcal{D}_{Y/Z, y}$ as in Lemma 2.5.15. Since $\partial_{l+1}, \dots, \partial_l$ vanish on t_1, \dots, t_l , they induce derivations on Y' . Clearly, these derivations generate a sheaf of derivations on Y'/Z which is separating at y' . 

2.8. Relative logarithmic order of ideals.

monomsatsec

2.8.1. *Monomial saturation.* By *monomial saturation* $\mathcal{M}(\mathcal{I})$ of an ideal \mathcal{I} on a log DM stack Y we mean the minimal monomial ideal containing \mathcal{I} .

difsatur


2.8.2. *Differential saturation.* Let $Y \rightarrow Z$ be a log regular morphism and $\mathcal{F} \subseteq \mathcal{D}_{Y/Z}$ an \mathcal{O}_Y -submodule. We say that an ideal $\mathcal{I} \subseteq \mathcal{O}_Y$ is \mathcal{F} -saturated if $\mathcal{F}(\mathcal{I}) = \mathcal{I}$. For example, any monomial ideal is \mathcal{F} -saturated. The ideal, $\mathcal{F}^{(\leq \infty)}(\mathcal{I})$ will be called the \mathcal{F} -saturation of \mathcal{I} . Obviously, it is the minimal \mathcal{F} -saturated ideal containing \mathcal{I} . The ideal $\mathcal{M}_{Y/Z}(\mathcal{I}) := \mathcal{D}_{Y/Z}^{(\leq \infty)}(\mathcal{I})$ will be called the \mathcal{D} -saturation of \mathcal{I} over Z .

2.8.3. *Log clean ideals.* We say that \mathcal{I} is *log clean* at a point $y \in Y$ over Z if $\mathcal{M}_{Y/Z}(\mathcal{I})_y = \mathcal{M}(\mathcal{I})_y$, and \mathcal{I} is *log clean* over Z if $\mathcal{M}_{Y/Z}(\mathcal{I}) = \mathcal{M}(\mathcal{I})$.

Remark 2.8.4. A basic intuition beyond this definition is illustrated by the following model case: $Z = \text{Spec}(O)$ with a log structure $P \rightarrow O$ and $Y = \text{Spec}(A)$ with $A = O[u^Q][t_1, \dots, t_n]$. If $\mathcal{I} \subseteq A$ is generated by $f_i = \sum_{q \in Q, l \in \mathbb{N}^n} b_{iql} u^q t^l$, then it is easy to see (and will be shown in Section 3) that $\mathcal{M}_{A/O}(\mathcal{I})$ is the ideal generated by the elements $b_{iql} u^q$. In particular, if these elements are monomial then \mathcal{I} is log clean, and this can be achieved by a modification of the base, and even just by enlarging P . Quite surprisingly, extending this to more general log regular morphisms, and even log smooth ones, is substantially more difficult. This will be the central topic of Section 3.

logcleanlem

Lemma 2.8.5. *Let $f: Y \rightarrow Z$ be a log regular morphism, $\mathcal{I} \subseteq \mathcal{O}_Y$ an ideal, and $\mathcal{F} \subseteq \mathcal{D}_{Y/Z}$ an \mathcal{O}_Y -submodule. If $\mathcal{M}_{\mathcal{F}}(\mathcal{I})$ is monomial, then \mathcal{I} is log clean over Z and $\mathcal{M}_{\mathcal{F}}(\mathcal{I}) = \mathcal{M}_{Y/Z}(\mathcal{I}) = \mathcal{M}(\mathcal{I})$.*

Proof. If $\mathcal{M}_{\mathcal{F}}(\mathcal{I})$ is monomial, then it is the smallest monomial ideal containing \mathcal{I} . 

2.8.6. *The log order.* Let $Y \rightarrow Z$ be a log regular morphisms and \mathcal{I} an ideal on Y . Similarly to [ATW17a, §3.6.1], if Y is a scheme then by the (relative) log order $\text{logord}_y(\mathcal{I})$ of \mathcal{I} at a point $y \in Y$ we mean the usual order of $\mathcal{I}|_S$ at y , where $S = S_y$ is the log fiber of y . Since log order is compatible with strict étale morphisms $Y' \rightarrow Y$, this definition extends to the case when Y is a log DM stack. In particular, we obtain a function $\text{logord}_{\mathcal{I}/Z}: |Y| \rightarrow \mathbb{N} \cup \{\infty\}$. By $\text{logord}(\mathcal{I}) = \max_{x \in X} \text{logord}_x(\mathcal{I})$ we denote the (*maximal*) log order of \mathcal{I} on Y .

balancedsec

2.8.7. *Clean ideals.* Let $Y \rightarrow Z$ be a log regular morphism. An ideal \mathcal{I} is called *Z-clean* at a point $y \in |Y|$ if $\text{logord}_y(\mathcal{I}) < \infty$, and \mathcal{I} is called *Z-clean* or simply *clean* if it is *Z-clean* at all points of Y .

2.8.8. *Relation to derivations.* As in the absolute case, the relative logarithmic order can be computed using derivations.

logordlem

Lemma 2.8.9. *Let $Y \rightarrow Z$ be a log regular morphism, $\mathcal{F} \subseteq \mathcal{D}_{Y/Z}$ a separating \mathcal{O}_Y -submodule, \mathcal{I} an ideal, and $y \in |Y|$ a point. Then*

$$\text{logord}_y(\mathcal{I}) = \min\{a \in \mathbb{N} \mid \mathcal{F}^{(\leq a)}(\mathcal{I})_y = \mathcal{O}_{Y,y}\},$$

where $\min(\emptyset) = \infty$ by convention. In particular, \mathcal{I} is clean at y if and only if $y \notin V(\mathcal{M}_{\mathcal{F}}(\mathcal{I}))$, and \mathcal{I} is clean if and only if $\mathcal{M}_{\mathcal{F}}(\mathcal{I}) = 1$.

Proof. Let S be the log fiber of y , and set $\mathcal{I}_S = \mathcal{I}|_S$ and $\mathcal{F}_S = \mathcal{F}|_S$. Since $\mathcal{F}^{(\leq a)}(\mathcal{I})|_S = (\mathcal{F}_S)^{(\leq a)}(\mathcal{I}_S)$, we should check that $\text{ord}_y(\mathcal{I}_S)$ is the minimal a such that $\mathcal{F}_S^{(\leq a)}(\mathcal{I}_S)_y = \mathcal{O}_{S,y}$. It suffices to check the latter in the formal completion $\widehat{\mathcal{O}}_{S,y}$. Fix a family of regular parameters t_1, \dots, t_n . Then $\widehat{\mathcal{O}}_{S,y} \simeq k(y)[[t_1, \dots, t_n]]$, and by Lemma 2.5.15, \mathcal{F}_S contains derivations ∂_i such that $\partial_i(t_j) = \delta_{ij}$ (though the action on $k(y)$ can be non-trivial). This makes the assertion obvious. ♣

Remark 2.8.10. Assume that $\mathcal{D}_{Y/Z}$ is separating. Then by the above lemma, any clean ideal is log clean.

2.8.11. *Balanced ideals.* An ideal $\mathcal{I} \subseteq \mathcal{O}_Y$ is called *balanced* if it is log clean and $\mathcal{M}(\mathcal{I})$ is invertible.

balancedlem

Lemma 2.8.12. *An ideal \mathcal{I} is balanced if and only if it is of the form $\mathcal{I} = \mathcal{M} \cdot \mathcal{I}^{\text{cln}}$, where \mathcal{M} is invertible monomial and \mathcal{I}^{cln} is clean.*

Proof. Only the direct implication needs a proof. If \mathcal{I} is balanced, then $\mathcal{M} = \mathcal{M}_{Y/Z}(\mathcal{I})$ is invertible, hence an ideal $\mathcal{I}^{\text{cln}} = \mathcal{M}^{-1}\mathcal{I}$ is defined. Since $\mathcal{M} = \mathcal{M}_{Y/Z}(\mathcal{M}\mathcal{I}^{\text{cln}}) = \mathcal{M}\mathcal{M}_{Y/Z}(\mathcal{I}^{\text{cln}})$, we obtain that $\mathcal{M}_{Y/Z}(\mathcal{I}^{\text{cln}}) = 1$, and hence \mathcal{I}^{cln} is clean by Lemma 2.8.9. ♣

The ideal $\mathcal{I}^{\text{cln}} = \mathcal{M}^{-1}\mathcal{I}$ will be called the *clean part* of \mathcal{I} .

2.8.13. *Functoriality.* We conclude this section with studying functoriality of \mathcal{D} -saturation and log order. We start with base changes.

functorlogorder1

Lemma 2.8.14. *Let $f: Y \rightarrow Z$ be a log regular morphism of log DM stacks, $\mathcal{I} \subseteq \mathcal{O}_Y$ an ideal, $\mathcal{F} \subseteq \mathcal{D}_{Y/Z}$ an \mathcal{O}_Y -submodule, $g: Z' \rightarrow Z$ a morphism of log DM stacks with base changes $g': Y' \rightarrow Y$ and $f': Y' \rightarrow Z'$, and $\mathcal{I}' = g'^{-1}\mathcal{I}$. Then*

(i) $\text{logord}_{\mathcal{I}/Z} = \text{logord}_{\mathcal{I}'/Z'} \circ |f'|$. In particular, if \mathcal{I} is *Z-clean*, then \mathcal{I}' is *Z'-clean*.

- (ii) If $\mathcal{M}_{\mathcal{F}}(\mathcal{I})$ is monomial, then $g'^{-1}\mathcal{M}_{\mathcal{F}}(\mathcal{I}) = \mathcal{M}_{g'^*\mathcal{F}}(g'^{-1}\mathcal{I})$.
- (iii) If \mathcal{I} is log clean over Z , then \mathcal{I}' is log clean over Z' .

Proof. Claim (i) follows from Lemma 2.3.13. Claim (ii) can be checked étale-locally using global sections $h \in \Gamma(\mathcal{I})$ and $\partial \in \Gamma(\mathcal{F})$, and then it becomes a tautology. Taking $\mathcal{F} = \mathcal{D}_{Y/Z}$ and applying Lemma 2.8.5 we obtain (iii). ♣

Log regular functoriality is checked similarly.

functorlogorder2

Lemma 2.8.15. *Let $g: X \rightarrow Y$ and $f: Y \rightarrow Z$ be log regular morphism of log DM stacks, $\mathcal{I} \subseteq \mathcal{O}_Y$ an ideal with $\mathcal{I}' = g^{-1}\mathcal{I}$, and $\mathcal{F} \subseteq \mathcal{D}_{Y/Z}$ a submodule. Then,*

- (i) $\text{logord}_{\mathcal{I}/Z} = \text{logord}_{\mathcal{I}'/Z} \circ |g|$. In particular, if \mathcal{I} is Z -clean, then \mathcal{I}' is Z -clean too.
- (ii) If $\mathcal{M}_{\mathcal{F}}(\mathcal{I})$ is monomial, then $g^{-1}\mathcal{M}_{\mathcal{F}}(\mathcal{I}) = \mathcal{M}_{g^*\mathcal{F}}(g^{-1}\mathcal{I})$.
- (iii) If \mathcal{I} is log clean over Z , then \mathcal{I}' is log clean over Z and $\mathcal{M}(\mathcal{I}') = g^{-1}(\mathcal{M}(\mathcal{I}))$.

Proof. Claim (i) follows from [MT19] and properties of the usual order. Claim (ii) can be checked étale-locally using global sections $h \in \Gamma(\mathcal{I})$ and $\partial \in \Gamma(\mathcal{F})$, and then it becomes a tautology. Taking $\mathcal{F} = \mathcal{D}_{Y/Z}$ and applying Lemma 2.8.5 we obtain (iii). ♣

3. MONOMIALIZATION OF \mathcal{D} -SATURATED IDEALS

monomsec

We keep our assumptions on $f: X \rightarrow B$ as in Section 2.1.

intsec

3.1. Integral homomorphisms.

3.1.1. *Some motivation.* Elements of $A[u^{Q/P}]$ and $\widehat{A}[[u^{Q/P}]]$ can be expressed as polynomials and power series in u^q with coefficients in \mathcal{A} , but, unless $P = 1$, the presentation is not unique. This explains why it is technically difficult to work with arbitrary base changes $A[u^{Q/P}]$ and $\widehat{A}[[u^{Q/P}]]$. The standard way to circumvent this issue is to stick to the case when $P \rightarrow Q$ is integral, which, as we will later see, is a rather harmless assumption.

3.1.2. *Definitions.* Recall that a homomorphism $\phi: P \rightarrow Q$ of integral monoids is called *integral* if for any homomorphism of integral monoids $P \rightarrow P'$ the monoid $Q \otimes_P P'$ is integral. Integrality can be characterized in many equivalent ways. In particular, by [Kat89, Proposition 4.1] ϕ is integral if and only if $\mathbb{Z}[P] \rightarrow \mathbb{Z}[Q]$ is flat, and this is perhaps the best explanation why this notion is very important. Note that in this case ϕ is automatically injective. Also, by [Ogu16, Corollary 4.6.11] one has the following analogue of Lazard's theorem on flatness: Let $\widetilde{Q} = Q/P^{\text{gp}}$ denote the image of the homomorphism $Q \rightarrow Q^{\text{gp}}/P^{\text{gp}}$, then ϕ is integral if and only if the fibers of the map $\lambda: Q \rightarrow \widetilde{Q}$ are filtered unions of free P -orbits. The following remark is only given for the sake of completeness.

Remark 3.1.3. Proofs and actual computations are based on another criterion provided by [Kat89, Proposition 4.1]: ϕ is integral if and only if for any $p_1, p_2 \in P$ and $q_1, q_2 \in Q$ such that $q_1 + \phi(p_1) = q_2 + \phi(p_2)$, there exist $q \in Q$ and $p'_1, p'_2 \in P$ such that $q_i = q + \phi(p'_i)$ for $i = 1, 2$ and $p_1 + p'_1 = p_2 + p'_2$.

3.1.4. *The canonical section.* In the fine case one can simplify the last criterion as follows: if Q is fine, then ϕ is integral if and only if each fiber $\lambda^{-1}(\tilde{q})$ is a free P -orbit $q_0 + P$. In particular, in this case $\mathbb{Z}[Q]$ is even a free $\mathbb{Z}[P]$ -module. If, moreover, P is sharp then the choice of q_0 is unique, and we obtain a canonical set-theoretical section $i: \tilde{Q} \hookrightarrow Q$ of λ and a bijection $Q = \tilde{Q} \times P$, which are not, however, homomorphisms. By abuse of language, we will identify in this case Q/P^{gp} with the corresponding subset of Q . In particular, this convention is used in the following obvious result:

intbasechangelem

Lemma 3.1.5. *Assume that $P \rightarrow Q$ is an integral homomorphism of fine monoids and $u: P \rightarrow A$ is a logarithmic ring. Then any element $h \in A[u^{Q/P}]$ possesses a unique presentation as a polynomial $h = \sum_{q \in Q/P^{\text{gp}}} h_q u^q$ with $h_q \in A$, and hence $A[u^{Q/P}]$ acquires a natural $Q^{\text{gp}}/P^{\text{gp}}$ -grading $A[u^{Q/P}] = \bigoplus_{q \in Q/P^{\text{gp}}} u^q A$.*

In the local case, an explicit description of the formal completion follows immediately from [AT18, Proposition 4.5.6]. Notice that a formal grading is a product rather than a sum, and it makes sense only when all non-trivially graded components lie in the maximal ideal.

formalgradedcor

Corollary 3.1.6. *In addition to assumptions of the lemma assume that A is local and Q is sharp, and let \hat{A} be the completion of A . Then $\hat{A}[[u^{Q/P}]]$ acquires a formal grading $\hat{A}[[u^{Q/P}]] = \prod_{q \in Q/P^{\text{gp}}} u^q \hat{A}$. In particular, any element $h \in \hat{A}[[u^{Q/P}]]$ possesses a unique presentation as a power series $h = \sum_{q \in Q/P^{\text{gp}}} h_q u^q$ with $h_q \in \hat{A}$.*

3.2. Abundance of derivations.

3.2.1. *Formal description.* The classical formal-local description of regular morphisms in the characteristic zero case (e.g. see [AT18, Remark 2.2.12]) extends to logarithmically regular morphism as follows:

logregchart

Proposition 3.2.2. *Assume that $g: Y \rightarrow Z$ is a morphism of noetherian logarithmic schemes of characteristic zero, $y \in Y$ is a point with $z = g(y)$, $Q = \overline{M}_y$ and $P = \overline{M}_z$. Then the following conditions are equivalent:*

- (1) g is logarithmically regular at y .
- (2) The homomorphism $P \rightarrow Q$ is injective and lifts to a chart $Q \rightarrow \mathcal{O}_Y(Y_0)$, $P \rightarrow \mathcal{O}_Z(Z_0)$ of g at y such that the induced morphism $Y_0 \rightarrow Z_0 \otimes_P Q$ is regular at y .
- (3) There exist elements $t_1, \dots, t_n \in \hat{\mathcal{O}}_y$, a chart $Q \rightarrow \mathcal{O}_Y(Y_0)$, $P \rightarrow \mathcal{O}_Z(Z_0)$ of g at y and a choice of fields of coefficients $k(z) \hookrightarrow \hat{\mathcal{O}}_z$ and $k(y) \hookrightarrow \hat{\mathcal{O}}_y$ such that

$$(\hat{\mathcal{O}}_z \hat{\otimes}_{k(z)} k(y))[[u^{Q/P}]][[t_1, \dots, t_n]] = \hat{\mathcal{O}}_y.$$

Proof. No problems expected, fill later. ♣

3.2.3. *Formal derivations.* Let us also describe a formal picture. So, assume that $P \rightarrow Q$ is an embedding of sharp fs monoids, O is a complete noetherian local ring with a logarithmic structure $u: P \rightarrow O$, and l is a field extension of $k = O/m_O$. Let $A = (O \hat{\otimes}_k l)[[t_1, \dots, t_n]][[u^{Q/P}]]$. We will work with the usual module of logarithmic derivations $\mathcal{D}_{A/O}$ because any derivation ∂ on A takes m_A^n to m_A^{n-1} and hence is continuous. In particular, this implies that a derivation $\partial \in \mathcal{D}_{A/O}$ is determined by its action to l , Q and t_1, \dots, t_n .

Consider the following three types of derivations, whose uniqueness follows from the above and existence is an easy exercise:

- (1) Ordinary derivations: ∂_i is defined by $\partial_i(t_j) = \delta_{ij}$, $\partial_i(l) = \partial(u^Q) = 0$.
- (2) Monomial derivations: for any $\phi \in \text{Hom}_{\mathbb{Z}}(Q^{\text{gp}}/P^{\text{gp}}, A)$ ∂_ϕ is defined by $\partial_\phi(u^q) = \phi(q)u^q$, $\partial_\phi(t_j) = \partial_\phi(l) = 0$.
- (3) Constant derivations: any $\partial_l \in \mathcal{D}_{l/k}$ uniquely extends to $\mathcal{D}_{A/O}$ so that $\partial_l(t_j) = \partial_l(u^Q) = 0$.

formderlem

Lemma 3.2.4. *If $A = (O \widehat{\otimes}_k l)[[t_1, \dots, t_n]][[u^{Q/P}]]$ is as above, then*

$$\mathcal{D}_{A/O} = (\mathcal{D}_{l/k} \otimes_l A) \oplus \text{Hom}_{\mathbb{Z}}(Q^{\text{gp}}/P^{\text{gp}}, A) \oplus (\oplus_{i=1}^n A \partial_i).$$

In particular, $\mathcal{D}_{A/O}$ is free, and if l/k is perfect and the torsion order of $Q^{\text{gp}}/P^{\text{gp}}$ is invertible in O , then $\text{rk}(\mathcal{D}_{A/O}) = \text{tr.deg.}(l/k) + \text{rk}_{\mathbb{Q}}(Q^{\text{gp}}/P^{\text{gp}}) + n$.

Proof. Restricting derivations onto l , Q and t_1, \dots, t_n one obtains projections of $\mathcal{D}_{A/O}$ to the direct summands. The induced map to the direct sum is injective because derivations are determined by the restrictions. The surjectivity follows because any element of the direct sum lifts to the corresponding linear combination of derivations $\partial_i, \partial_\phi$ and ∂_l . \clubsuit

3.2.5. *Abundance of derivations.* Assume that $g: Y \rightarrow Z$ is logarithmically regular, $y \in Y$ a point with $z = g(y)$, $Q = \overline{M}_y$ and $P = \overline{M}_z$. We say that g has *abundance of derivations at y* if the natural map $\mathcal{D}_{Y/Z, y}^1 \otimes_{\mathcal{O}_y} \widehat{\mathcal{O}}_y \rightarrow \mathcal{D}_{\widehat{\mathcal{O}}_y/\widehat{\mathcal{O}}_z}^1$ is an isomorphism.

abundrem

Remark 3.2.6. (1) If g is logarithmically smooth, then it has abundance of derivations. Indeed, this follows from Lemmas 2.5.2 and 3.2.4 by comparing the ranks.

(2) In general, abundance of derivations takes place if and only if the natural embeddings $\mathcal{D}_{Y/Z, y} \hookrightarrow \mathcal{D}_{\mathcal{O}_y/\mathcal{O}_z}^1$ and $\mathcal{D}_{\mathcal{O}_y/\mathcal{O}_z}^1 \otimes_{\mathcal{O}_y} \widehat{\mathcal{O}}_y \hookrightarrow \mathcal{D}_{\widehat{\mathcal{O}}_y/\widehat{\mathcal{O}}_z}^1$ are isomorphisms. Since $\mathcal{D}_{Y/Z}$ does not have to be quasi-coherent, both conditions are non-trivial.

(3) Loosely speaking, the abundance condition means that locally at y there are enough derivations to distinguish, regular coordinates of the logarithmic fiber through y , the monomial coordinates of the fiber at y , and the transcendental elements of $k(y)/k(z)$.

stablesec

3.2.7. *Stability of abundance.*

3.3. \mathcal{D} -saturated and logarithmically fibral ideals.

3.3.1. *Logarithmically fibral ideals.* We say that an ideal $\mathcal{I} \subseteq \mathcal{O}_X$ is *logarithmically fibral over B at a point $x \in X$* if \mathcal{I}_x is a sum of ideals $u^q \mathcal{I}_q \mathcal{O}_x$, where \mathcal{I}_q are ideals of $\mathcal{O}_{f(x)}$. An ideal is *logarithmically fibral over B* if it is so at all points of X .

logfibrem

Remark 3.3.2. (1) Let us say that an ideal $\mathcal{I} \subseteq \mathcal{O}_X$ is *fibral* if it is logarithmically fibral with respect to the trivial logarithmic structures. This happens if and only if locally on Y the ideal \mathcal{I} is induced from B . The two notions are related as follows: \mathcal{I} is logarithmically fibral over B if and only if it is fibral over $\mathbf{Log}(B)$.

(2) The property of being a logarithmically fibral ideal is preserved under sums of ideals and base changes $B' \rightarrow B$. It is not easy to see that it is also preserved by intersections, but this follows from Corollary 3.3.10 below. However, being logarithmically fibral is not preserved under étale descent even in the case of trivial logarithmic structures. For example, let B be a neighborhood of the origin $b \in \mathbf{A}^2$

such that there exists a finite étale double covering $X \rightarrow B$ that separates the branches of the nodal curve $C = V(x^2 - y^2 - y^3)$. The preimage \tilde{C} of C in X defines an ideal $\mathcal{I}_{\tilde{C}}$, which is obviously non-fibral over B . However, the base change $B' = X$ splits the cover, and so any ideal on $X' = X \times_B B'$, including $\mathcal{I}_{\tilde{C}}\mathcal{O}_{X'}$, are fibral over B' .

3.3.3. *\mathcal{D} -saturatedness.* We say that an ideal $\mathcal{I} \subseteq \mathcal{O}_X$ is *\mathcal{D} -saturated over B at a point $x \in X$* if $\mathcal{D}_{X/B}^{(\leq 1)}(\mathcal{I}_x) = \mathcal{I}_x$, and \mathcal{I} is *\mathcal{D} -saturated over B* if it is so at all points of X . When B is clear from the context we will simply say that \mathcal{I} is *\mathcal{D} -saturated*. In particular, if $X \rightarrow B$ is logarithmically étale then any ideal is \mathcal{D} -saturated. The following obvious result will be our main source of producing \mathcal{D} -saturated ideals:

infstylem

Lemma 3.3.4. *For any ideal $\mathcal{I} \subseteq \mathcal{O}_X$, the ideal $\mathcal{D}_{X/B}^\infty(\mathcal{I})$ is the minimal \mathcal{D} -saturated ideal that contains \mathcal{I} .*

dstablerem

Remark 3.3.5. It follows from the definition that \mathcal{D} -saturated ideals are closed under sums, intersections and base changes. In addition, it is easy to see that this property is preserved by étale descent.

3.3.6. *Relation between the two notions.* A direct inspection shows that any logarithmically fibral ideal is \mathcal{D} -saturated. We will not do it, but using a theory we develop below one can show that in the case of trivial logarithmic structures, any \mathcal{D} -saturated ideal is fibral étale-locally on B . In view of Remarks 3.3.2(3) and 3.3.5, this is essentially the best one might hope for. Surprisingly, the question if this extends to general logarithmic structures (with respect to the Kummer-étale topology) turns out to be substantially more difficult, and we do not know the answer. For example, reduction to the non-logarithmic case using the map $X \rightarrow \mathbf{Log}(B)$ seems to be impossible because $\mathbf{Log}(B)$ is an Artin stack, and its étale topology is too coarse. As a main result of §3.3 we will obtain a rather special criterion for a \mathcal{D} -saturated ideal to be logarithmically fibral.

assumsec

3.3.7. *Notation and assumptions.* We will work locally, so fix a point $x \in X$ and set $b = f(x)$, $P = \overline{M}_b$, $Q = \overline{M}_x$, and a \mathcal{D} -saturated ideal $\mathcal{J} \subseteq \mathcal{O}_x$. We will always assume that $Q^{\text{gp}}/P^{\text{gp}}$ has no torsion and the homomorphism $P \rightarrow Q$ is integral, see §3.1. Given a ring homomorphism $\phi: O \rightarrow A$ we will say that an ideal $J \subseteq A$ is *defined over O* if it is generated by an ideal $I \subseteq O$. In this case, one can take $I = J \cap O$, and if ϕ is flat then $J \cap O$ is the only such ideal of O .

formalDsec

3.3.8. *A formal description.* We start with studying the problem formally-locally. So, fix a presentation $\hat{\mathcal{O}}_x = O[[t_1, \dots, t_r]][[u^{Q/P}]]$ as in Proposition 3.2.2(3), where $O = \hat{\mathcal{O}}_b \hat{\otimes}_{k(b)} k(x)$. Recall that $\hat{\mathcal{O}}_x$ is formally graded by Corollary 3.1.6.

formalstabprop

Proposition 3.3.9. *Keep the above notation and let $\hat{\mathcal{J}} \subseteq \hat{\mathcal{O}}_x$ be an ideal. Then the following conditions are equivalent:*

- (1) $\hat{\mathcal{J}}$ is formally graded over O , that is $\hat{\mathcal{J}} = \prod_{q \in Q/P^{\text{gp}}} u^q \hat{\mathcal{J}}_q$, and the ideals $\hat{\mathcal{J}}_q \subseteq O[[t_1, \dots, t_r]]$ are defined over O .
- (2) $\hat{\mathcal{J}}$ is formally logarithmically fibral over O , that is, it is generated by elements $c_q u^q$ with $c_q \in O$.
- (3) $\hat{\mathcal{J}}$ is \mathcal{D} -saturated over O , that is, $\mathcal{D}_{\hat{\mathcal{O}}_x/O}(\hat{\mathcal{J}}) \subseteq \hat{\mathcal{J}}$.

Proof. (1) \implies (2) Since $\widehat{\mathcal{O}}_x$ is noetherian, its ideals are closed and hence $\widehat{\mathcal{J}}$ is generated by the graded components $u^q \widehat{\mathcal{J}}_q$. It follows that $\widehat{\mathcal{J}}$ is formally logarithmically fibral.

(2) \implies (3) By Lemma 3.2.4, $\mathcal{D}_{\widehat{\mathcal{O}}_x/O}$ is generated by ordinary derivations $\partial_1, \dots, \partial_n$ and monomial derivations ∂_ϕ . It remains to note that $\partial_i(c_q u^q) = 0$ and $\partial_\phi(c_q u^q) = \phi(q) c_q u^q$.

(3) \implies (1) Since $\widehat{\mathcal{J}}$ is closed, it suffices to prove that if f is an element of $\widehat{\mathcal{J}}$ and $\sum_{q \in Q/p^{\text{gp}}, n \in \mathbb{N}^r} f_{q,n} u^q t^n$ is its power series presentation, then $f_{q,n} u^q \in \widehat{\mathcal{J}}$. For shortness, we will often denote the monomials $f_{q,n} u^q$ by $f_m m$. By the *degree* of m we mean the maximal number d such that $m \in m_x^d$. Clearly, there exist only finitely many monomials of degree bounded by d .

Note that $t_1 \partial_1, \dots, t_n \partial_n$ and ∂_ϕ generate a commutative \mathbb{Z} -subalgebra \mathcal{A} of $\mathcal{D}_{\widehat{\mathcal{O}}_x/O}^\infty$ which acts on each monomial $f_m m$ through a character $\chi_m : \mathcal{A} \rightarrow \mathbb{Z}$ determined by the equalities $\chi_m(\partial_\phi) = \phi(q)$ and $\chi_m(t_i \partial_i) = n_i$. Since $Q^{\text{gp}}/P^{\text{gp}}$ is torsion free and the characteristic is zero, all monomials have different characters, and it follows that using operators of \mathcal{A} we can separate any finite set of monomials. In particular, for each m of degree at most d there exists an operator $\delta_m \in \mathcal{A}$ such that $\delta_m(m) = m$ and $\delta_m(m') = 0$ for any other monomial of degree at most d . Thus,

$$f_m m + \sum_{\deg(m') > d} \chi_{m'}(\delta_m) f_{m'} m' = \delta_m(f) \in \widehat{\mathcal{J}},$$

where $a_{m'} \in \mathbb{Z}$. Since $\widehat{\mathcal{J}}$ is closed and d is arbitrary, we obtain that $f_m m \in \widehat{\mathcal{J}}$. Assume that $m = u^q t^n$, and set $\delta = \prod_{i=1}^r \frac{1}{n_i!} \partial_i^{n_i}$. Then $f_m u^q = \delta(f_m m) \in \widehat{\mathcal{J}}$, as required. \clubsuit

logfibralcor

Corollary 3.3.10. *An ideal $\mathcal{J} \subseteq \mathcal{O}_x$ is logarithmically fibral over b if and only if the completion $\widehat{\mathcal{J}} = \widehat{\mathcal{J}}_{\widehat{\mathcal{O}}_x}$ is formally logarithmically fibral and for each homogeneous component $u^q \widehat{\mathcal{J}}_q$ the ideal $\widehat{\mathcal{J}}_q \subseteq O[[t_1, \dots, t_n]]$ is defined over \mathcal{O}_b .*

Proof. The direct implication is obvious. Conversely, that assumption on $\widehat{\mathcal{J}}$ means that it is generated by elements $c_q u^q$ with $c_q \in \mathcal{O}_b$. The ideal \mathcal{J}' these elements generate in \mathcal{O}_b lies in \mathcal{J} and satisfies $\mathcal{J}' \widehat{\mathcal{O}}_x = \widehat{\mathcal{J}} = \widehat{\mathcal{J}}_{\widehat{\mathcal{O}}_x}$. Since $\widehat{\mathcal{O}}_x$ is flat over \mathcal{O}_x , this implies that \mathcal{J} coincides with the logarithmically fibral ideal \mathcal{J}' . \clubsuit

3.3.11. *Descent through the field extension.* Naturally, we will use $k(b)$ -derivations of $k(x)$ to descend the ideals $\widehat{\mathcal{J}}_q$ to $\widehat{\mathcal{O}}_b$. This will be done by use of the following result, whose proof is based on the same ideas as earlier.

descentfield

Lemma 3.3.12. *Assume that O is a noetherian complete local ring with a field of coefficients $k \hookrightarrow O$ of characteristic zero, l/k is a field extension such that k is algebraically closed in l , and $I \subseteq A := O \widehat{\otimes}_k l$ is a $\mathcal{D}_{A/O}$ -saturated ideal. Then I is defined over O .*

Proof. Fix a basis Q of l over k . Then any element $f \in A$ possesses a unique presentation $f = \sum_{i \in \mathbb{N}} a_i x_i$, where x_i are distinct elements of Q and $a_i \in m_O^{c_i}$ for a monotonic unbounded sequence c_i (if the sum ends at n , then we take $a_i = 0$ and $c_i = \infty$ for $i > n$). It suffices to prove that if $f \in I$, then each a_i lies in I .

Fix $i \in \mathbb{N}$. By Lemma 3.3.13 below, for any $n \in \mathbb{N}$ there exists a differential operator $\partial_n \in \mathcal{D}_{l/k}^\infty$ such that $\partial(x_j) = \delta_{ij}$ for any j such that $0 \leq j \leq n$. Denoting the extension to $\mathcal{D}_{A/O}^\infty$ by ∂_n too, we obtain that $\partial_n(f) - a_i = \sum_{j>n} \partial(x_j)a_j \in m_O^{c_n}$, and hence $\partial_n(f) \in I$ converge to a_i . Since I is closed, $a_i \in I$. ♣

fieldlem

Lemma 3.3.13. *Assume that l/k is an extension of fields of characteristic zero such that k is algebraically closed in l . Then for any finite k -linearly independent subset $\{x_1, \dots, x_n\} \subset l$ and $i \in \{1, \dots, n\}$ there exists a differential operator $\partial \in \mathcal{D}_{l/k}^\infty$ such that $\partial(x_j) = \delta_{ij}$.*

Proof. By Noether normalization there exist transcendently independent elements $t_1 = x_i, t_2, \dots, t_m \in A := k[x_1, \dots, x_n]$ such that A is finite over $O = k[t_1, \dots, t_m]$. Moreover, shifting t_i by elements of k we can assume that A/O is strictly étale over the origin $V(t_1, \dots, t_m)$. Then A embeds into $\widehat{O} = k[[t_1, \dots, t_m]]$, and one can easily construct $\partial = \sum a_i \partial_1^{i_1} \dots \partial_m^{i_m}$ as required.¹³ ♣

←13

3.3.14. *Algebraization.* Finally, we are in a position to obtain a criterion for an ideal to be logarithmically fibral. It is easy to see that if \mathcal{J} contains a power of m_x , then one can replace the elements \widehat{c}_q by good enough approximations $c_q \in \mathcal{O}_b$, and the elements $u^q c_q$ still generate $\widehat{\mathcal{J}}$ and hence also \mathcal{J} . We will later need the following more general claim which is designed to run noetherian induction on B and its modifications.

algebrth

Proposition 3.3.15. *Let $g: X \rightarrow B$ and $b = g(x)$ be as in §3.3.7, and let $\mathcal{J} \subseteq \mathcal{O}_x$ be a \mathcal{D} -saturated ideal with decomposition $\widehat{\mathcal{J}} = \prod_{q \in \mathbb{Q}/p^{\mathbb{N}}} u^q \widehat{\mathcal{J}}_q$ as in Proposition 3.3.9. Assume that the base B is logarithmically regular¹⁴, the logarithmic structure of B is Zariski, and there exists an ideal $\mathcal{I} \subset \mathcal{O}_b$ such that the ideal \mathcal{J} is monomial over the complement of $V(\mathcal{I})$ and the ideals $\widehat{\mathcal{J}}_q$ of $\widehat{\mathcal{O}}_b \widehat{\otimes}_{k(b)} k(x)[[t_1, \dots, t_n]]$ are defined over the completion $\widehat{\mathcal{O}}_{b, \mathcal{I}}$ of \mathcal{O}_b along \mathcal{I} . Then \mathcal{J} is logarithmically fibral at x .*

←14

Proof. Let $T = V(\mathcal{J})$ be the closed subscheme defined by \mathcal{J} . Set $B_0 = B \setminus V(\mathcal{I})$, $X_0 = X \times_B B_0$ and $T_0 = T \times_X X_0$, and let $T' = V(\mathcal{J}')$ be the schematic closure of T_0 in X . In particular, $\mathcal{J} \subseteq \mathcal{J}'$ and \mathcal{J}'/\mathcal{J} is annihilated by a power \mathcal{I}^l of \mathcal{I} . Since T_0 is monomial by our assumption and X is logarithmically regular by ...¹⁵, T' is also monomial.

←15

Consider the decomposition $\widehat{\mathcal{J}}' = \prod_{q \in \mathbb{Q}/p^{\mathbb{N}}} u^q \widehat{\mathcal{J}}'_q$ of the completion of the monomial ideal \mathcal{J}' at x . Each $\widehat{\mathcal{J}}'_q$ is a monomial ideal of $\widehat{\mathcal{O}}_b \widehat{\otimes}_{k(b)} k(x)[[t_1, \dots, t_n]]$, that is, it is generated by elements of u^P , and hence is defined over $\widehat{\mathcal{O}}_{b, \mathcal{I}}$, and even over \mathcal{O}_b . Clearly, $\widetilde{\mathcal{J}}_q = \widehat{\mathcal{O}}_{b, \mathcal{I}} \cap \widehat{\mathcal{J}}_q$ is contained in $\widetilde{\mathcal{J}}'_q = \widehat{\mathcal{O}}_{b, \mathcal{I}} \cap \widehat{\mathcal{J}}'_q$, and $\widetilde{\mathcal{J}}'_q/\widetilde{\mathcal{J}}_q$ is annihilated by \mathcal{I}^l .

¹³(Michael) Is this convincing? Is there a shorter argument or, better, a reference?

¹⁴(Michael) A strong assumption on the base. Certainly can be weakened. Maybe enough to require that $B \rightarrow \text{Spec}(\mathbb{Q}[P])$ is flat, *Tor*-independent for maps $\mathbb{Q}[P] \rightarrow \mathbb{Q}[Q]$, or something of this kind.

¹⁵(Michael) In the section on logarithmic regularity should add lemma that logarithmically regular over logarithmically regular is logarithmically regular.

Let f_1, \dots, f_m be generators of $\tilde{\mathcal{J}}_q$. Since $\tilde{\mathcal{J}}_q$ is contained in the ideal $\tilde{\mathcal{J}}'_q$ defined over \mathcal{O}_b , each f_i is of the form $\sum g_j h_j$ with $h_j \in \tilde{\mathcal{J}}'_q \cap \mathcal{O}_b$ and $g_j \in \hat{\mathcal{O}}_{b, \mathcal{I}}$. Choose $g'_j \in \mathcal{O}_b$ such that $g_j - g'_j \in \mathcal{I}^{l+1} \hat{\mathcal{O}}_{b, \mathcal{I}}$ and set $f'_i = \sum h_j g'_j \in \mathcal{O}_b$. Then $f_i - f'_i = \sum h_j (g_j - g'_j) \in \mathcal{I}^{l+1} \tilde{\mathcal{J}}'_q \subseteq \mathcal{I} \tilde{\mathcal{J}}_q$. Therefore f'_1, \dots, f'_m is also a generating set of $\tilde{\mathcal{J}}_q$, and hence the latter is defined over \mathcal{O}_b . Thus, \mathcal{J} is logarithmically fibral by Corollary 3.3.10. \clubsuit

algebracor

Corollary 3.3.16. *Let $g: X \rightarrow B$ and $b = g(x)$ be as in §3.3.7 and let $\mathcal{J} \subseteq \mathcal{O}_x$ be a \mathcal{D} -saturated ideal on X . Assume that B is local and b is its closed point, the logarithmic structure of B is Zariski, and $k(b)$ is algebraically closed. Let $B' \rightarrow B$ be a modification such that B' is logarithmically regular and the pullback $\mathcal{J}' = \mathcal{J} \mathcal{O}'_X$ on $X' = X \times_B B'$ becomes monomial once restricted onto the preimage of $B \setminus \{b\}$. Then \mathcal{J}' is logarithmically fibral over B' .*

Proof. Monomial ideals are logarithmically fibral, hence it suffices to check that \mathcal{J}' is monomial at a point $x' \in X'$ belonging to the preimage of b . Let $b' \in B'$ and $x \in X$ denote the images of x' . We claim that \mathcal{J}' satisfies the assumptions of Proposition 3.3.15 at x' with $\mathcal{I} = m_b \mathcal{O}_{b'}$, and hence \mathcal{J}' is logarithmically fibral over B' at x' .

Since $V(\mathcal{I} \mathcal{O}_{X'})$ is the preimage of $B \setminus \{b\}$, we see that \mathcal{J}' is monomial over the complement of $V(\mathcal{I})$. So, we should only check that each $\hat{\mathcal{J}}'_q$ is defined already over $\hat{\mathcal{O}}_{b', \mathcal{I}}$, and this follows from the observations that the decomposition $\hat{\mathcal{J}}' = \prod_q \hat{\mathcal{J}}'_q$ in $\hat{\mathcal{O}}_{x'}$ is compatible with the decomposition $\hat{\mathcal{J}} = \prod_q \hat{\mathcal{J}}_q$ in $\hat{\mathcal{O}}_x$, each $\hat{\mathcal{J}}_q$ is defined over $\hat{\mathcal{O}}_b$ by Proposition 3.3.9 and Lemma 3.3.12, and the homomorphism $\hat{\mathcal{O}}_b \rightarrow \hat{\mathcal{O}}_{b'}$ factors through $\hat{\mathcal{O}}_{b', \mathcal{I}}$. \clubsuit

monomthsec

3.4. Monomialization of \mathcal{D} -saturated ideals.

3.4.1. *Monomializable ideals.* We say that a morphism $g: B' \rightarrow B$ *monomializes* an ideal $\mathcal{J} \subseteq \mathcal{O}_X$ if the pullback $\mathcal{J}' = \mathcal{J} \mathcal{O}_{X'}$ of \mathcal{J} to the saturated base change $X' = X \times_B B'$ is monomial. If there exists a monomializing blow up g then \mathcal{J} is called *monomializable* over B . Finally, we say that \mathcal{J} is *monomializable* over B at a point $x \in X$ if the ideal $\mathcal{J} \mathcal{O}_x$ on the localization of X at x is monomializable.

3.4.2. *First criteria of monomializability.* The main goal of §3 is to prove that any \mathcal{D} -saturated ideal is monomializable. We will achieve this by proving that the class of monomializable ideals is large enough. Let us first systematize from this perspective what we have already done.

firstproplem

Lemma 3.4.3. *Let $f: X \rightarrow B$ and $\mathcal{J} \subseteq \mathcal{O}_X$ be as above, then*

- (1) *If \mathcal{J} is logarithmically fibral at x then it is monomializable at x .*
- (2) *If f is Kummer logarithmically étale then \mathcal{J} is monomializable.*
- (3) *\mathcal{J} is monomializable if and only if it is monomializable at all points of X .*

Proof. (1) If \mathcal{J} is logarithmically fibral at x and $b = f(x)$, then $\mathcal{J}_x = \sum_{q \in Q} u^q \mathcal{J}_q$ for ideals \mathcal{J}_q of \mathcal{O}_b . Hence the blow up of the product $\prod_q \mathcal{J}_q$ monomializes \mathcal{J}_x .

(2) There exist modifications $h: X' \rightarrow X$ such that $h^{-1}(\mathcal{J})$ is monomial, for example, the blow up along \mathcal{J} . So, the assertion follows by applying Lemma 2.2.14

to h . (In fact, $\mathcal{D}_{X/B} = 0$, hence any ideal is \mathcal{D} -saturated, and so (2) is equivalent to Lemma 2.2.14.)

(3) If \mathcal{J} is monomializable at x , then the center of the monomializing blow up can be extended to a neighborhood of x , and using that being monomial is an open condition, we obtain that \mathcal{J} is monomializable in a neighborhood of x . It remains to use that X is noetherian and the fact that blow ups form a filtered family and can be extended from an open subscheme. ♣

3.4.4. *Descent on B .* We proceed with few more results that extend the pool of monomializable ideals further.

moreredlem

Lemma 3.4.5. *Let $f: X \rightarrow B$ and $\mathcal{J} \subseteq \mathcal{O}_X$ be as above, and assume that $B_1 \rightarrow B$ is a composition of Kummer logarithmically étale covers and blow ups of solid logarithmic schemes such that the pullback \mathcal{J}_1 of \mathcal{J} to $X_1 = X \times_B B_1$ is monomializable over B_1 . Then \mathcal{J} is monomializable over B .*

Proof. It suffices to check this when $B_1 \rightarrow B$ is either a blow up or a Kummer logarithmically étale cover. In the first case, any monomialization $B' \rightarrow B_1$ of \mathcal{J}_1 induces a monomialization $B' \rightarrow B$ of \mathcal{J} . Thus, we can assume that $B_1 \rightarrow B$ is Kummer logarithmically étale. Let $g_1: B'_1 \rightarrow B_1$ be a modification that monomializes \mathcal{J}_1 . By Lemma 2.2.14 there exists a modification $g: B' \rightarrow B$ such that $B'_1 = (B' \times_B B_1)^{\text{sat}} \rightarrow B_1$ factors through g_1 , and we claim that $\mathcal{J}' = \mathcal{J}\mathcal{O}_{B'}$ is monomial, that is, g monomializes \mathcal{J} . Indeed, the pullback of \mathcal{J}' with respect to the Kummer étale covering $B'_1 \rightarrow B_1$ is the monomial ideal $\mathcal{J}\mathcal{O}_{B'_1} = \mathcal{J}_1\mathcal{O}_{B'_1}$. Hence, \mathcal{J}' is monomial by Lemma 2.2.6. ♣

henscor

Corollary 3.4.6. *Let $f: X \rightarrow B$ and \mathcal{J} be as above. Assume that $B = \text{Spec}(\mathcal{O}_b)$ is local, and let $B_1 = \text{Spec}(\mathcal{O}_b^{sh})$ be the strict henselization of B with the pullback logarithmic structure and $X_1 = X \times_B B_1$. If $\mathcal{J}_1 = \mathcal{J}\mathcal{O}_{B_1}$ monomializable over B_1 then \mathcal{J} is monomializable over B .*

Proof. By definition, \mathcal{O}_b^{sh} is the union of étale \mathcal{O}_b -subalgebras A_i . The center $I \subseteq \mathcal{O}_b^{sh}$ of a monomializing blow up $B'_1 \rightarrow B_1$ is defined already over some A_i . So, there exists a strict étale cover $B_0 \rightarrow B$ such that $B'_1 \rightarrow B_1$ is the pullback of a blow up $B'_0 \rightarrow B_0$. In the same way, one sees that replacing B_0 by a larger étale covering of B one can also achieve that $\mathcal{J}\mathcal{O}_{B'_0}$ is monomial. (In fact, this latter action is redundant, but proving that would take more space.) This shows that $\mathcal{J}\mathcal{O}_{B_0}$ is monomializable, and hence \mathcal{J} is monomializable by Lemma 3.4.5. ♣

3.4.7. *The monomialization theorem.* Finally, we can prove the main result of §3.

monomialization

Theorem 3.4.8. *Assume that $f: X \rightarrow B$ is a logarithmically regular morphism with abundance of derivations and any blow up of B is dominated by a logarithmically regular one. Then any \mathcal{D} -saturated ideal $\mathcal{J} \subseteq \mathcal{O}_X$ is monomializable over B .*

Proof. We start with a few reductions. First, recall that by [IKN05, A.4.4, A.4.3] there exists a logarithmic blow up $B_1 \rightarrow B$ and a Kummer covering $B_2 \rightarrow B$ such that the saturated pullback $f_2: X_2 \rightarrow B_2$ of f is a saturated morphism of logarithmic schemes.¹⁶ By Lemma 3.4.5, it suffices to prove that $\mathcal{J}\mathcal{O}_{X_2}$ is monomializable

¹⁶Some arguments in [IKN05] are written in the context of complex logarithmic spaces, but they apply to schemes without changes.

over B_2 . This reduces the claim to the particular, case of a saturated logarithmically regular morphism, and to simplify the notation we assume in the sequel that f is saturated. note that this condition is preserved further by fs base changes.

Next, by Lemma 3.4.3(3) it suffices to prove the theorem locally on X , in particular, we can assume that B is local with closed point b . Set $d = \dim(\mathcal{O}_b)$. By induction we can assume that the theorem holds true when B is local of smaller dimension. Finally, by Corollary 3.4.6 it suffices to prove the theorem over the strict henselization of B , hence we can assume that $k(b)$ is algebraically closed.

Set $B_0 = B \setminus \{b\}$ and $X_0 = X \times_B B_0$, then $\mathcal{J}_0 = \mathcal{J}|_{X_0}$ is monomializable at any point of X_0 by induction assumption. By Lemma 3.4.3(3), \mathcal{J}_0 is monomializable by a blow up $B'_0 \rightarrow B_0$, and we consider an arbitrary extension of the latter to a blow up $B' \rightarrow B$. By the construction, if $g': X' \rightarrow B'$ denotes the base change of g then the restriction of $\mathcal{J}' = \mathcal{J}\mathcal{O}_{X'}$ onto $X'_0 = X' \times_{B'} B'_0$ is monomial. Moreover, assumption of the theorem, blowing up B' further we can assume that it is logarithmically regular.

It suffices to prove that \mathcal{J}' is monomializable at any point $x' \in X'$. In fact, we claim that \mathcal{J}' is even logarithmically fibral at x' , and hence monomializable by Lemma 3.4.3(1). We can assume that x' sits over x and hence $b' = g'(x')$ sits over b , since otherwise \mathcal{J}' is already monomial at x' . We claim that \mathcal{J}' satisfies the assumptions of Proposition 3.3.15 at x' with $\mathcal{I} = m_b\mathcal{O}_b$, and hence is logarithmically fibral at x' . Since $V(\mathcal{I}\mathcal{O}_{X'}) = X'_0$, we see that \mathcal{J}' is monomial over the complement of \mathcal{I} . So, we should only check that each $\widehat{\mathcal{J}}'_q$ is defined already over $\widehat{\mathcal{O}}_{b',\mathcal{I}}$, and this follows from the observations that the decomposition $\widehat{\mathcal{J}}' = \prod_q \widehat{\mathcal{J}}'_q$ in $\widehat{\mathcal{O}}_{x'}$ is compatible with the decomposition $\widehat{\mathcal{J}} = \prod_q \widehat{\mathcal{J}}_q$ in $\widehat{\mathcal{O}}_x$, each $\widehat{\mathcal{J}}_q$ is defined over $\widehat{\mathcal{O}}_b$ by Proposition 3.3.9 and Lemma 3.3.12, and the homomorphism $\widehat{\mathcal{O}}_b \rightarrow \widehat{\mathcal{O}}_{b'}$ factors through $\widehat{\mathcal{O}}_{b',\mathcal{I}}$. \clubsuit

4. BLOW UPS AND TRANSFORMS

4.1. Submonomial blow ups.

submonomialsec

4.1.1. *Submonomial blow ups.* Let $f: Y \rightarrow Z$ be a log regular morphism. Then by a *Z-submonomial ideal* on Y (relatively to Z) we mean any ideal $\mathcal{I} \subseteq \mathcal{O}_Y$ locally generated by a monomial ideal and a Z -submanifold ideal. Equivalently, one has that locally $V(\mathcal{I})$ is a monomial subscheme of a Z -submanifold of Y . By a *Z-submonomial blow up* of Y we mean a morphism $g: Y' \rightarrow Y$ such that on the level of schemes g is a blow up along a Z -submonomial ideal \mathcal{I} and $M_{Y'}$ is the *saturated* submonoid of $\mathcal{O}_{Y'}$ locally generated by $g^{-1}M_Y$ and the generator of $g^{-1}(\mathcal{I})$. We will denote the exceptional divisor $E = E_g$ and the corresponding ideal $\mathcal{I}_E = g^{-1}(\mathcal{I})$.

chartsec

4.1.2. *Charts of submonomial blow ups.* Let us now describe charts over $y \in Y$ of a Z -submonomial blow up $Y' \rightarrow Y$ along $\mathcal{I} + \mathcal{J}$, where \mathcal{I} is generated by a partial family t_1, \dots, t_d of parameters at y , and \mathcal{J} is generated by monomials u_1, \dots, u_r . Working étale-locally we can assume that $Y = \text{Spec}(A)$ is affine, $t_i \in A$, and Y possesses a global chart $Y \rightarrow \mathbf{A}_Q$ with $u_j = u^{q_j}$ for $q_1, \dots, q_r \in Q$. Then Y' is covered by charts Y'_s for $s \in S = \{t_1, \dots, t_d, u_1, \dots, u_r\}$, and each Y'_s is the

saturation of the log scheme $\text{Spec } A[\frac{S}{s}]$, which possesses a global chart with monoid Q_s as follows:

- (1) If $s = t_i$, then Q_s is the submonoid of $Q \oplus \mathbb{Z}s$ generated by Q and $s, q_1 - s, \dots, q_r - s$.
- (2) If $s = q_j$, then Q_s is the submonoid of Q^{gp} generated by Q and $q_1 - q_j, \dots, q_r - q_j$. (Equivalently, it is the quotient of a monoid defined as in (1) by the equivalence relation generated by $s \sim q_j$.)

4.1.3. *Log regularity of blow ups.* A key property of submonomial blow ups is that they preserve the log regularity. As in the absolute case, see [ATW17b, §5.2.2], the simplest proof is via increasing/decreasing the log structure.

submonblowlem

Lemma 4.1.4. *Let $f: Y \rightarrow Z$ be a log regular morphism and let $g: X \rightarrow Y$ be a Z -submonomial blow up. Then the morphism $f \circ g: X \rightarrow Z$ is log regular. Furthermore, if f is a relative log manifold (resp. log orbifold) then so is $f \circ g$.*

Proof. The claim easily reduces to the case when Y and Z are schemes. The main part is to prove that $f \circ g$ is log regular. We can work locally at a point $y \in Y$. In particular, we can assume that the center of g is of the form $\mathcal{I} + \mathcal{J}$, where \mathcal{I} is a Z -submanifold ideal and \mathcal{J} is monomial. Furthermore, by Lemma 2.4.10 there exists an enlargement M'_Y of the log structure such that $Y' = (Y, M'_Y)$ is log regular over Z and \mathcal{I} becomes monomial too. Then the saturated blow up $g': X' \rightarrow Y'$ along $\mathcal{I} + \mathcal{J}$ is log smooth, and hence X' is log regular over Z . We have a natural morphism $h: X' \rightarrow X$ arising from the morphism between the unsaturated blow ups.

We claim that $\overline{M}_{X',x} = \overline{M}_{X,x} \oplus \mathbb{N}^l$ for any $x' \in g'^{-1}(y)$ and its image $x \in X$. It will follow then that X is obtained from X' by decreasing the log structure (in particular, $X' = X$ on the level of schemes), and hence $X \rightarrow Z$ is log regular as in Lemma 2.4.11. We can work étale-locally, and then charts of the blow ups were described in §4.1.2. This reduces our claim to a purely combinatorial claim about monoids which can be checked directly. (Note also that in a slightly implicit way it was proved already when dealing with the absolute case, see the second part of the proof of [ATW17b, Lemma 5.2.3].)

Finally, we notice that if f is a relative log manifold then $f \circ g$ is a relative log manifold by the above and Theorem 2.7.6. ♣

4.2. Submonomial Kummer blow ups.

4.2.1. *Kummer topology and ideals.* First, we recall generalities that apply to an arbitrary log scheme Y and a morphism $Y \rightarrow Z$. Kummer étale morphisms and covers form a Kummer étale topology $Y_{\text{két}}$, that we simply call the Kummer topology of Y , e.g. see [ATW17b, §5.3] or [ATW17a]. The presheaf $\mathcal{O}_{Y_{\text{két}}}$ is a sheaf and its finitely generated ideals will be called *Kummer ideals* on Y . This and Proposition 2.7.4(ii) easily imply that the presheaf of derivations $\mathcal{D}_{Y_{\text{két}}/Z}$, which assigns to a Kummer étale morphism $Y' \rightarrow Y$ the $\mathcal{O}_{Y'}$ -module $\mathcal{D}_{Y'/Z}(Y')$, is also a sheaf. Often, we will view an ordinary ideal $\mathcal{I} \subseteq \mathcal{O}_Y$ as the Kummer ideal $\mathcal{I}_{\text{két}}$ it generates. This preserves arithmetic operations and derivations: $(\mathcal{I}\mathcal{J})_{\text{két}} = \mathcal{I}_{\text{két}} + \mathcal{J}_{\text{két}}$, $(\mathcal{D}_{Y/Z}\mathcal{I})_{\text{két}} = \mathcal{D}_{Y_{\text{két}}/Z}(\mathcal{I}_{\text{két}})$, etc, and in the sequel we will safely write \mathcal{I} instead of $\mathcal{I}_{\text{két}}$ and $\mathcal{D}_{Y/Z}$ instead of $\mathcal{D}_{Y_{\text{két}}/Z}$. With this conventions, any Kummer ideal \mathcal{I} is an ideal Kummer-locally: there exists a Kummer covering $Y' \rightarrow Y$ such that $\mathcal{I}|_{Y'}$

is an ordinary ideal. By the vanishing locus $V(\mathcal{J})$ we denote the minimal closed set such that \mathcal{J} is trivial on its complement.

Finally, we note that all above definitions are local with respect to strict étale (and even Kummer étale) morphisms and hence extend to log DM stacks.

4.2.2. *Submonomial Kummer ideals.* Now, assume that $Y \rightarrow Z$ is log regular. A Kummer ideal \mathcal{J} is called *Z-submonomial* (resp. *monomial* if Kummer locally it is a *Z-submonomial* (resp. *monomial*) ideal.

4.2.3. *Integral closure.* Integral closure \mathcal{J}^{nor} of a Kummer ideal \mathcal{J} is defined via Kummer étale sheafification of the usual integral closure, see [ATW17a, §4.3.1]. This operation will only be used in the following special case: for any submonomial ideal \mathcal{J} we set $\mathcal{J}^{(a)} = (\mathcal{J}^a)^{\text{nor}}$ for shortness of notation.

norrem

Remark 4.2.4. It may happen that \mathcal{J} is integrally closed (even monomial and saturated) but \mathcal{J}^a is not, and it will be convenient to use the ideals $\mathcal{J}^{(a)}$ in the definition of admissibility because for any ideal \mathcal{I} on X , if $x^n \subseteq \mathcal{I}^{(an)}$ then $x \subseteq \mathcal{I}^{(a)}$. This property is an immediate consequence of integral closedness.

Part (2) of the following lemma is the only other result about $\mathcal{J}^{(a)}$ we will need. However, to prove this we have to describe this ideal precisely.

Jnorlem

Lemma 4.2.5. *Assume that $Y \rightarrow Z$ is a log regular morphism of reduced¹⁷ log DM stacks, $\mathcal{J} \subseteq \mathcal{O}_Y$ is a *Z-submonomial ideal*, and $a \geq 0$.*

(i) *Fix a presentation $\mathcal{J} = \mathcal{I} + \mathcal{N}$, where \mathcal{N} is a monomial Kummer ideal and \mathcal{I} is a *Z-submanifold ideal*. Then*

$$\mathcal{J}^{(a)} = \sum_{j=0}^a (\mathcal{N}^j)^{\text{sat}} \cdot \mathcal{I}^{a-j}.$$

(ii) *For any i with $0 \leq i \leq a$ the equality $\mathcal{D}_{Y/Z}^{(\leq i)}(\mathcal{J}^{(a)}) \subseteq \mathcal{J}^{(a-i)}$ holds.*

Proof. The argument from [ATW17a, Lemma 4.3.2] applies here too: (2) follows from (1) because $\mathcal{D}_{Y/Z}^{(\leq i)}$ preserves $(\mathcal{N}^j)^{\text{sat}}$ and takes \mathcal{I}^b to \mathcal{I}^{b-i} , and (1) is proved by reducing to the monomial case as follows. First, one increases the log structure via Lemma 2.4.10 so that \mathcal{J} becomes monomial. Then a direct computation shows that the sum in (1) describes the saturation of \mathcal{J} , and it remains to use Lemma 2.2.9. ♣

submonomblowsec

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4.2.6. *Submonomial Kummer blow ups.*¹⁸ For any *Z-submonomial Kummer ideal* \mathcal{J} there exists a universal log DM *Y-stack* Y' such that $\mathcal{J}' = \mathcal{J}\mathcal{O}_{Y'/\text{két}}$ is an invertible (ordinary) ideal. One calls $Y' = \text{Bl}_{\mathcal{J}}(Y)$ and the morphism $g: Y' \rightarrow Y$ the *Z-submonomial Kummer blow up with center \mathcal{J}* . The invertible pullback will be also denoted $\mathcal{J}' = \mathcal{I}_E$. Note that g is an isomorphism over the complement of $V(\mathcal{J})$.

norcenterlem

Lemma 4.2.7. *Let $Y \rightarrow Z$ be a log regular morphism of reduced log DM stacks and $\sigma: Y' \rightarrow Y$ a submonomial Kummer blow with center \mathcal{J} . Then $\sigma^{-1}(\mathcal{J}^{(a)}) = \mathcal{I}_E^a$.*

¹⁷(Michael) Here and in Lemma 4.2.7 use Lemma 2.2.9, hence should respect its assumption, or somehow avoid using it.

¹⁸(Michael) Proofs in this subsection and Lemma 4.2.9 will be added later. Maybe using the Proj construction.

Proof. Clearly, $\mathcal{I}_E^a = \sigma^{-1}(\mathcal{J}^a) \subseteq \sigma^{-1}(\mathcal{J}^{(a)}) \subseteq (\mathcal{I}_E^a)^{\text{nor}}$. Since \mathcal{I}_E^a is an invertible monomial ideal, it is saturated, and hence $(\mathcal{I}_E^a)^{\text{nor}} = \mathcal{I}_E^a$ by Lemma 2.2.9. ♣

4.2.8. *Basic properties.* Submonomial Kummer blow ups preserve relative log regularity and are compatible with base changes and log regular morphisms.

kummerblowlem

Lemma 4.2.9. *Let $Y \rightarrow Z$ be a log regular morphism of DM log stacks and let $Bl_{\mathcal{J}}(Y) \rightarrow Y$ be a Z -submonomial Kummer blow up with center \mathcal{J} . Then,*

(i) $Bl_{\mathcal{J}}(Y)$ is log regular over Z .

(ii) For any morphism $Z' \rightarrow Z$, the pullback of \mathcal{J} to $Y' = Y \times_Z Z'$ is a Z -submonomial Kummer ideal \mathcal{J}' and $Bl_{\mathcal{J}'}(Y') = Y \times_Z Z'$.

(iii) For any log regular morphism $T \rightarrow Y$, the pullback of \mathcal{J} to T is a Z -submonomial Kummer ideal \mathcal{I} and $Bl_{\mathcal{I}}(T) = T \times_Y Y'$.

Proof. ♣

4.2.10. *Sequences of submonomial Kummer blow ups.* Since log regularity is preserved by submonomial Kummer blow ups, we can define sequences of such blow ups. Typically such a sequence

$$Y' =: Y_n \xrightarrow{\sigma_n} Y_{n-1} \xrightarrow{\sigma_{n-1}} \dots \xrightarrow{\sigma_2} Y_1 \xrightarrow{\sigma_1} Y_0 := Y.$$

will be denoted $\sigma_k: Y_k \rightarrow Y_{k-1}$, $1 \leq k \leq n$, or just $\sigma: Y' \dashrightarrow Y$ in short notation.

4.2.11. *Strict transforms.* As in the classical case, by the *strict transform* H' of a closed substack $H \hookrightarrow Y$ under a Z -submonomial Kummer blow up $g: Y' = Bl_{\mathcal{J}}(Y) \rightarrow Y$ we mean the schematic closure of $H \setminus V(\mathcal{J})$ in Y' .

strictlem

Lemma 4.2.12. *Keep the above notation and assume that H is a log Z -submanifold such that $\mathcal{I} = \mathcal{J}\mathcal{O}_{H_{\text{két}}}$ is a Z -submonomial ideal on H . Then H' underlies a Z -submanifold of Y' and $H' \rightarrow H$ is the Z -submonomial Kummer blow up along \mathcal{I} .*

Proof. Opposite isomorphisms between H' and $H'' = Bl_{\mathcal{I}}(H)$ are constructed by use of the universal property of Kummer blow ups as follows. Since $\mathcal{J}\mathcal{O}_{H''_{\text{két}}} = \mathcal{I}\mathcal{O}_{H''_{\text{két}}}$ is an invertible ideal, we obtain a morphism $H'' \rightarrow Y'$, which clearly factors through H' . Conversely, $\mathcal{I}\mathcal{O}_{H'_{\text{két}}} = \mathcal{J}'|_{H'}$ for the invertible ideal $\mathcal{J}' = \mathcal{J}\mathcal{O}_{Y'_{\text{két}}}$. Since $H' \setminus V(\mathcal{J}')$ is schematically dense in H' , we obtain that $\mathcal{I}\mathcal{O}_{H'_{\text{két}}}$ is an invertible ideal on H' , yielding a morphism $H' \rightarrow H''$. ♣

pushsec

4.2.13. *Pushforwards from submanifolds.* For any log Z -submanifold $i: H \hookrightarrow Y$ and a Kummer ideal $\mathcal{J} \subseteq \mathcal{O}_{H_{\text{két}}}$ we denote its preimage in $\mathcal{O}_{Y_{\text{két}}}$ by $i_*(\mathcal{J})$.

pushlem

Lemma 4.2.14. *Assume that $Y_0 \rightarrow Z$ is a log regular morphism of log DM stacks, $H_0 \hookrightarrow Y_0$ is a log Z -submanifold, and $g_k: H_k \rightarrow H_{k-1}$, $1 \leq k \leq n$ a sequence of Z -submonomial Kummer blow ups with centers \mathcal{J}_k . Then there exists a unique sequence of Z -submonomial Kummer blow ups $h_k: Y_k \rightarrow Y_{k-1}$, $1 \leq k \leq n$ with centers \mathcal{I}_k and log Z -submanifolds $i_k: H_k \hookrightarrow Y_k$ such that $\mathcal{I}_k = (i_k)_*(\mathcal{J}_k)$ for any $k \in \{0, \dots, n-1\}$.*

Proof. These conditions define h_0 , and since $\mathcal{I}_0|_{H_0} = \mathcal{J}_0$, Lemma 4.2.12 yields that $H_1 = Y_1 \times_{Y_0} H_0$. The rest follows by induction on k . ♣

In the situation described by Lemma 4.2.14 we call the blow up sequence h the *pushforward* of g and use the notation $h = i_*(g)$.

admisssec

4.2.15. *Admissibility.* We say that a Z -submonomial Kummer ideal \mathcal{J} is *a-admissible with respect to* an ideal $\mathcal{I} \subseteq \mathcal{O}_Y$ if $\mathcal{I} \subseteq \mathcal{J}^{(a)}$. In this case, the submonomial Kummer blow up $\sigma: Y' \rightarrow Y$ along \mathcal{J} is also called *a-admissible with respect to* \mathcal{I} .

Remark 4.2.16. In the classical Hironaka's algorithm $\mathcal{J}^a = \mathcal{J}^{(a)}$ automatically, but in our case it is more convenient to use $\mathcal{J}^{(a)}$ in this definition.

4.3. Pullbacks.

4.3.1. *Serre's twist.* If $\sigma: Y' \rightarrow Y$ is a blow up along an ideal \mathcal{J} , then $\mathcal{I}_E = g^{-1}(\mathcal{J})$ is the inverse Serre's twisting sheaf $\mathcal{O}_{Y'}(-1)$ for $Y' = \text{Proj}_Y(\oplus_n \mathcal{J}^n)$. Informally, we will view the invertible sheaf \mathcal{I}_E in this way also in the case of submonomial Kummer blow ups. Transforms of various objects under blow ups often involve multiplication by an appropriate power \mathcal{I}_E^d with $d \in \mathbb{Z}$. This operation can be viewed as Serre's twisting, so we will use the usual notation $(-d)$.

twistedpullback

4.3.2. *Ideals.* Assume that σ is *a-admissible* with respect to an ideal \mathcal{I} . Then $\sigma^{-1}\mathcal{I} \subseteq \mathcal{I}_E^a$ by Lemma 4.2.7, and hence the twisted pullback $\sigma^{-1}(\mathcal{I})(a)$ is defined as an ideal on Y' . We will usually shorten the notation as $\sigma^{-1}(\mathcal{I})(a) = \sigma^{-1}(\mathcal{I}, a)$.

4.3.3. *Derivations.* As in the absolute case (see [ATW17a, Lemma 4.2.1]), differential operators can be pulled back with an opposite twist.

transformderivlem

Lemma 4.3.4. *Assume that $Y \rightarrow Z$ is a log regular morphism of DM log stacks and $\sigma: Y' \rightarrow Y$ is a submonomial Kummer blow up. Then for any $i \in \mathbb{N}$ there is a natural embedding $\mathcal{I}_E^i \mathcal{D}_{Y'/Z}^{(\leq i)}(Y') \hookrightarrow \mathcal{D}_{Y'/Z}^{(\leq i)}$ given by a unique extension of differential operators from \mathcal{O}_Y to $\mathcal{O}_{Y'}$.*

Proof. Since the characteristic is zero, we can assume that $i = 1$. In addition, the claim is Kummer-local on Y , hence it reduces to the particular case when $Y = \text{Spec}(A)$ and $Z = \text{Spec}(C)$ are affine log schemes and σ is a submonomial blow up along an ideal J , see the proof of [ATW17a, Lemma 4.2.1] for details. One can now proceed as in the cited proof, but a direct computation as follows seems to be the shortest argument.

Let $t = t_1, \dots, t_n$ be generators of J . It suffices to study the situation on a chart $Y'_t = \text{Spec}(A[\frac{J}{t}])$ of σ . We should show that for a log C -derivation $\partial: A \rightarrow A'$, the log derivation $t\partial$ uniquely extends to A' . It suffices to deal with the A -generators $\frac{t_i}{t}$ of $A' = A[\frac{J}{t}]$. Clearly, the formula $t\partial(\frac{t_i}{t}) = \partial(t_i) - \frac{t_i}{t}\partial(t)$ gives rise to a unique extension of $t\partial$ to a C -derivation $\partial': A' \rightarrow A'$. It is a log derivation since $\frac{\partial'(t)}{t} = \partial(t) \in A$, and for each t_i , which is a monomial, we have that $\frac{\partial(t_i)}{t_i} \in A$ and hence $\partial'(\frac{t_i}{t})/\frac{t_i}{t} \in A'$. ♣

By Lemma 2.5.4 we obtain the following corollary for derivations:

transformderivcor

Corollary 4.3.5. *In the situation of Lemma 4.3.4, $\mathcal{D}_{Y'/Y} = 0$ and*

$$\mathcal{I}_E \mathcal{D}_{Y/Z}(Y') \subseteq \mathcal{D}_{Y'/Z} \subseteq \mathcal{D}_{Y/Z}(Y').$$

4.3.6. *Derived ideals.* For any \mathcal{O}_Y -submodule $\mathcal{F} \subseteq \mathcal{D}_{Y/Z}$ let $\sigma^*\mathcal{F} \subseteq \mathcal{D}_{Y'/Z}$ be as in §2.5.5. Then $\sigma^c(\mathcal{F}) := \mathcal{I}_E\sigma^*(\mathcal{F})$ is an $\mathcal{O}_{Y'}$ -submodule of $\mathcal{D}_{Y'/Z}$ by Lemma 4.3.4. More generally, for a sequence of submonomial Kummer blow ups $\sigma: Y' \dashrightarrow Y$ we define the *controlled transform* $\sigma^c(\mathcal{F}) \subseteq \mathcal{D}_{Y'/Z}$ by induction on the length, and will usually use abbreviations $\mathcal{F}_\sigma = \sigma^*(\mathcal{F})$ and $\mathcal{F}_\sigma^{(\leq i)} = (\mathcal{F}_\sigma)^{(\leq i)}$. The following result describes compatibility of pullbacks and derived ideals. It extends and makes more precise [ATW17a, Lemma 4.3.11], but the argument is the same.

Flem

Lemma 4.3.7. *Let $Y \rightarrow Z$ be a log regular morphism of DM stacks, $\sigma: Y' \rightarrow Y$ a submonomial Kummer blow up, and $\mathcal{F} \subseteq \mathcal{D}_{Y/Z}$ an \mathcal{O}_Y -submodule. If $0 \leq i < a$ are integers and σ is a -admissible with respect to an ideal $\mathcal{I} \subseteq \mathcal{O}_Y$, then*

$$\mathcal{F}_\sigma^{(\leq i)}(\sigma^{-1}(\mathcal{I}, a)) = \sum_{j=0}^i \sigma^{-1}(\mathcal{F}^{(\leq j)}(\mathcal{I}, a - j)).$$

Proof. Set $\mathcal{I}' = \sigma^{-1}(\mathcal{I}, a)$. The fact that $\mathcal{F}_\sigma^{(\leq i+1)} = \mathcal{F}_\sigma^{(\leq 1)}(\mathcal{F}_\sigma^{(\leq i)})$ enables a direct induction on i , reducing the claim to the case of $i = 1$:

$$\mathcal{F}_\sigma(\mathcal{I}') + \mathcal{I}' = \mathcal{F}_\sigma^{(\leq 1)}(\mathcal{I}') = \sigma^{-1}(\mathcal{F}^{(\leq 1)}(\mathcal{I}, a - 1)) + \mathcal{I}' = \sigma^{-1}(\mathcal{F}(\mathcal{I}, a - 1)) + \mathcal{I}',$$

where we use that $\sigma^{-1}(\mathcal{I}, a - 1) \subset \mathcal{I}'$. Étale-locally $\mathcal{F}_\sigma(\mathcal{I}')$ and $\sigma^{-1}(\mathcal{F}(\mathcal{I}, a - 1))$ are generated by global sections $y\partial(y^{-a}h)$ and $y^{1-a}\partial(h)$, respectively, where $h \in \Gamma(\mathcal{I})$ and $\partial \in \Gamma(\mathcal{F})$, and $y \in \Gamma(\mathcal{I}_E)$ is a generator. It remains to note that

$$y^{1-a}\partial(h) - y\partial(y^{-a}h) = a\partial(y) \cdot y^{-a}h \in \Gamma(\mathcal{I}').$$

♣

5. MARKED IDEALS AND ADMISSIBILITY

basicfactssec

5.1. Basic facts.

5.1.1. *The definition.* A *marked ideal* $\underline{\mathcal{I}} = (\mathcal{I}, a)$ for a log regular morphism $Y \rightarrow Z$ consists of an ideal $\mathcal{I} \subseteq \mathcal{O}_Z$ and a positive integer a . The *support* of $\underline{\mathcal{I}}$ is the set $\text{supp}(\underline{\mathcal{I}})$ of points $y \in |Y|$ such that $\text{logord}_{Y/Z, y}(\mathcal{I}) \geq a$, and $\underline{\mathcal{I}}$ is *resolved* if $\text{supp}(\underline{\mathcal{I}}) = \emptyset$. At few places it will also be convenient to consider *generalized marked ideals*, where a can equal 0.

Remark 5.1.2. Probably the best way of thinking about marked ideals is as about a kind of weighted ideals. In particular, this logic is consistent with operations we define on marked ideals below. We prefer the word “marked” since it is traditional in the area.

invarsec

5.1.3. *Weighted invariants.* Invariants of ideals possess the following weighted analogues. Given a marked ideal $\underline{\mathcal{I}} = (\mathcal{I}, a)$ we define its *weighted log order* as follows: $\mu_y(\underline{\mathcal{I}}) = 0$ if $\text{logord}_y(\mathcal{I}) < a$, and $\mu_y(\underline{\mathcal{I}}) = \text{logord}_y(\mathcal{I})/a$ otherwise. Also, we define the *weighted differential saturation* $\mathcal{W}(\underline{\mathcal{I}}) = \mathcal{W}_{Y/Z}(\underline{\mathcal{I}})$ as follows: $\mathcal{W}(\underline{\mathcal{I}}) = 0$ if \mathcal{I} is not log clean, and $\mathcal{W}(\underline{\mathcal{I}}) = (\mathcal{M}(\mathcal{I})^{\text{sat}})^{1/a}$ is a monomial Kummer ideal otherwise.

Remark 5.1.4. (i) The value $\mathcal{W}(\underline{\mathcal{I}}) = 0$ is just an indicator; an alternative would be to say that $\mathcal{W}(\underline{\mathcal{I}})$ is not defined. Note also that the equality $\mathcal{W}(\underline{\mathcal{I}}) = 0$ is not of local nature, it just says that $\mathcal{M}_{Y/Z}(\mathcal{I})$ is not monomial at some point of Y . On the other hand, if $\mathcal{W}(\underline{\mathcal{I}}) \neq 0$ then its definition is local on Z .

(ii) The notation μ_y follows its analogue in [BM08, Section 6]. Note that $\mu_y(\underline{\mathcal{I}}) \in \mathbb{Q}_{\geq 1} \cup \{0, \infty\}$ and $\mu_y(\underline{\mathcal{I}}) = 0$ if and only if $y \notin \text{supp}(\underline{\mathcal{I}})$.

5.1.5. *Cleaning blow up.* A log clean ideal can be made clean by a single blow up along $\mathcal{W}(\underline{\mathcal{I}})$. This extends the absolute case, in which any ideal is log clean, see [ATW17a, §4.4].

cleaningth

Theorem 5.1.6. *Assume that $Y \rightarrow Z$ is a log regular morphism of log DM stacks and $\underline{\mathcal{I}} = (\mathcal{I}, a)$ is a marked ideal on Y such that $\mathcal{I} \subseteq \mathcal{O}_Y$ is log clean over Z . Then the monomial Kummer blow up $\sigma: Y' \rightarrow Y$ along $\mathcal{W}(\underline{\mathcal{I}})$ is $\underline{\mathcal{I}}$ -admissible and the ideal $\sigma^{-1}\mathcal{I}(a)$ is clean.*

Proof. The blow up is admissible because $\mathcal{I} \subseteq \mathcal{M}(\mathcal{I}) \subseteq \mathcal{W}(\underline{\mathcal{I}})^a$. Furthermore, σ is log étale, hence $\mathcal{M}(\sigma^{-1}\mathcal{I}) = \sigma^{-1}(\mathcal{M}(\mathcal{I})) = \mathcal{I}_E^a$ by Lemma 2.8.15(iii). Therefore, $\mathcal{M}(\sigma^{-1}\mathcal{I}(a)) = \mathcal{I}_E^a \mathcal{I}_E^{-a} = \mathcal{O}_{Y'}$, and we obtain that $\sigma^{-1}\mathcal{I}(a)$ is clean by Lemma 2.8.9. \clubsuit

5.1.7. *Arithmetic operations.* Given n marked ideals $\underline{\mathcal{I}}_i = (\mathcal{I}_i, a_i)$ one usually sets $\underline{\mathcal{I}}_1 + \dots + \underline{\mathcal{I}}_n = (\mathcal{I}_1^{a/a_1} + \dots + \mathcal{I}_n^{a/a_n}, a)$, $\underline{\mathcal{I}}_1 \dots \underline{\mathcal{I}}_n = (\mathcal{I}_1 \dots \mathcal{I}_n, a_1 + \dots + a_n)$, where $a = a_1 \dots a_n$. However, we prefer to replace the definition of sums by the following homogenized one

$$\underline{\mathcal{I}}_1 + \dots + \underline{\mathcal{I}}_n = \left(\sum_{l \in \mathbb{N}^n \mid l_1 a_1 + \dots + l_n a_n \geq a} \mathcal{I}_1^{l_1} \dots \mathcal{I}_n^{l_n}, a \right)$$

In addition, we write $\underline{\mathcal{I}}_1 \subseteq \underline{\mathcal{I}}_2$ if $a_1 = a_2$ and $\mathcal{I}_1 \subseteq \mathcal{I}_2$.

sumrem

Remark 5.1.8. (i) Similarly to the situation with the usual definition, the addition is commutative but not associative. However, it is associative up to an equivalence relation introduced in §5.2 below and this is enough for applications, see also [ATW17a, §5.2].

(ii) In fact, our definition and the usual one produce equivalent marked ideals. The usual definition is lighter, so it may be preferable for computations. Homogenized sums are critical to have Theorem 5.3.6 below.

5.1.9. *Derivations.* Action of differential operators on $\underline{\mathcal{I}}$ is also defined in a weighted way: if $\underline{\mathcal{I}} = (\mathcal{I}, a)$ and \mathcal{F} is an \mathcal{O}_Y -submodule of $\mathcal{D}_{Y/Z}^{(\leq i)}$, where $0 \leq i \leq a$, then we automatically provide \mathcal{F} with weight $-i$ and set $\mathcal{F}(\underline{\mathcal{I}}) = (\mathcal{F}(\mathcal{I}), a - i)$ accordingly. Note that for $i = a$ we obtain a generalized marked ideal.

5.1.10. *Admissible sequences and transforms.* Let $\underline{\mathcal{I}}$ be a generalized marked ideal. A Z -submonomial Kummer blow up $\sigma: Y' \rightarrow Y$ is called $\underline{\mathcal{I}}$ -admissible if it is a -admissible with respect to \mathcal{I} , and in this case we define the *controlled transform* $\sigma^c \underline{\mathcal{I}} = (\sigma^{-1}\mathcal{I}(a), a)$, see §4.3.2. These definitions straightforwardly extend to the case when σ is a sequence of Z -submonomial Kummer blow ups of length n . Namely, σ is $\underline{\mathcal{I}}$ -admissible if each $\sigma_i: Y_i \rightarrow Y_{i-1}$ for $0 \leq i \leq n-1$ is $\underline{\mathcal{I}}_{i-1}$ -admissible, where $\underline{\mathcal{I}}_i = \sigma_i^c \underline{\mathcal{I}}_{i-1}$, and the *controlled transform* under σ is $\sigma^c \underline{\mathcal{I}} = \underline{\mathcal{I}}_n$. Usually, we will simply say an $\underline{\mathcal{I}}$ -admissible sequence for shortness. Note that any sequence of Z -submonomial Kummer blow ups is $(\mathcal{I}, 0)$ -admissible and the controlled transform is the usual pullback in this case.

5.1.11. *Order reduction.* By an *order reduction* of a marked ideal $\underline{\mathcal{I}}$ we mean an admissible sequence $\sigma: X' \dashrightarrow X$ such that $\sigma^c(\underline{\mathcal{I}})$ is resolved.

5.1.12. *Transforms and operations.* Compatibility relations between transforms and operations are the same as in the classical case, see [Wlo05, §3], [BM08, §3], and the absolute logarithmic case, see [ATW17a, §5]. Arguments are also the same, so we just sketch them.

operlem

Lemma 5.1.13. *Assume that $Y \rightarrow Z$ is a log regular morphism of log DM stacks, $\sigma: Y' \dashrightarrow Y$ is a sequence of Z -submonomial Kummer blow ups, and $\underline{\mathcal{I}} = \underline{\mathcal{I}}_1, \underline{\mathcal{I}}_2 \dots \underline{\mathcal{I}}_n$ are n marked ideals on Y , and set $\underline{\mathcal{P}} = \underline{\mathcal{I}}_1 \dots \underline{\mathcal{I}}_n$ and $\underline{\mathcal{S}} = \underline{\mathcal{I}}_1 + \dots + \underline{\mathcal{I}}_n$.*

(i) *If σ is $\underline{\mathcal{I}}_i$ -admissible for $1 \leq i \leq n$, then σ is $\underline{\mathcal{P}}$ -admissible and $\sigma^c(\underline{\mathcal{P}}) = \sigma^c(\underline{\mathcal{I}}_1) \dots \sigma^c(\underline{\mathcal{I}}_n)$.*

(ii) *Let k be a positive integer. Then σ is $\underline{\mathcal{I}}$ -admissible if and only if it is $\underline{\mathcal{I}}^k$ -admissible, and in this case $(\sigma^c \underline{\mathcal{I}})^k = \sigma^c(\underline{\mathcal{I}}^k)$.*

(iii) *σ is $\underline{\mathcal{I}}_i$ -admissible for each $i \in \{1, \dots, n\}$ if and only if it is $\underline{\mathcal{S}}$ -admissible, and in this case $\sigma^c(\underline{\mathcal{S}}) = \sigma^c(\underline{\mathcal{I}}_1) + \dots + \sigma^c(\underline{\mathcal{I}}_n)$.*

(iv) *If σ is $\underline{\mathcal{I}}$ -admissible, $\mathcal{F} \subseteq \mathcal{D}_{Y/Z}$ is an \mathcal{O}_Y -submodule, and $0 \leq i \leq a$, where $\underline{\mathcal{I}} = (\mathcal{I}, a)$, then σ is $\mathcal{F}^{(\leq i)}(\underline{\mathcal{I}})$ -admissible and $\sigma^c(\mathcal{F}^{(\leq i)}(\underline{\mathcal{I}})) \subseteq \mathcal{F}_\sigma^{(\leq i)}(\sigma^c \underline{\mathcal{I}})$.*

Proof. In all claims a straightforward induction on the length of σ reduces the proof to the case when σ is a single Kummer blow up along \mathcal{J} . The argument is the same for each $n \geq 2$, and we consider the case of $n = 2$ to simplify the notation. We start with the admissibility claims. Admissibility is obvious in (i), and follows from Remark 4.2.4 in (ii). In (iii), if $\mathcal{I}_i \subseteq \mathcal{J}^{(a_i)}$ for $i = 1, 2$ then $\mathcal{I}_1^m \mathcal{I}_2^l \subseteq \mathcal{J}^{(a_1 a_2)}$ whenever $ma_1 + la_2 \geq a_1 a_2$ and hence \mathcal{J} is $(\underline{\mathcal{I}}_1 + \underline{\mathcal{I}}_2)$ -admissible. Conversely, if \mathcal{J} is $(\underline{\mathcal{I}}_1 + \underline{\mathcal{I}}_2)$ -admissible, then it is admissible for $\underline{\mathcal{I}}_1^{a_2}$ and $\underline{\mathcal{I}}_2^{a_1}$, and it remains to use (ii). Finally, in (iv) we apply $\mathcal{F}^{(\leq i)}$ to the inclusion $\mathcal{I} \subseteq \mathcal{J}^{(a)}$ and use that $\mathcal{F}^{(\leq i)}(\mathcal{J}^{(a)}) \subseteq \mathcal{J}^{(a-i)}$ by Lemma 4.2.5(ii).

Concerning the relations between the transforms, the equalities in (i), (ii) and (iii) are obtained by unwinding the definitions, and the equality in (iv) follows from Lemma 4.3.7. ♣

5.1.14. *Maximal order.* As usually, a marked ideal $\underline{\mathcal{I}} = (\mathcal{I}, a)$ is said to be of *maximal order* if $\log \text{ord}_{Y/Z, y}(\mathcal{I}) \leq a$ (or $\mu_y(\underline{\mathcal{I}}) \leq 1$) for any $y \in Y$.

maxorderlem

Lemma 5.1.15. *Let $Y \rightarrow Z$ be a log regular morphism of log DM stacks, $\underline{\mathcal{I}} = \underline{\mathcal{I}}_1, \dots, \underline{\mathcal{I}}_n$ marked ideals, and k a positive integer.*

(i) *$\underline{\mathcal{I}}$ is of maximal order if and only if $\underline{\mathcal{I}}^k$ is of maximal order.*

(ii) *If $\underline{\mathcal{I}}_1$ is of maximal order, then $\underline{\mathcal{I}}_1 + \dots + \underline{\mathcal{I}}_n$ is of maximal order.*

Proof. Arithmetic operations commute with restriction onto the log fibers, hence this reduces to basic properties of the usual order of ideals on regular schemes. ♣

Probably the following result holds for arbitrary log regular morphisms, but proving this would involve a direct computation with completed log fibers. So we prefer to deal with the slightly less general case, where derivations provide a nice short argument.

maxordertransform

Lemma 5.1.16. *Let $X \rightarrow B$ be a relative log orbifold and $\underline{\mathcal{I}} = (\mathcal{I}, a)$ a marked ideal. Then*

(i) *$\underline{\mathcal{I}}$ is of maximal order if and only if $\mathcal{D}_{X/B}^{(\leq a)}(\mathcal{I}) = \mathcal{O}_X$.*

(ii) *If $\underline{\mathcal{I}}$ is of maximal order and $\sigma: X' \dashrightarrow X$ is an $\underline{\mathcal{I}}$ -admissible sequence, then $\underline{\mathcal{I}}' = \sigma^c(\underline{\mathcal{I}})$ is of maximal order too.*

Proof. The first claim follows from Lemma 2.8.9. Using (i) and Lemma 5.1.13(v) we obtain in (ii) that

$$(\mathcal{O}_{X'}, 0) = \sigma^c \left(\mathcal{D}_{X/B}^{(\leq a)}(\mathcal{I}) \right) \subseteq \mathcal{D}_{X'/B'}^{(\leq a)}(\underline{\mathcal{I}}').$$

So $\mathcal{D}_{X'/B'}^{(\leq a)}(\underline{\mathcal{I}}') = \mathcal{O}_{X'}$, and hence $\underline{\mathcal{I}}'$ is of maximal order by (i). \clubsuit

5.1.17. *Balanced marked ideals.* We say that a marked ideal $\underline{\mathcal{I}}$ is balanced if so is \mathcal{I} , see §2.8.7. By the *clean part* of $\underline{\mathcal{I}}$ we mean the marked ideal $\underline{\mathcal{I}}^{\text{cln}} = (\mathcal{I}^{\text{cln}}, b)$, where $b = \max(a, \log \text{ord}(\mathcal{I}))$. Note that $\underline{\mathcal{I}}^{\text{cln}}$ is of maximal order, and if $\underline{\mathcal{I}}$ is of maximal order, then $\underline{\mathcal{I}}^{\text{cln}}$ is obtained from $\underline{\mathcal{I}}$ by the maximal increase of the weight that keeps it of maximal order.

balancedcor

Corollary 5.1.18. *Let $X \rightarrow B$ be a relative log orbifold and $\underline{\mathcal{I}}$ a balanced marked ideal. Then any $\underline{\mathcal{I}}^{\text{cln}}$ -admissible sequence $\sigma: X' \dashrightarrow X$ is also $\underline{\mathcal{I}}$ -admissible and $\sigma^c(\underline{\mathcal{I}})$ is balanced with the clean part equal to the clean part of $\sigma^c(\underline{\mathcal{I}}^{\text{cln}})$.*

Proof. Let $\underline{\mathcal{I}} = (\mathcal{I}, a)$ and $\underline{\mathcal{I}}^{\text{cln}} = (\mathcal{I}^{\text{cln}}, b)$, in particular, $\mathcal{I} = \mathcal{M} \cdot \mathcal{I}^{\text{cln}}$, where $\mathcal{M} = \mathcal{M}(\mathcal{I})$ is an invertible monomial ideal. By induction on the length it suffices to consider a single blow up and then the admissibility claim is obvious.

By Lemma 5.1.16, $\sigma^c(\underline{\mathcal{I}}^{\text{cln}}) = (\mathcal{I}', b)$ is of maximal order, in particular, \mathcal{I}' is clean. It remains to note that $\sigma^c(\underline{\mathcal{I}}) = (\sigma^{-1}(\mathcal{M}) \cdot \mathcal{I}'^{b-a}, a)$. \clubsuit

equivsec

5.2. Equivalence and domination of marked ideals.

5.2.1. *The definition.* Let $Y \rightarrow Z$ be a log regular morphism and let $\underline{\mathcal{I}}_1$ and $\underline{\mathcal{I}}_2$ be marked ideals on Y . We say that $\underline{\mathcal{I}}_1$ is *dominated* by $\underline{\mathcal{I}}_2$ if any $\underline{\mathcal{I}}_2$ -admissible sequence of Z -submonomial Kummer blow ups is also $\underline{\mathcal{I}}_1$ -admissible. If $\underline{\mathcal{I}}_1$ and $\underline{\mathcal{I}}_2$ dominate each other, then we say that the marked ideals are *equivalent*. Furthermore, we say that $\underline{\mathcal{I}}_1$ and $\underline{\mathcal{I}}_2$ are *functorially equivalent* (resp. *functorially dominated*) if they stay equivalent (resp. dominated) after any base change $Z' \rightarrow Z$ and after pullback to any Y' for a log regular morphism $Y' \rightarrow Y$. We will use notation $\underline{\mathcal{I}}_1 \preceq \underline{\mathcal{I}}_2$ and $\underline{\mathcal{I}}_1 \approx \underline{\mathcal{I}}_2$ to denote functorial domination and functorial equivalence.

equivrem

Remark 5.2.2. (i) Our definition of equivalence extends that of [ATW17a, §5.1]. Similarly to the approach of Bierstone-Milman, our proof of independence of maximal contact is based on equivalence.

(ii) Non-functorial equivalence is not informative in general since non-resolved marked ideals may admit no non-trivial admissible blow ups, see Example 1.3.7.

(iii) In fact, we will only need base changes $Z' \rightarrow Z$ with $Z' = \text{Spec}(O)$, where O is either $k(Z)$ or a DVR with $\text{Frac}(O) = k(Z)$, and pullbacks with respect to morphisms $Y' \rightarrow Y$ which are either strict étale or localizations. We prefer not to restrict to these cases in the definition for aesthetical reasons. In fact, in the absolute case with Z the spectrum of a field, the usual equivalence covers our needs.

(iv) In the classical algorithm there are much fewer choices of admissible blow ups, so Hironaka and Bierstone-Milman had to consider additional test morphisms, see [Hir03, Definition 1.3] and [BM08, §1.2]. In our case, there are additional test morphisms too – base changes and log regular morphisms $Y' \rightarrow Y$, but their role is very clear. Additional test morphisms in the classical case may look artificial, but logarithmic geometry clarifies their role too: these test morphisms $(Y', E') \rightarrow (Y, E)$ are log smooth, so including them just makes the equivalence (at least partially) functorial in the logarithmic sense.

5.2.3. *Main examples of equivalence.* Main cases of functorial equivalence of marked ideals follow from Lemma 5.1.13.

equivlem

Lemma 5.2.4. *Assume that $f: Y \rightarrow Z$ is a log regular morphism of log DM stacks, $\sigma: Y' \dashrightarrow Y$ is a sequence of Z -submonomial Kummer blow ups, and $\underline{\mathcal{I}} = \underline{\mathcal{I}}_1, \dots, \underline{\mathcal{I}}_n$ are marked ideals on Y .*

(o) *If $\underline{\mathcal{I}}_1 \subseteq \underline{\mathcal{I}}_2$, then $\underline{\mathcal{I}}_1 \preceq \underline{\mathcal{I}}_2$. In particular, if $\underline{\mathcal{I}}_1 \subseteq \underline{\mathcal{I}}_2 \subseteq \underline{\mathcal{I}}_3$ and $\underline{\mathcal{I}}_1 \approx \underline{\mathcal{I}}_3$, then $\underline{\mathcal{I}}_1 \approx \underline{\mathcal{I}}_2 \approx \underline{\mathcal{I}}_3$.*

(i) *If $(\mathcal{I}_i, a_i) \preceq (\mathcal{I}_1, a_1)$ for $2 \leq i \leq n$, then $(\mathcal{I}_2, a_2) \cdots (\mathcal{I}_m, a_m) \preceq (\mathcal{I}_1, a_1)$.*

(ii) *$\underline{\mathcal{I}} \approx \underline{\mathcal{I}}^k$ for any positive natural k .*

(iii) *If $(\mathcal{I}_i, a_i) \preceq (\mathcal{I}_1, a_1)$ for $2 \leq i \leq n$, then $(\mathcal{I}_1, a_1) + \dots + (\mathcal{I}_m, a_m) \approx (\mathcal{I}_1, a_1)$.*

(iv) *$\underline{\mathcal{I}} \approx \underline{\mathcal{I}} + \mathcal{F}_1 \underline{\mathcal{I}} + \dots + \mathcal{F}_{a-1} \underline{\mathcal{I}}$ for any \mathcal{O}_Y -submodules $\mathcal{F}_i \subseteq \mathcal{D}_{Y/Z}^{(\leq i)}$.*

Proof. Claim (o) is trivial. Claims (i)–(iv) follow from their counterparts in Lemma 5.1.13 and the previous claims. For example, (iv) follows from (iii) and the fact that $\mathcal{F}_i \underline{\mathcal{I}} \preceq \underline{\mathcal{I}}$ by Lemma 5.1.13(iv). \clubsuit

5.3. Coefficients ideals.

5.3.1. *The definition.* As in [ATW17a, §6.1], given a marked ideal $\underline{\mathcal{I}} = (\mathcal{I}, a)$ and an \mathcal{O}_Y -submodule $\mathcal{F} \subseteq \mathcal{D}_{Y/Z}$ we define the (homogenized) \mathcal{F} -coefficient marked ideal of $\underline{\mathcal{I}}$ by the formula $\underline{\mathcal{C}}_{\mathcal{F}}(\underline{\mathcal{I}}) := \sum_{i=0}^{a-1} \mathcal{F}^{(\leq i)} \underline{\mathcal{I}}$. In particular, for $\mathcal{F} = \mathcal{D}_{Y/Z}$ one obtains the homogenized relative logarithmic version $\underline{\mathcal{C}}_Y(\underline{\mathcal{I}})$ of the usual coefficient ideal.

Remark 5.3.2. Usually, one defines the coefficient ideal using non-homogenized sums. Our definition is an analogue of Kollar’s tuning ideal $W_{a!}(\mathcal{I})$, see [Kol07].

5.3.3. *Equivalence.* Lemmas 5.2.4(v) and 5.1.15 immediately yield the following result.

coefcor

Lemma 5.3.4. *Assume that $Y \rightarrow Z$ is a log regular morphism of log DM stacks, $\underline{\mathcal{I}}$ is a marked ideal on Y and $\mathcal{F} \subseteq \mathcal{D}_{Y/Z}$ is an \mathcal{O}_Y -submodule. Then $\underline{\mathcal{I}} \approx \underline{\mathcal{C}}_{\mathcal{F}}(\underline{\mathcal{I}})$, and $\underline{\mathcal{C}}_{\mathcal{F}}(\underline{\mathcal{I}})$ is of maximal order whenever $\underline{\mathcal{I}}$ is of maximal order.*

5.3.5. *Transforms.* The following result is an analogue of [ATW17a, Proposition 6.1.3], which makes it more precise. Somewhat surprisingly, compatibility of our coefficient ideal with transforms is as good as possible.

coefprop

Theorem 5.3.6. *Assume that $Y \rightarrow Z$ is a log regular morphism of log DM stacks with a marked ideal $\underline{\mathcal{I}}$ of maximal order, $\mathcal{F} \subseteq \mathcal{D}_{Y/Z}$ is an \mathcal{O}_Y -submodule, and $\sigma: Y' \dashrightarrow Y$ is an $\underline{\mathcal{I}}$ -admissible sequence of Z -submonomial Kummer blow ups. Then*

$$\underline{\mathcal{C}}_{\sigma^c(\mathcal{F})}(\sigma^c(\underline{\mathcal{I}})) = \sigma^c(\underline{\mathcal{C}}_{\mathcal{F}}(\underline{\mathcal{I}})).$$

Proof. The inverse inclusion follows from Lemma 5.1.13, so let us prove the direct one. Induction on the length reduces the claim to the case of a single blow up. We should prove that if $n_0, \dots, n_{a-1} \in \mathbb{N}$ satisfy $\sum_{i=0}^{a-1} n_i(a-i) \geq a!$, then

$$\mathcal{I}_n := \prod_{i=0}^{a-1} \left(\mathcal{F}_\sigma^{(\leq i)}(\sigma^{-1}(\underline{\mathcal{I}}, a)) \right)^{n_i} \subseteq \sigma^{-1}(\mathcal{C}_{\mathcal{F}}(\underline{\mathcal{I}}, a!)).$$

By Lemma 4.3.7 we have that

$$\mathcal{I}_n = \prod_{i=0}^{a-1} \left(\sum_{j=0}^i \sigma^{-1} \left(\mathcal{F}^{(\leq j)}(\underline{\mathcal{I}}, a-j) \right) \right)^{n_i} = \sum_{(n_{ij})} \prod_{0 \leq j \leq i \leq a-1} \left(\sigma^{-1} \left(\mathcal{F}^{(\leq j)}(\underline{\mathcal{I}}, a-j) \right) \right)^{n_{ij}},$$

where the right hand sum is over the sets of partitions $n_i = n_{i0} + \dots + n_{ii}$ for $0 \leq i \leq a-1$. For each $j \in \{0, \dots, a-1\}$ set $l_j = \sum_{i=0}^{a-1} n_{ij}$. Then

$$\sum_{j=0}^{a-1} l_j(a-j) = \sum_{0 \leq j \leq i \leq a} n_{ij}(a-j) \geq \sum_{0 \leq j \leq i \leq a} n_{ij}(a-i) = \sum_{i=0}^{a-1} n_i(a-i) \geq a!$$

and hence

$$\prod_{0 \leq j \leq i \leq a-1} \left(\sigma^{-1} \left(\mathcal{F}^{(\leq j)}(\underline{\mathcal{I}}, a-j) \right) \right)^{n_{ij}} = \prod_{j=0}^{a-1} \left(\sigma^{-1} \left(\mathcal{F}^{(\leq j)}(\underline{\mathcal{I}}, a-j) \right) \right)^{l_j} = \sigma^{-1} \left(\prod_{j=0}^{a-1} \left(\mathcal{F}^{(\leq j)}(\underline{\mathcal{I}}) \right)^{l_j}, \sum_{j=0}^{a-1} l_j(a-j) \right) \subseteq \sigma^{-1} \left(\prod_{j=0}^{a-1} \left(\mathcal{F}^{(\leq j)}(\underline{\mathcal{I}}) \right)^{l_j}, a! \right) \subseteq \sigma^{-1}(\mathcal{C}_{\mathcal{F}}(\underline{\mathcal{I}}, a!)).$$

♣

contactsec

6. THE THEORY OF MAXIMAL CONTACT

6.1. Existence.

6.1.1. *Hypersurfaces of maximal contact.* Let $\underline{\mathcal{I}} = (\underline{\mathcal{I}}, a)$ be of maximal order on a log regular morphism $Y \rightarrow Z$. A *hypersurface of maximal contact* or simply a *maximal contact* to $\underline{\mathcal{I}}$ is a closed suborbifold $H \hookrightarrow Y$ of pure codimension 1 such that its ideal \mathcal{I}_H is contained in $\mathcal{T}(\underline{\mathcal{I}}) := \mathcal{D}_{Y/Z}^{(\leq a-1)}(\underline{\mathcal{I}})$.

6.1.2. *Local existence.* In the classical situation, maximal contact exists locally. For DM stacks one should use the étale topology instead.

contactexists

Theorem 6.1.3. *Assume that $X \rightarrow B$ is a relative log orbifold and $\underline{\mathcal{I}}$ is a marked ideal on X of maximal order. Then there exists an étale covering $g: X' \rightarrow X$ with a hypersurface of maximal contact $H' \hookrightarrow X'$ to $\underline{\mathcal{I}}' = g^{-1}\underline{\mathcal{I}}$.*

Proof. Since $\mathcal{D}_{X/B}^{(\leq a)}(\underline{\mathcal{I}}) = 1$ by Lemma 5.1.16(i), one can find an étale covering $X' \rightarrow X$ and global sections $h \in \Gamma(X', \mathcal{T}(\underline{\mathcal{I}}))$ and $\partial \in \Gamma(\mathcal{D}_{X'/B})$ such that $\partial(h)$ is a unit. By Lemma 2.8.9 $\text{logord}(h) \leq 1$ on X' , hence $H' = V(h)$ is a log B -suborbifold as required. ♣

6.2. Equivalence of marked ideals on suborbifolds. In order to show that restriction to maximal contact preserves equivalence of certain marked ideals we should first extend the notion of equivalence that was defined in §5.2. Our goal is to compare marked ideals \mathcal{I}_i , $i = 1, 2$ on log Z -orbifolds Y_i via embeddings of Y_i as Z -suborbifolds into a log Z -orbifold Y . Naturally, we will push forward Kummer blow up sequences of Y_i and compare the obtained Kummer blow up sequences of Y .

6.2.1. H -admissibility. Given a log relative morphism of log DM stacks $Y \rightarrow Z$ with a Z -suborbifold $i: H \hookrightarrow Y$ let $\mathcal{I}_H \subseteq \mathcal{O}_Y$ be the ideal defining H and $\underline{\mathcal{I}}_H = (\mathcal{I}, 1)$. A submonomial Kummer blow up sequence $\sigma: Y' \dashrightarrow Y$ is called H -admissible if it is $\underline{\mathcal{I}}_H$ -admissible.

Hadmisslem

Lemma 6.2.2. *Keep the above notation. Then a sequence $\sigma: Y' \dashrightarrow Y$ is H -admissible if and only if it is a pushforward $i_*(\tau)$ of a submonomial Kummer blow up sequence $\sigma: H' \dashrightarrow H$. Furthermore, in this case $\underline{\mathcal{I}}_{H'} = \sigma^c(\underline{\mathcal{I}}_H)$.*

Proof. Assume first that σ is a single blow up and let \mathcal{J} be its center. Then both H -admissibility and $\underline{\mathcal{I}}_H$ -admissibility mean that $\mathcal{I} \subseteq \mathcal{J}$. To check that $\underline{\mathcal{I}}_{H'} = \sigma^c(\underline{\mathcal{I}}_H)$ we can work étale-locally on Y , and then one can assume that there exist regular parameters t_1, \dots, t_n such that $H = V(t_1, \dots, t_l)$ and $\mathcal{J} = (t_1, \dots, t_m, u_1, \dots, u_s)$ for $l \leq m \leq n$ and monomials u_1, \dots, u_s . In this case, for any $y \in \{t_1, \dots, t_m, u_1, \dots, u_s\}$ the restrictions of both $\underline{\mathcal{I}}_{H'}$ and $\sigma^{-1}(\underline{\mathcal{I}}_H)(1)$ to the y -chart equal $(y^{-1}t_1, \dots, y^{-1}t_l)$. (In particular, both are trivial when $y \in \{t_1, \dots, t_l\}$.) The case of an arbitrary σ now follows by induction on the length. ♣

6.2.3. $(H, \underline{\mathcal{I}})$ -admissibility. If $H \hookrightarrow Y$ is a suborbifold and $\underline{\mathcal{I}}$ is a marked ideal on H , then a sequence $Y' \dashrightarrow Y$ is called $(H, \underline{\mathcal{I}})$ -admissible if it is H -admissible and the induced sequence $H' \dashrightarrow H$ is $\underline{\mathcal{I}}$ -admissible. Thus, pushforward induces a bijection between $\underline{\mathcal{I}}$ -admissible sequences $H' \dashrightarrow H$ and $(H, \underline{\mathcal{I}})$ -admissible sequences $Y' \dashrightarrow Y$.

equivsubsec

6.2.4. Equivalence on suborbifolds. As in §5.2, a pair $(H_1, \underline{\mathcal{I}}_1)$ dominates $(H_2, \underline{\mathcal{I}}_2)$ if $(H_1, \underline{\mathcal{I}}_1)$ -admissibility implies $(H_2, \underline{\mathcal{I}}_2)$ -admissibility, and equivalence is defined as mutual domination. Equivalence or domination is *functorial* if it is preserved under base changes $Z' \rightarrow Z$ and pullbacks with respect to log regular morphisms $Y' \rightarrow Y$. The functorial equivalence class of the pair $(H, \underline{\mathcal{I}})$ will be denoted $[H, \underline{\mathcal{I}}]$, and we say that a sequence $Y' \dashrightarrow Y$ is $[H, \underline{\mathcal{I}}]$ -admissible if it is $(H, \underline{\mathcal{I}})$ -admissible.

markedpushsec

6.2.5. Pushforwards. If $i: H \hookrightarrow Y$ is a suborbifold and $\underline{\mathcal{I}} = (\mathcal{I}, 1)$ is a marked ideal on H of weight 1, then we set $i_*\underline{\mathcal{I}} = (i_*\mathcal{I}, 1)$. It is easy to see (and can be deduced from Lemma 6.3.2(ii) below) that $(H, \underline{\mathcal{I}}) \approx (Y, i_*\underline{\mathcal{I}})$.

6.3. Restriction to maximal contact.

6.3.1. Basic compatibility. Operations on marked ideals and restriction onto a suborbifold are related as follows.

restrictlem

Lemma 6.3.2. *Assume that $Y \rightarrow Z$ is a log regular morphism of log DM stacks and $i: H \hookrightarrow Y$ a Z -suborbifold.*

(i) *If \mathcal{J} is an H -admissible submonomial Kummer ideal on Y and $a \geq 1$, then $\mathcal{J}^{(a)}|_H \subseteq (\mathcal{J}|_H)^{(a)}$.*

(ii) If $\underline{\mathcal{I}}$ is a marked ideal on Y and $\tau: H' \dashrightarrow H$ is an $\underline{\mathcal{I}}|_H$ -admissible sequence of submonomial Kummer blow ups, then the pushforward $\sigma = i_*(\tau): Y' \dashrightarrow Y$ is $\underline{\mathcal{I}}$ -admissible and $\sigma^c(\underline{\mathcal{I}})|_{H'} = \tau^c(\underline{\mathcal{I}}|_H)$.

Proof. Straightforward. ♣

6.3.3. *H-contracting modules of derivations.* Let H be a Z -suborbifold of pure codimension 1 in Y . As in [ATW17a, §6.2.3], a module $\mathcal{F} \subseteq \mathcal{D}_{Y/Z}$ is called *H-contracting* if $\mathcal{F}(\mathcal{I}_H) = \mathcal{O}_Y$. This property is preserved under submonomial Kummer blow ups:

Hcontract

Lemma 6.3.4. *If \mathcal{F} is H-contracting, $\sigma: Y' \rightarrow Y$ an H-admissible sequence of submonomial Kummer blow up, and H' is the strict transform of H , then $\mathcal{F}' = \sigma^c(\mathcal{F})$ is H' -contracting.*

Proof. The proof is the same as in [ATW17a, Lemma 6.2.4]. By étale localization and induction on the length this reduces to the case of a single blow up along $\mathcal{J} = (t_1, \dots, t_n, u_1, \dots, u_s)$ such that $\mathcal{I}_H = (t_1)$ and there exists $\partial \in \Gamma(\mathcal{F})$ with $\partial(t_1)$ a unit. On a y -chart one has that $\mathcal{I}_{H'} = (t'_1)$ for $t'_1 = y^{-1}t_1$ and $\partial' := y\partial \in \Gamma(\mathcal{F}')$. Then it remains to note that $\partial'(t'_1) = \partial(t_1) - t'_1\partial(y)$ is a unit along H' because $\partial(t_1)$ is a unit on H and hence also on H' . ♣

6.3.5. *Lift of admissibility.* Despite a relatively simple proof the following proposition is the only result where a simple induction on the order of differential operators does not work, and one has to use Taylor series.

liftlem

Proposition 6.3.6. *Let $f: Y \rightarrow Z$ be a log regular morphism of log DM stacks and let $\underline{\mathcal{I}}$ be a marked ideal on Y of maximal order. Assume that H is a maximal contact to $\underline{\mathcal{I}}$ and $\mathcal{F} \subseteq \mathcal{D}_{Y/Z}$ is an H-contracting submodule. Let \mathcal{J} be a submonomial Kummer ideal. Then \mathcal{J} is $\underline{\mathcal{I}}$ -admissible if and only if it is $(H, \underline{\mathcal{C}}_{\mathcal{F}}(\underline{\mathcal{I}})|_H)$ -admissible.*

Proof. Set $\underline{\mathcal{C}} = \underline{\mathcal{C}}_{\mathcal{F}}(\underline{\mathcal{I}})$ and $\mathcal{C}_i := \mathcal{F}^{(\leq i)}(\underline{\mathcal{I}})^{a!/(a-i)}$, where $0 \leq i \leq a-1$. In particular, $\underline{\mathcal{C}} = (\mathcal{C}, a!)$, where $\mathcal{C} = \sum_{i=0}^{a-1} \mathcal{C}_i$. Recall that $\underline{\mathcal{I}} \approx \underline{\mathcal{C}}$ by Lemma 5.3.4, hence \mathcal{J} is $\underline{\mathcal{I}}$ -admissible if and only if $\mathcal{C} \subseteq \mathcal{J}^{(a)}$. If \mathcal{J} is $\underline{\mathcal{I}}$ -admissible, then

$$\mathcal{C}|_H \subseteq \mathcal{J}^{(a)}|_H \subseteq (\mathcal{J}|_H)^{(a)}$$

by Lemma 6.3.2(i), so $\mathcal{J}|_H$ is $\underline{\mathcal{C}}|_H$ -admissible. In addition, $\mathcal{I}_H^{a!} \subseteq \mathcal{C}_{a-1} \subseteq \mathcal{J}^{(a)}$, hence $\mathcal{I}_H \subseteq \mathcal{J}^{\text{nor}}$ and \mathcal{J} is H-admissible.

Conversely, we should prove that if $\mathcal{C}_i|_H \subseteq (\mathcal{J}|_H)^{(a)}$ for $0 \leq i \leq a-1$ and $\mathcal{I}_H \subseteq \mathcal{J}^{\text{nor}}$, then $\underline{\mathcal{I}} \subseteq \mathcal{J}^{(a)}$. This can be checked étale-locally, so we can assume that Y is a scheme with Zariski log structure, $H = V(h)$, and $\partial \in \Gamma(\mathcal{D}_{Y/Z})$ is such that $u = \partial(h)$ is a unit. Moreover, by flatness of completions we can work formally-locally at a point $y \in Y$, and here comes the crucial ingredient: we fix an isomorphism $\widehat{\mathcal{O}}_{Y,y} = \widehat{\mathcal{O}}_{H,y}[[h]]$, which exists by Proposition 3.2.2.¹⁹ For any $\phi \in \mathcal{I}_y$ we obtain a Taylor series presentation $\phi = \sum_{i=0}^{\infty} c_i h^i$ with $\frac{1}{i!u^i} \partial^i(\phi)|_H = c_i$. Then $c_i \in (\widehat{\mathcal{J}}_{H,y}^{a-i})^{\text{nor}} \subset (\widehat{\mathcal{J}}_y^{a-i})^{\text{nor}}$ and $h^a \in \mathcal{J}^{(a)}$, and hence $\phi \in (\widehat{\mathcal{J}}_y^a)^{\text{nor}}$. ♣

19→

¹⁹(Michael) Should update the proposition, including the non-integral case.

6.3.7. *Equivalence of the restriction.* Now, we are ready to prove the main theorem of the theory of maximal contact.

Htheorem

Theorem 6.3.8. *Assume that $Y \rightarrow Z$ is a log regular morphism of log DM stacks, $\underline{\mathcal{I}}$ is a marked ideal of maximal order on Y , $i: H \hookrightarrow Y$ is a hypersurface of maximal contact to $\underline{\mathcal{I}}$, and $\mathcal{F} \subseteq \mathcal{D}_{Y/Z}$ is an H -contractible \mathcal{O}_Y -submodule. Then $(Y, \underline{\mathcal{I}}) \approx (H, \underline{\mathcal{C}}_{\mathcal{F}}(\underline{\mathcal{I}})|_H)$.*

Proof. Recall that $\underline{\mathcal{I}} \approx \underline{\mathcal{C}} := \underline{\mathcal{C}}_{\mathcal{F}}(\underline{\mathcal{I}})$ by Lemma 5.3.4, hence $(Y, \underline{\mathcal{I}}) \preccurlyeq (H, \underline{\mathcal{C}}_H)$, where we set $\underline{\mathcal{C}}_H = \underline{\mathcal{C}}|_H$. The main part is to prove the opposite domination. By induction on the length it suffices to prove the following assertion:

Assume that $\tau: H' \dashrightarrow H$ is $\underline{\mathcal{C}}_H$ -admissible and such that its pushforward $\sigma = i_*(\tau): Y' \rightarrow Y$ is $\underline{\mathcal{I}}$ -admissible, and assume that \mathcal{J} is an H' -admissible submonomial Kummer ideal on Y' such that $\mathcal{J}|_{H'}$ is $\tau^c(\underline{\mathcal{C}}_H)$ -admissible. Then \mathcal{J} is $\underline{\mathcal{I}}'$ -admissible, where $\underline{\mathcal{I}}' = \sigma^c(\underline{\mathcal{I}})$.

To prove the assertion, recall that by Theorem 5.3.6 $\underline{\mathcal{C}}_{\mathcal{F}'}(\underline{\mathcal{I}}') = \sigma^c(\underline{\mathcal{C}})$, where $\mathcal{F}' = \sigma^*(\mathcal{F})$. Restricting this onto H' and applying Lemma 6.3.2(ii) we obtain that

$$\underline{\mathcal{C}}_{\mathcal{F}'}(\underline{\mathcal{I}}')|_{H'} = \sigma^c(\underline{\mathcal{C}})|_{H'} = \tau^c(\underline{\mathcal{C}}_H),$$

and hence $\mathcal{J}|_{H'}$ is $\underline{\mathcal{C}}_{\mathcal{F}'}(\underline{\mathcal{I}}')|_{H'}$ -admissible. Thus, \mathcal{J} is $\underline{\mathcal{I}}'$ -admissible by Lemma 6.3.4 and Proposition 6.3.6. \clubsuit

6.4. **Functoriality.** We conclude Section 6 with a brief explanation of the fact that all constructions with marked ideals are compatible with log regular morphisms and base changes. If $\underline{\mathcal{I}} = (\mathcal{I}, a)$ is a marked ideal on Y and $g: Y' \rightarrow Y$ is a morphism, then the *pullback* marked ideal is $g^{-1}(\underline{\mathcal{I}}) = (g^{-1}(\mathcal{I}), a)$.

functormarked1

Lemma 6.4.1. *Let $f: Y \rightarrow Z$ be a log regular morphism of log DM stacks, $\underline{\mathcal{I}}$ a marked ideal on Y , $\mathcal{F} \subseteq \mathcal{D}_{Y/Z}$ an \mathcal{O}_Y -submodule, $g: Z' \rightarrow Z$ a morphism of log DM stacks with base changes $g': Y' \rightarrow Y$ and $f': Y' \rightarrow Z'$, and $\underline{\mathcal{I}}' = g'^{-1}\underline{\mathcal{I}}$, $\mathcal{F}' = g^*(\mathcal{F})$. Then*

(i) *Assume that H is a maximal contact to $\underline{\mathcal{I}}$. Then $H' = H \times_Y Y'$ is a maximal contact to $\underline{\mathcal{I}}'$. In addition, if \mathcal{F} is H -contracting, then \mathcal{F}' is H' -contracting.*

(ii) $g'^{-1}(\underline{\mathcal{C}}_{\mathcal{F}}(\underline{\mathcal{I}})) = \underline{\mathcal{C}}_{\mathcal{F}'}(\underline{\mathcal{I}}')$.

(iii) $g'^{-1}(\mathcal{W}(\underline{\mathcal{I}})) = \mathcal{W}(\underline{\mathcal{I}}')$ and $\mu_{\underline{\mathcal{I}}} \circ g' = \mu_{\underline{\mathcal{I}}}'$.

(iv) *If a submonomial Kummer blow up sequence $Y_n \dashrightarrow Y$ is admissible for (resp. an order reduction of) $\underline{\mathcal{I}}$, then the pullback sequence $Y'_n \dashrightarrow Y'$ is admissible for (resp. an order reduction of) $\underline{\mathcal{I}}'$.*

Proof. Claim (iii) follows from Lemma 2.8.14, and claims (i) and (ii) are proved by unravelling the definitions. For example, if \mathcal{F} is H -contracting, then $\mathcal{F}(\mathcal{I}_H) = \mathcal{O}_Y$ and applying g^{-1} yields $\mathcal{F}'(\mathcal{I}'_H) = \mathcal{O}_{Y'}$, that is, \mathcal{F}' is H' -contracting. \clubsuit

In the same way, but using Lemma 2.8.15 as an input one proves

functormarked2

Lemma 6.4.2. *Assume that $g: Y \rightarrow Y$ and $f: Y \rightarrow Z$ are log regular morphism of log DM stacks, $\mathcal{I} \subseteq \mathcal{O}_Y$ an ideal with $\mathcal{I}' = g^{-1}\mathcal{I}$, and $\mathcal{F} \subseteq \mathcal{D}_{Y/Z}$ a submodule with $\mathcal{F}' = g^*(\mathcal{F})$. Then,*

(i) *Assume that H is a maximal contact to $\underline{\mathcal{I}}$. Then $H' = H \times_Y Y'$ is a maximal contact to $\underline{\mathcal{I}}'$. In addition, if \mathcal{F} is H -contracting, then \mathcal{F}' is H' -contracting.*

(ii) $g'^{-1}(\underline{\mathcal{C}}_{\mathcal{F}}(\underline{\mathcal{I}})) = \underline{\mathcal{C}}_{\mathcal{F}'}(\underline{\mathcal{I}}')$.

(iii) $g'^{-1}(\mathcal{W}(\underline{\mathcal{I}})) = \mathcal{W}(\underline{\mathcal{I}}')$ and $\mu_{\underline{\mathcal{I}}} \circ g' = \mu_{\underline{\mathcal{I}}}'$.

(iv) If a submonomial Kummer blow up sequence $Y_n \dashrightarrow Y$ is admissible for (resp. an order reduction of) $\underline{\mathcal{I}}$, then the pullback sequence $Y'_n \dashrightarrow Y$ is admissible for (resp. an order reduction of) $\underline{\mathcal{I}}'$.

princsec

7. RELATIVE LOGARITHMIC ORDER REDUCTION

In this section we will construct the (logarithmic) relative principalization method and prove Theorem 1.2.3.

7.1. Reduction to marked ideals. By an *order reduction method* we mean a method which assigns either an order reduction or the empty output to marked ideals on relative log orbifolds. As in the classical case, relative log principalization of \mathcal{I} is nothing else but a relative order reduction of the marked ideal $(\mathcal{I}, 1)$. Thus Theorem 1.2.3 is a particular case of the following

mainth

Theorem 7.1.1 (Order reduction). *There exists an order reduction method \mathcal{F} satisfying the following properties:*

(i) *Existence: if $f: X \rightarrow B$ is a relative log orbifold with a marked ideal $\underline{\mathcal{I}}$ and B is universally resolvable, then there exists a modification $g: B' \rightarrow B$ such that g is an isomorphism over the complement to the closure of the image of $\text{supp}(\underline{\mathcal{I}})$ in B and $\mathcal{F}(f', \underline{\mathcal{I}}') \neq \emptyset$, where $f': X' \rightarrow B'$ is the saturated pullback of f and $\underline{\mathcal{I}}' = \underline{\mathcal{I}}\mathcal{O}_{X'}$ is the pullback of $\underline{\mathcal{I}}$.*

(ii) *Compatibility with base change: if $\mathcal{F}(f, \underline{\mathcal{I}}) \neq \emptyset$ and $g: B' \rightarrow B$ is any morphism of logarithmic stacks with saturated base changes $f': X' \rightarrow B'$ and $g': X' \rightarrow X$, and $\underline{\mathcal{I}}' = g'^{-1}\underline{\mathcal{I}}$, then the sequence $\mathcal{F}(f', \underline{\mathcal{I}}')$ is obtained from the saturated pullback sequence $\mathcal{F}(f, \underline{\mathcal{I}}) \times_B B'$ by removing Kummer blowings up with empty centers.*

(iii) *Functoriality: if $\mathcal{F}(f, \underline{\mathcal{I}}) \neq \emptyset$ and $\underline{\mathcal{I}}' = g^{-1}\underline{\mathcal{I}}$ for a logarithmically regular morphism $g: X' \rightarrow X$ such that $X' \rightarrow B$ is a relative log orbifold, then $\mathcal{F}(f', \underline{\mathcal{I}}')$ is obtained from the saturated pullback sequence $\mathcal{F}(f, \underline{\mathcal{I}}) \times_X X'$ by removing Kummer blowings up with empty centers.*

(iv) *Dependence on the equivalence class: if $\bar{f}: \bar{X} \rightarrow B$ is another relative log orbifold and $i: X \hookrightarrow \bar{X}$ is a B -suborbifold embedding of pure codimension, then the sequence $i_*(\mathcal{F}(f \circ i, \underline{\mathcal{I}})): \bar{X}' \dashrightarrow \bar{X}$ depends only on \bar{f} and the functorial equivalence class $[X, \underline{\mathcal{I}}]$. In particular, if $\underline{\mathcal{I}}$ is of weight 1, then $i_*(\mathcal{F}(f \circ i, \underline{\mathcal{I}})) = \mathcal{F}(f, i_*\underline{\mathcal{I}})$.*

Remark 7.1.2. Since there are many functorially equivalent marked ideals on X , part (iv) provides a non-trivial addendum even when $X = \bar{X}$. However, it is important for our argument to have it for arbitrary i .

The rest of Section 7 is devoted to proving the theorem: we will first construct a method satisfying (iv), and then establish properties (i)–(iii).

7.2. Invariants of functorial equivalence classes. Similarly to its predecessors, our order reduction method runs by restricting to suborbifolds $i: H \hookrightarrow X$. In this paper, we prove independence of choices using equivalence of marked ideals on different suborbifolds as defined in §6.2.4. As a preparation, we are now going to study invariants of equivalence classes.

modelsec

7.2.1. *A model case.* We start with the following simple observation. Assume that X is a smooth scheme with a marked ideal $\underline{\mathcal{I}} = (\mathcal{I}, a)$ and a closed point $x \in X$. If $b = \text{ord}_x(\mathcal{I})$ and $\sigma: X' \rightarrow X$ is the blow up at x , then \mathcal{I}_E^b is the maximal power of \mathcal{I}_E that divides $\sigma^{-1}(\mathcal{I})$. Therefore, the sequence $X'(d) \rightarrow X' \rightarrow X$ with centers m_x and \mathcal{I}_E^d is $\underline{\mathcal{I}}$ -admissible if and only if $da \leq b - a$, that is $d \leq \mu_x(\underline{\mathcal{I}}) - 1$. In our approach we allow any $d \in \mathbb{Q}_{\geq 0}$, hence the equivalence class of $\underline{\mathcal{I}}$ determines the weighted order in a very elementary way. Note for comparison that in the classical approach one blows up only smooth centers, and the same goal is achieved by Hironaka's trick, where one first replaces X by $X \times \mathbf{A}^1$ with the log structure given by $X \times \{0\}$.

In general, we will have to solve a few minor technical problems: x can be non-closed, m_x does not have to be submonomial, and the codimension of H is not determined by $[H, \underline{\mathcal{I}}]$. The following result will be used to deal with the first two issues.

submonomiallem

Lemma 7.2.2. *Let $f: X \rightarrow B$ be a relative log orbifold and $x \in |X|$ a point. Then there exists a base change $B' \rightarrow B$, a localized étale morphism $X' \rightarrow X \times_B B'$ with X' a scheme, and a closed point x' with image $x \in |X|$ such that the ideal $m_{x'}$ is submonomial.*

Proof. It is easy to see that there always exists a morphism of log schemes $g: B' = \text{Spec}(O) \rightarrow B$ such that O is either a field or a DVR, the log structure is $O \setminus \{0\}$, and g takes the closed point $b' \in B'$ to $b := f(x)$. Find an étale cover $X' \rightarrow X \times_B B'$ with X' a scheme, and choose a point $x' \in X'$ over x . Then replacing X' by its localization at x' we can assume that x' is closed. Now, the fiber $X'_{b'}$ is a monomial subscheme, and it is easy to see that the log fiber $S = S_{x'}$ coincides with the log stratum of x' in $X'_{b'}$, hence $S_{x'}$ is monomial. Since x' is a closed point in $S_{x'}$, the ideal $m_{x'}$ is submonomial. ♣

submonomialrem

Remark 7.2.3. One could use other classes of base changes in the lemma. For example, one could replace B by the localization at B , increase the log structure so that m_b becomes monomial, and then apply a monomial blow up so that the base change of f becomes exact at x .

7.2.4. *Operations on equivalence classes.* By definition, functorial equivalence is preserved by base changes of B , pullbacks with respect to log regular morphisms $g: X' \rightarrow X$, and controlled transforms under admissible blow ups. Hence all these operations are defined on equivalence classes $[H, \underline{\mathcal{I}}]$. For example, $g^{-1}([H, \underline{\mathcal{I}}])$ is defined to be $[H \times_X X', (g|_H)^{-1}\underline{\mathcal{I}}]$.

7.2.5. *Codimensions.* The codimension of H at a point $x \in |X|$ will be denoted $c_x(H)$. Let \mathcal{C} be a functorial equivalence class of marked ideals on suborbifolds of X . By the *codimension* $c_x(\mathcal{C})$ of \mathcal{C} at $x \in |X|$ we mean the maximal possible codimension $c_x(H')$, where $g: X' \rightarrow X$ is an étale neighborhood of x and $(H', \underline{\mathcal{I}}')$ is a representative of $g^{-1}(\mathcal{C})$. Finally, if \mathcal{J} is a submonomial Kummer ideal of the form $\mathcal{I}_H + \mathcal{N}$, where \mathcal{N} is Kummer monomial and H is a suborbifold, then the number $c_x(H)$ is easily seen to depend only on \mathcal{J} and we call it the *monomial codimension* $c_x(\mathcal{J})$ of \mathcal{J} at x . Naturally, the monomial codimension of centers of admissible sequences must be at least the maximal codimension of equivalence classes:

codimlem

Lemma 7.2.6. *Assume that \mathcal{C} is a functorial equivalence class on X and a sequence $\sigma: X_n \dashrightarrow X_0 = X$ with centers $\mathcal{J}_i \subseteq \mathcal{O}_{(X_i)_{\text{két}}}$ is \mathcal{C} -admissible. Let $x_i \in \text{supp}(\mathcal{J}_i)$ and let $x \in |X|$ be its image. Then $c_{x_i}(\mathcal{J}_i) \geq c_x(\mathcal{C})$.*

Proof. Replacing X be an étale neighborhood of x and replacing σ by its pullback we can assume that \mathcal{C} contains a representative (H, \mathcal{I}) with $c_x(H) = c_x(\mathcal{C})$. Then the strict transform $H_i \hookrightarrow X_i$ satisfies $c_{x_i}(H_i) = c_x(H)$ and $\mathcal{I}_{H_i} \subseteq \mathcal{J}_i$. Therefore, $c_{x_i}(\mathcal{J}_i) \geq c_{x_i}(H_i) = c_x(\mathcal{C})$, as required. \clubsuit

Now we can show that codimensions detect whether a marked ideal is of maximal order.

codimth

Theorem 7.2.7. *Let $X \rightarrow B$ be a relative log orbifold, $i: H \hookrightarrow X$ a suborbifold, $\underline{\mathcal{I}}$ a marked ideal on H with equivalence class $\mathcal{C} = [H, \underline{\mathcal{I}}]$, and $x \in \text{supp}(\underline{\mathcal{I}})$ a point. Then $c_x(\mathcal{C}) > c_x(H)$ if and only if $\underline{\mathcal{I}}$ is of maximal order at x .*

Proof. Assume that $\underline{\mathcal{I}}$ is of maximal order at x . We can replace X by an étale neighborhood of x , hence by Theorem 6.1.3 we can assume that there exists a maximal contact $H' \hookrightarrow H$ to $\underline{\mathcal{I}}$. By Theorem 6.3.8, $\underline{\mathcal{C}}(\underline{\mathcal{I}})|_{H'}$ is another representative of \mathcal{C} , and its codimension is larger than that of $\underline{\mathcal{I}}$.

Conversely, it suffices to obtain a contradiction to the following assumption: $\underline{\mathcal{I}} = (\mathcal{I}, a)$ is not of maximal order at x , but $c_x(\mathcal{C}) > c_x(H)$. This situation persists after base changes and pullbacks with respect to étale localizations of X , hence by Lemma 4.1.4 we can assume that x is closed and $m_{X,x}$ is submonomial. Then $m_{H,x}$ is submonomial too, say, $m_{H,x} = (t, u)$, where $t = (t_1, \dots, t_n) \subset m_{H,x}$ is a family of regular parameters at x and $u = (u_1, \dots, u_r)$ monomials generating the maximal ideal of \overline{M}_x .

By our assumption $\text{logord}_x(\underline{\mathcal{I}}) \geq d := a + 1$. Since $\mathcal{I}_x \subseteq (t)^d + (u) \subseteq (t, u^{1/d})^d$, the blow up $\tau: H' \rightarrow H$ along $(t, u^{1/d})$ is (\mathcal{I}, d) -admissible, and hence, \mathcal{I}_E divides $\tau^{-1}\mathcal{I}(a)$. Let $H'(1/a) \rightarrow H'$ be the blow up along $\mathcal{I}_E^{1/a}$. Then the sequence $H'(1/a) \rightarrow H' \rightarrow H$ is $\underline{\mathcal{I}}$ -admissible, and its pushforward $X'(1/a) \rightarrow X' \rightarrow X$ is \mathcal{C} -admissible. Since the center of $X'(1/a) \rightarrow X'$ is of monomial codimension $c_x(H)$, Lemma 7.2.6 yields a contradiction. \clubsuit

7.2.8. Weighted invariants. One can define analogues of weighted invariants for equivalence classes $[H, \underline{\mathcal{I}}]$. Since H is not determined by the class, it is natural to take the weighted invariants of $\underline{\mathcal{I}}$ and push them forward to X . Namely, by $\mu_{[H, \underline{\mathcal{I}}]}: |X| \rightarrow \mathbb{N}$ we denote the weighted log order function $\mu_{\underline{\mathcal{I}}}: |H| \rightarrow \mathbb{N}$ extended by zero outside of $|H|$, and by $\mathcal{W}_{X/B}([H, \underline{\mathcal{I}}])$ we denote the Kummer ideal $i_*(\mathcal{W}_{H/B}(\underline{\mathcal{I}}))$. We have slightly abused notation because these invariants certainly depend on c_H . On the positive side, our main result about equivalence classes will be that they depend only on $[H, \underline{\mathcal{I}}]$ and c_H .

Remark 7.2.9. It is important to push forward the invariants of $\underline{\mathcal{I}}$, instead of taking invariants of $i_*(\underline{\mathcal{I}})$. Indeed, if H is of positive codimension, then the log order of $i_*(\underline{\mathcal{I}})$ does not exceed one. In fact, $\text{logord}_{i_*(\underline{\mathcal{I}})}$ is just the test function of $\text{supp}(\underline{\mathcal{I}})$. In addition, $\mathcal{W}_{X/B}([H, \underline{\mathcal{I}}])$ is submonomial while $\mathcal{W}_{X/B}(i_*(\underline{\mathcal{I}}))$ is monomial.

7.2.10. Hironaka's trick: the logarithmic version. The following theorem is an analogue of [BM08, Theorems 6.1, 6.2]. The argument is in the same venue but more straightforward.

equivth

Theorem 7.2.11. *Assume that $f: X \rightarrow B$ is a relative log orbifold, $i: H \hookrightarrow X$ is a suborbifold of pure codimension c_H , and $\underline{\mathcal{I}} = (\mathcal{I}, a)$ is a marked ideal on H . Then the function $\mu_{[H, \underline{\mathcal{I}}]}: |X| \rightarrow \mathbb{N}$ and the Kummer ideal $\mathcal{W}_{X/B}([H, \underline{\mathcal{I}}])$ depend only on $\mathcal{C} = [H, \underline{\mathcal{I}}]$ and c_H rather than on the choice of the pair $(H, \underline{\mathcal{I}}) \in \mathcal{C}$.*

Proof. First, let us show that for any $x \in |X|$ the number $\mu = \mu_x(\underline{\mathcal{I}})$ is determined by \mathcal{C} and c_H . Since the log order is compatible with regular morphisms and base changes by Lemmas 2.8.14 and 2.8.15, as in the proof Theorem 7.2.7 we can use Lemma 7.2.2 to reduce to the case when X is a scheme, x is closed, and $m_{H,x} = (t, u)$ is submonomial.

Case 0: $\mu = 0$. The vanishing locus of $\mu_{[H, \underline{\mathcal{I}}]}$ is precisely $X \setminus \text{supp}(\underline{\mathcal{I}})$, and $\mu_x(\underline{\mathcal{I}}) = 0$ if and only if $d := \text{logord}_x(\mathcal{I}) < a$ if and only if $m_{H,x}$ is not $\underline{\mathcal{I}}$ -admissible if and only if $m_{X,x}$ is not \mathcal{C} -admissible. Thus, this case (and $\text{supp}(\underline{\mathcal{I}})$) is detected by \mathcal{C} only.

Case 1: $\mu = 1$. By Theorem 7.2.7 and Case 0, this happens if and only if $c_H > c_x(\mathcal{C})$ and $m_{X,x}$ is \mathcal{C} -admissible.

Case 2: $\mu = \infty$. This happens if and only if there exists an $\underline{\mathcal{I}}$ -admissible monomial ideal $\mathcal{N} \subset \mathcal{O}_H$ with $x \in V(\mathcal{N})$ (in fact, $\mathcal{N} = (u^{1/a})$ will work). The latter happens if and only if x lies in the support of a \mathcal{C} -admissible submonomial ideal of monomial index c_H , the property depending on \mathcal{C} and c_H only.

Case 3: $1 < \mu < \infty$. By the above cases, this situation is also detected by \mathcal{C} and c_H . Let us show how to find μ . Note that $a < d < \infty$ and for any $l \geq d$ we have that $\mathcal{I}_x \subseteq (t)^d + (u) \subseteq (t, u^{1/l})^d$ and the blow up $\tau_l: H_l \rightarrow H$ along $(t, u^{1/l})$ is (\mathcal{I}, d) -admissible at x . For $n \in \mathbb{Q}_{>0}$ let $\tau_{l,n}: H_l(n) \rightarrow H_l$ denote the blow up along the monomial Kummer ideal \mathcal{I}_E^n , and let $X_l(n) \xrightarrow{\sigma_{l,n}^*} X_l \xrightarrow{\sigma_l} X$ be the pushforward. By Corollary 5.1.18 $\tau_l^c(\underline{\mathcal{I}})$ is balanced with monomial part \mathcal{I}_E^{d-a} , hence the sequence σ is \mathcal{C} -admissible if and only if $na \leq d - a$. The latter happens if and only if $n \leq \mu - 1$. Taking $l \gg 0$ this characterizes μ once we show that the sequence $X_l(\mu - 1) \rightarrow X_l \rightarrow X$ depends only on \mathcal{C} and c_H (and not on H via \mathcal{I}_E). Clearly, $X_l \rightarrow X$ depends only on x and l , and the center of $\sigma_{l, \mu - 1}$ is the maximal submonomial $\sigma_l^c(\mathcal{C})$ -admissible Kummer ideal on X_l of monomial codimension c_H , the property which depends only on \mathcal{C} and c_H .

Finally, let us deal with the weighted monomial parts. If $\mathcal{W}_{H/B}(\underline{\mathcal{I}}) \neq 0$, then it is the maximal monomial $\underline{\mathcal{I}}$ -admissible Kummer ideal. Therefore, $\mathcal{W}_{X/B}(\mathcal{C})$ is the maximal submonomial Kummer ideal which is \mathcal{C} -admissible and of monomial codimension c_H . This description only depends on \mathcal{C} and c_H . It remains to show that $\mathcal{W}_{H_1/B}(\underline{\mathcal{I}}_1) \neq 0$ for any other representative $(H_1, \underline{\mathcal{I}}_1) \in \mathcal{C}$ of codimension c_H . If $\tau: H' \rightarrow H$ is the blow up along $\mathcal{W}_{H/B}(\underline{\mathcal{I}})$, then $\tau^c(\underline{\mathcal{I}})$ is a clean ideal by Theorem 5.1.6. Let $\tau_1: H'_1 \rightarrow H_1$ be the blow up corresponding to τ via the equivalence. Being clean is equivalent to the condition $\mu < \infty$, hence $\tau^c(\underline{\mathcal{I}}_1)$ is also clean by the first part of the theorem. This proves that $\tau_1^{-1} \mathcal{M}_{H_1/B}(\underline{\mathcal{I}}_1)$ is monomial, and since τ_1 is log étale we obtain that $\mathcal{M}_{H_1/B}(\underline{\mathcal{I}}_1)$ is monomial by ...²⁰ ♣ ←20

Combining the above results with Corollary 5.1.18 one easily obtains the following result:

²⁰(Michael) Should extend Lemma 2.2.6 to the general log étale case.

equivcor

Corollary 7.2.12. *Assume that $(H_1, \underline{\mathcal{I}}_1) \approx (H_2, \underline{\mathcal{I}}_2)$ with $\underline{\mathcal{I}}_1$ balanced and H_1, H_2 of pure codimension c in X . Then $\underline{\mathcal{I}}_2$ is balanced too and $(H_1, \underline{\mathcal{I}}_1^{\text{cln}}) \approx (H_2, \underline{\mathcal{I}}_2^{\text{cln}})$.*

ordersec

7.3. The method. Now, let us construct the method \mathcal{F} whose existence is asserted by Theorem 7.1.1. It is very close to (and a generalization of) the absolute logarithmic order reduction in [ATW17a, §2.11].

7.3.1. Induction scheme. The method runs by induction on $n = \log \dim(X/B)$, as defined in §2.4.6. The case of $n = 0$ is allowed, so the basis of induction is empty. We will also prove that $i_*(\mathcal{F}(f, \underline{\mathcal{I}}))$ depends only on the class $\mathcal{C} = [X, \underline{\mathcal{I}}]$ in \overline{X} , but on step n we will only compare $(X, \underline{\mathcal{I}})$ to representatives $(\tilde{X}, \tilde{\underline{\mathcal{I}}})$ of \mathcal{C} with $\log \dim(\tilde{X}/B) \leq n$.

maxordercase

7.3.2. The maximal order case. First, let us construct $\sigma = \mathcal{F}(f, \underline{\mathcal{I}})$ in the particular case when $\underline{\mathcal{I}}$ is of maximal order. The general line is very simple: étale-locally $\underline{\mathcal{I}}$ is equivalent to a marked ideal $\underline{\mathcal{I}}_0$ on a maximal contact H_0 . By induction, the order reduction of $\underline{\mathcal{I}}_0$ is defined and depends only on the equivalence class of $\underline{\mathcal{I}}_0$. This implies that the construction is independent of choices and hence descends to an order reduction of $\underline{\mathcal{I}}$.

Now, let us work out details. By Theorem 7.2.7 there exists an étale covering $p: X_0 \rightarrow X$ such that the equivalence class $[X_0, p^{-1}(\underline{\mathcal{I}})]$ contains a representative $(H_0, \underline{\mathcal{I}}_0)$ with $i_0: H_0 \hookrightarrow X_0$ of pure codimension 1. If $g_0 = f \circ p \circ i_0$ is the relative log orbifold $H_0 \rightarrow X$, then $\mathcal{F}(g_0, \underline{\mathcal{I}}_0)$ is already defined by induction on n . If it is empty, then we set $\mathcal{F}(f, \underline{\mathcal{I}})$ to fail. Otherwise we define $\sigma_0: X'_0 \dashrightarrow X_0$ to be the pushforward of $\tau_0 = \mathcal{F}(g_0, \underline{\mathcal{I}}_0): H'_0 \dashrightarrow H_0$, and we claim that this sequence descends to a Kummer blow up sequence $X' \dashrightarrow X$ (despite the fact that H_0 does not have to descend to a suborbifold of X). Once we prove this, the latter sequence will be denoted σ and its independence of the covering p is easy: given another covering p' consider a mutual refinement and use functoriality of \mathcal{F} with respect to étale (even log regular) morphisms.

Let $p_i, i \in \{1, 2\}$ be the two projections of $X_1 := X_0 \times_X X_0$ onto X_0 . By étale descent we should prove that the two pullbacks $\sigma_i = p_i^{-1} \sigma_0$ coincide. Consider the suborbifolds $H_i = p_i^{-1}(H_0)$ of X_1 with sequences $\tau_i: H'_i \dashrightarrow H_i$ pulled back from τ_0 , morphisms $g_i: H_i \rightarrow B$ and marked ideals $\underline{\mathcal{I}}_i = p_i^{-1}(\underline{\mathcal{I}}_0)$. Clearly, σ_i is the pushforward of τ_i , and $\tau_i = \mathcal{F}(g_i, \underline{\mathcal{I}}_i)$ by the functoriality of \mathcal{F} . In addition, H_i are of the same log dimension over B as H_0 , and $(H_1, \underline{\mathcal{I}}_1) \approx (H_2, \underline{\mathcal{I}}_2)$ because both are equivalent to the pullback of $\underline{\mathcal{I}}$ to X_1 via functoriality of the equivalence $(H_0, \underline{\mathcal{I}}_0) \approx (X_0, p^{-1}\underline{\mathcal{I}})$ with respect to p_i . Therefore, $\sigma_1 = \sigma_2$ by the induction assumption applied to $\underline{\mathcal{I}}_i$ and $H_i \hookrightarrow X_1$, and we obtain the required order reduction σ .

indeprem

Remark 7.3.3. The above paragraph provides a critical gluing argument, where equivalence on suborbifolds of different codimensions is used to prove independence of the étale local choice of a maximal contact.

Finally, let us explain why the pushforward $\bar{\sigma}: \overline{X}' \dashrightarrow \overline{X}$ of σ depends only on \mathcal{C} . So, let $\tilde{X} \hookrightarrow \overline{X}$ be another B -suborbifold of pure codimension $c_{\tilde{X}} \geq c_X$ and let $\tilde{\underline{\mathcal{I}}}$ be a marked ideal on \tilde{X} such that $(X, \underline{\mathcal{I}}) \approx (\tilde{X}, \tilde{\underline{\mathcal{I}}})$. We should check that $\bar{\sigma}$ coincides with the pushforward of $\tilde{\sigma} = \mathcal{F}(f, \tilde{\underline{\mathcal{I}}})$. This can be done étale-locally on \overline{X} , and since the étale topology of \overline{X} induces the étale topologies of X and \tilde{X} ,

passing to a fine enough étale covering of \overline{X} we can assume that there exist $H \hookrightarrow X$ of pure codimension one, $\tilde{H} \hookrightarrow \tilde{X}$ of pure codimension 1 if $c_X = c_{\tilde{X}}$ and $\tilde{H} = \tilde{X}$ otherwise, and marked ideals $\underline{\mathcal{I}}_0$ and $\tilde{\underline{\mathcal{I}}}_0$ on them such that $(H, \underline{\mathcal{I}}_0) \approx (X, \underline{\mathcal{I}}) \approx (\tilde{H}, \tilde{\underline{\mathcal{I}}}_0)$ in X . Then $(H, \underline{\mathcal{I}}_0) \approx (\tilde{H}, \tilde{\underline{\mathcal{I}}}_0)$ also in \overline{X} and by the induction assumption the pushforwards of $\mathcal{F}(H \rightarrow B, \underline{\mathcal{I}}_0)$ and $\mathcal{F}(\tilde{H} \rightarrow B, \tilde{\underline{\mathcal{I}}}_0)$ to \overline{X} coincide. By the construction, the latter are precisely the pushforwards of σ and $\tilde{\sigma}$.

generalcase

7.3.4. *The general case.* The method outputs a sequence

$$\sigma: X_0 \xrightarrow{\sigma_0} X_{\mu_1} \xrightarrow{\sigma_{\mu_1}} X_{\mu_{l-1}} \xrightarrow{\sigma_{\mu_{l-2}}} \dots \xrightarrow{\sigma_{\mu_0}} X_{\mu_0} = X_\infty \xrightarrow{\sigma_\infty} X,$$

which will be constructed in three steps below. For $\mu = \mu_i$ the controlled transform of $\underline{\mathcal{I}}$ to X_μ will be denoted $\underline{\mathcal{I}}_\mu$ and we will have $\mu = \mu(\underline{\mathcal{I}}_\mu^{\text{cln}})$. It is convenient to label this sequence by weighted log orders because they only depend on the equivalence class. We will also denote the pushout sequence $\bar{\sigma}: \overline{X}_0 \rightarrow \overline{X}$ and consider the induced embeddings $i_\mu: X_{\mu_i} \hookrightarrow \overline{X}_\mu$ and classes $\mathcal{C}_\mu = [X_\mu, \underline{\mathcal{I}}_\mu]$ in \overline{X}_μ .

Step 1. *Initial cleaning.* If $\mathcal{W}_{X/B}(\underline{\mathcal{I}}) = 0$, then the algorithm outputs the fail value \emptyset and stops. Otherwise the step outputs the blow up $\sigma_\infty: X_\infty \rightarrow X$ along the monomial Kummer ideal $\mathcal{W}_{X/B}(\underline{\mathcal{I}})$. The controlled transform $\underline{\mathcal{I}}_\infty$ is balanced, even clean, by Theorem 5.1.6. The induced blow up $\bar{\sigma}_\infty: \overline{X}_\infty \rightarrow \overline{X}$ is along $\mathcal{W}_{\overline{X}/B}(\mathcal{C}) = i_* \mathcal{W}_{X/B}(\underline{\mathcal{I}})$, so it only depends on \mathcal{C} and c_H by Theorem 7.2.11.

Step 2. *Reducing the order of the clean part.* This step composes sequences $\sigma_\mu := \mathcal{F}(X_\mu, \underline{\mathcal{I}}_\mu^{\text{cln}})$, which are defined by §7.3.2. Starting with $\mu_0 = \mu(\underline{\mathcal{I}}_\infty)$ and $X_{\mu_0} = X_\infty$, we inductively define σ_{μ_i} as above and label its target by $\mu_{i+1} := \mu((\sigma_{\mu_i}^c \underline{\mathcal{I}}_{\mu_i})^{\text{cln}})$.

By induction on i one obtains from Corollary 5.1.18 that each σ_μ is $\underline{\mathcal{I}}_\mu$ -admissible and $\sigma_\mu^c(\underline{\mathcal{I}}_\mu)$ is balanced with the clean part being $\sigma_\mu^c(\underline{\mathcal{I}}_\mu^{\text{cln}})$. In particular, the log order of the clean part drops on each step: $\mu_0 > \mu_1 > \dots$. Since $\mu_i \in \frac{1}{a}\mathbb{N}$, after finitely many steps we arrive at $\mu_l = 0$, obtaining $\underline{\mathcal{I}}_{\mu_l}$ with a resolved clean part.

In addition, the class $\mathcal{C}_\mu^{\text{cln}} = [X_\mu, \underline{\mathcal{I}}_\mu^{\text{cln}}]$ in \overline{X}_μ is determined by \mathcal{C}_μ and c_X by Corollary 7.2.12. By induction on i , the class \mathcal{C}_μ is determined by \mathcal{C} and c_X , so it remains to note that $(i_\mu)_*(\sigma_\mu)$ is determined by $\mathcal{C}_\mu^{\text{cln}}$ and c_X by §7.3.2.

Step 3. *Final cleaning.* At this step, the ideal is balanced with a resolved clean part, so σ_0 is simply the Kummer blow up of $\mathcal{W}(\underline{\mathcal{I}}_{\mu_l})$. The same argument as in step 1, shows that the pushforward to \overline{X} only depends on \mathcal{C} and c_X .

7.3.5. *Addenda.* We conclude with a couple of remarks about the algorithm.

uniquerem

Remark 7.3.6. In the classical case, intensive study of resolution led to different descriptions of essentially the same algorithm, with the only variations in combinatorial parts of the algorithm. A natural question, that interested us before starting this project, is whether the algorithm is indeed essentially unique or this happened just because of flow of ideas between different approaches. We expect that the first possibility is true and show that our algorithm, which has a simpler structure and can be easily analysed, is essentially unique.

It follows from the construction that our order reduction method is uniquely characterized by the following properties: it only depends on the functorial equivalence class, it treats $\underline{\mathcal{I}}$ and $\underline{\mathcal{I}}^{\text{cln}}$ in the same way, and it starts and finishes with blowing up $\mathcal{W}(\underline{\mathcal{I}})$. The first property seems to be rather necessary for functorial

algorithms that use maximal contact to induct on dimension. The other two properties are not necessary, for example, one can first blow up $\mathcal{W}(\underline{\mathcal{I}})^{1/2}$ and then the pullback of $\mathcal{W}(\underline{\mathcal{I}})^{1/2}$, but it seems that avoiding them could only result in deteriorating the method by dealing with the monomial parts in a less efficient and superficial way.

invrem

Remark 7.3.7. Similarly, to [ATW17a, §2.11.4] one can assign to $\underline{\mathcal{I}}$ an invariant $\text{inv}_{\underline{\mathcal{I}}}: |X| \rightarrow \text{Inv}$, where Inv is the set of sequences (μ_0, \dots, μ_n) with $\mu_i \in \mathbb{Q}_{\geq 1}$ for $i < n$ and $\mu_n \in \mathbb{Q}_{\geq 1} \cup \{0, \infty\}$. Its definition follows loc.cit. without changes: if étale-locally over a point x , one denotes by $\underline{\mathcal{I}}_i$ the appropriate restriction onto the i -th maximal contact H_i , then $\mu_i(x) = \mu_x(\underline{\mathcal{I}}_i^{\text{cln}})$. In particular, $\text{inv}_{\underline{\mathcal{I}}}$ only depends on $[X, \underline{\mathcal{I}}]$.

7.4. Properties of the method.

7.4.1. *Functoriality.* Our order reduction method satisfies (ii) and (iii) because all constructions used in the process, including weighted invariants, Kummer blow ups, transforms, coefficient ideals, and maximal contacts, are compatible with log regular morphisms and base changes. This was proved/observed in Lemmas 2.8.14, 2.8.15, 4.2.9, 6.4.1, and 6.4.2.

7.4.2. *Existence.* It remains to establish claim (i) of Theorem 7.1.1. We, again, act by induction on $n = \log \dim(X/B)$. Assume first that $\mathcal{F}(\underline{\mathcal{I}})$ fails in step 1, that is, $\mathcal{M}_{X/B}(\underline{\mathcal{I}})$ is not monomial. By the monomialization Theorem 3.4.8 there exists a blow up $B' \rightarrow B$ with the base change $g: X' = X \times_B B' \rightarrow X$ such that $g^{-1}\mathcal{M}_{X/B}(\underline{\mathcal{I}})$ is monomial. Let f' denote the morphism $X' \rightarrow B'$ and let $\underline{\mathcal{I}}' = g^{-1}(\underline{\mathcal{I}})$. Then $g^{-1}\mathcal{M}_{X/B}(\underline{\mathcal{I}}) = \mathcal{M}_{X'/B'}(\underline{\mathcal{I}}')$ by Lemmas 2.8.14 and 2.8.5, and hence $\mathcal{F}(f', \underline{\mathcal{I}}')$ does not fail in step 1 and outputs a Kummer blow up $\sigma'_\infty: X'_\infty \rightarrow X'$.

Let us prove that after a blow up of B the algorithm also survives step 2. This will complete the proof since the algorithm blows up an invertible monomial ideal in step 3, and hence cannot fail there. Recall that step 2 of $\mathcal{F}(f', \underline{\mathcal{I}}')$ composes sequences σ'_{μ_i} with $0 \leq i \leq l-1$ and fails if one of those fails. For any $\mu \in \mathbb{Q}_{\geq 1}$ let $\mathcal{F}_{\geq \mu}(f', \underline{\mathcal{I}}')$ (resp. $\mathcal{F}_{> \mu}(f', \underline{\mathcal{I}}')$) be the compositions of σ'_∞ and σ'_{μ_i} with $\mu_i \geq \mu$ (resp. $\mu_i > \mu$). The length l is bounded by $\log \text{ord}(\mathcal{I}'_\infty)$, hence by decreasing induction on μ it suffices to prove that if $\mathcal{F}_{> \mu}(f', \underline{\mathcal{I}}')$ does not fail, then there exists a blow up $B'' \rightarrow B$ such that $\mathcal{F}_{\geq \mu}(f'', \underline{\mathcal{I}}'')$ does not fail too.

Let $\sigma'_{> \mu}: X'_\mu \dashrightarrow X'$ be the sequence $\mathcal{F}_{> \mu}(\underline{\mathcal{I}}')$. Then the clean part of $\underline{\mathcal{I}}'_\mu := (\sigma'_{> \mu})^c(\underline{\mathcal{I}}')$ is of weighted log order at most μ , and the sequence $\mathcal{F}_{> \mu}(\underline{\mathcal{I}}')$, if exists, is obtained by composing $\sigma'_{> \mu}$ with the order reduction of $(\underline{\mathcal{I}}'_\mu)^{\text{cln}}$. Moreover, the same is true for any blow up $B'' \rightarrow B$ dominating $B' \rightarrow B$, since all ingredients of \mathcal{F} , including $\mathcal{F}_{> \mu}$, are compatible with base changes.

The order reduction of $(\underline{\mathcal{I}}'_\mu)^{\text{cln}}$ was constructed by pushing forward the order reduction of a coefficient ideal from an étale-local maximal contact H_0 . By induction of relative log dimension, the latter order reduction does not fail after an appropriate blow up base change $B'' \rightarrow B'$. Increasing the blow up we can assume that $g': B'' \rightarrow B$ is also a blow up. Setting $\underline{\mathcal{I}}'' = g'^{-1}(\underline{\mathcal{I}}')$, $\sigma''_{> \mu} = \mathcal{F}_{> \mu}(\underline{\mathcal{I}}'')$ and $\underline{\mathcal{I}}''_\mu = (\sigma''_{> \mu})^c(\underline{\mathcal{I}}'')$, we have now achieved that the order reduction of $(\underline{\mathcal{I}}''_\mu)^{\text{cln}}$ does not fail, and hence $\mathcal{F}_{\geq \mu}(\underline{\mathcal{I}}'')$ does not fail too.

8. RELATIVE LOGARITHMIC DESINGULARIZATION

desingsec

This section is devoted to proving Theorem 1.2.8. As in the classical case, it is easy to give a local construction based on principalization and the main issue is to prove functoriality, including independence of the embedding. This will be easier than in [ATW17a] because we have developed the theory for general log regular morphisms, hence once an algorithm (depending on choices) is constructed all its properties can be checked formally-locally rather than étale locally. This is the stage where the qe assumption is used.

8.1. The local construction. As usually with Hironaka's approach, the idea is to stop just one step before blowing up the strict transform.

localdesingprop

Proposition 8.1.1. *Let $f: X \rightarrow B$ be a relative log orbifold and let $i: Z \hookrightarrow X$ be a strict closed immersion of constant codimension such that the morphism $g: Z \rightarrow B$ is generically log regular. Assume that the relative log principalization $\mathcal{F}(f, \mathcal{I}_Z)$ of the defining ideal $\mathcal{I}_Z \subseteq \mathcal{O}_X$ of Z is defined and denote it $\sigma: X_t \dashrightarrow X$. Then the generic points of Z are blown up at the same stage $\sigma_l: X_{l+1} \rightarrow X_l$, and the strict transform $Z_l \hookrightarrow X_l$ of Z is a union of connected components of the center of σ_l . In particular, $g_l: Z_l \rightarrow B$ is log regular and $Z_l \rightarrow Z$ is a relative log resolution of g that will be denoted $g_{\text{res}}: Z_{\text{res}} \rightarrow B$.*

Proof. First, the claim is étale local on X and B , hence we can assume that they are schemes. Recall that σ is the order reduction of $(\mathcal{I}_Z, 1)$ and let $(\mathcal{I}_i, 1)$ denote its controlled transform to X_i . By $Z_i \hookrightarrow X_i$ we denote the strict transform of Z .

Let z be a generic point of Z . Since g is log regular at z , its image is the generic point $b \in B$. Consider first how the algorithm behaves on the localization $Z_z = \text{Spec}(\mathcal{O}_z) \rightarrow B_b = \text{Spec}(\mathcal{O}_b)$. Let d be the codimension of Z in X . Since $z \rightarrow b$ is log regular, z is a suborbifold of Z_z of codimension d . Therefore the algorithm simply restricts d times to maximal contacts H_z^i , and blows up the d -th maximal contact $H_z^d = z$ at the initial cleaning step on H^d . In particular, the algorithm behaves similarly at all maximal points of Z , and they all are blown up at the same stage l . Moreover on this stage one works on a d -th maximal contact $H^d \hookrightarrow X_l$ and blows up H^d .

Each generic point of Z_l is a generic point of H^d , hence the reduction H of Z_l is the union of the connected components of H^d contained in Z_l . In particular, $\mathcal{I}_l \subseteq \mathcal{I}_{Z_l} \subseteq \mathcal{I}_H$. On the other hand, H^d is the iterated maximal contact to $(\mathcal{I}_l, 1)$, hence $\mathcal{I}_{H^d} \subseteq \mathcal{I}_l$. It follows that all inclusions become equalities when restricted onto $X_l \setminus (H^d \setminus H)$, and hence $H = Z_l$. ♣

Here are two complements concerning the proposition.

Remark 8.1.2. (i) The value of the invariant at step σ_l is $(1, \dots, 1, \infty)$ with d ones, see also [ATW17a, Proof of Theorem 1.2.4].

(ii) The assumption on codimension in the proposition is essential. The assumption on generic log regularity can be removed similarly to [ATW17a, §7.2.7]. We leave this to the interested reader.

8.2. Functoriality and independence of the embedding. Our next task is to prove that Z_{res} is functorial. The argument is formal local and we need some preparations.

8.2.1. *Minimal presentations.* Assume that $i: O \hookrightarrow A$ is an embedding of local log rings with a complete A . By a *log regular O -presentation* we mean a factorization $O \hookrightarrow C \twoheadrightarrow A$ such that C is a complete log ring, $O \hookrightarrow C$ is log regular, and $C \twoheadrightarrow A$ is strict. Factorizations correspond to strict closed immersions of $\mathrm{Spec}(A)$ into log regular over $\mathrm{Spec}(O)$ schemes $\mathrm{Spec}(C)$ with a complete local C .

reemblem

Lemma 8.2.2. *Let $i: O \hookrightarrow A$ be as above and let $x_1, \dots, x_n \in m_A$ be elements whose images form a basis of m_D/m_D^2 , where $D = A/m_O u^{m_Q} A$. Then any log regular O -presentation of A is of the form $O \hookrightarrow \widehat{O} \otimes_k l[[u^{Q/P}]][[t_1, \dots, t_m]] \xrightarrow{\phi} A$ with $m \geq n$, $\phi(t_i) = x_i$ for $i \leq n$ and $\phi(t_i) = 0$ for $i > n$. In particular, for any pair of log regular O -presentations of A one of them factors through the other one.*

21→ *Proof.*²¹

♣

liftpreslem

Lemma 8.2.3. *Let $C \twoheadrightarrow A$ be a strict surjective homomorphism of complete local rings. Then for any log regular homomorphism $A \rightarrow A'$ of complete local log rings can be lifted to a log regular homomorphism $C \rightarrow C'$ of complete local ring. Namely, the homomorphism $C \rightarrow A'$ factors as $C \rightarrow C' \twoheadrightarrow A'$ so that $C \rightarrow C'$ is log regular and $A' = A \otimes_C C'$.*

Proof. Let $l = A/m_A$ and $l' = A'/m_{A'}$, and let $Q = \overline{M}_A$ and $Q' = \overline{M}_{A'}$. Then $A' = A \otimes_l l'[[u^{Q'/Q}]][[t_1, \dots, t_m]]$, and one can take $C' = C \otimes_l l'[[u^{Q'/Q}]][[t_1, \dots, t_m]]$ with the natural morphism $C' \rightarrow A'$. ♣

8.2.4. *Functoriality.* We impose below a (very mild) assumption about formal fibres. It might be the case that it can be weakened or removed for log orbifolds, but we do not care.

indepprop

Proposition 8.2.5. *Assume that $f_j: X_j \rightarrow B$, $j = 1, 2$ are two log B -orbifolds and $i_j: Z_j \hookrightarrow X_j$ are strict closed immersions of constant codimensions such that $\mathcal{F}(f_j, \mathcal{I}_{Z_j})$ are defined and X_j are q_e . Then for any pair of log regular morphisms $Z \rightarrow Z_1$ and $Z \rightarrow Z_2$ with a common q_e source Z , the induced relative desingularizations of Z coincide. Namely, there is an isomorphism of Z -stacks $(Z_1)_{\mathrm{res}} \times_{Z_1} Z = (Z_2)_{\mathrm{res}} \times_{Z_2} Z$.*

Before proving the proposition let us record an immediate corollary.

indeppcor

Corollary 8.2.6. *The relative desingularization $Z_{\mathrm{res}} = Z_{\mathrm{res}}(i)$ defined in Proposition 8.1.1 depends only on the morphism $Z \rightarrow B$ and is independent of the embedding $i: Z \hookrightarrow X$. Moreover, if Z' is another log B -orbifold satisfying assumptions of the proposition and $Z' \rightarrow Z$ is a log regular B -morphism, then the desingularizations are compatible: $Z'_{\mathrm{res}} = Z_{\mathrm{res}} \times_Z Z'$.*

Proof of Proposition 8.2.5. By flat descent, it suffices to check the isomorphism of modifications after replacing Z by a flat covering. For example, we can replace Z by its étale covering, or we can replace X_1 by its étale covering X'_1 and replace Z_1 and Z by their base changes with respect to $X'_1 \rightarrow X$. In this way one easily reduces the claim to the case when B, Z_i, X_i and Z are schemes. Moreover, it suffices to

²¹(Michael) Add proof – should refer to section 3

prove that for any point $z \in Z$ both $\tau_j: (Z_j)_{\text{res}} \rightarrow Z_j$ are pulled back to the same modification of $\widehat{Z}_z = \text{Spec}(\widehat{\mathcal{O}}_{Z,z})$.

Let $z_j \in Z_j$ be the images of z . For shortness we denote $i_j(z_j)$ by z_j and set $\widehat{Z}_j = \text{Spec}(\widehat{\mathcal{O}}_{Z_j,z_j})$, $\widehat{X}_j = \text{Spec}(\widehat{\mathcal{O}}_{X_j,z_j})$. The morphism $\widehat{X}_j \rightarrow X$ is regular and $\widehat{i}_j: \widehat{Z}_j \hookrightarrow \widehat{X}_j$ is the base change of i , hence the principalization of \widehat{Z}_j in \widehat{X}_j is the pullback of the principalization of Z_j in X_j by the functoriality. In particular, the B -resolution of \widehat{Z}_j obtained from \widehat{i}_j is the base change of τ_j . This reduces the claim to the particular case when Z , Z_j and X_j are spectra of complete local rings. In addition, we can assume that $B = \text{Spec}(\mathcal{O}_b)$, where $b = g(z)$.

By Lemma 8.2.3, the embedding $\widehat{Z}_j \hookrightarrow \widehat{X}_j$ can be lifted to an embedding $\alpha_j: \widehat{Z}_z \hookrightarrow Y_j$, where Y_j is log regular over \widehat{X}_j and hence also over B . By functoriality of the principalization, the pullbacks of τ_j to \widehat{Z}_z coincide with the relative desingularizations of \widehat{Z}_z induced by the log regular B -presentations α_j . It remains to note that the latter coincide by Lemma 8.2.2 and the re-embedding principle from Theorem 1.2.3(iv). \clubsuit

desingmethodsec

8.3. The method. We say that a morphism $g: Z \rightarrow B$ is *locally embeddable* into a log orbifold (with abundance of derivations) if there exists an étale base change $u: B' \rightarrow B$ and an étale covering $v: Z' \rightarrow Z \times_B B'$ such that the morphism $g': Z' \rightarrow B'$ factors into the composition of a strict closed immersion $Z' \hookrightarrow X'$ and a relative log orbifold $Z' \rightarrow B'$ (with abundance of derivations). For any such g we have defined in Proposition 8.1.1 a resolution g'_{res} of g' , and showed in Corollary 8.2.6 that it is independent of the embedding $Z' \hookrightarrow X'$.

We claim that g'_{res} descends to a desingularization g_{res} of $Z \times_B B'$. To simplify the notation, assume that $B = B'$. The two pullbacks of g'_{res} to a desingularization of $Z'' = Z' \times_Z Z'$ coincide by Proposition 8.2.5, and hence g'_{res} descends to g_{res} . Descent with respect to a base change $B' \rightarrow B$ is done similarly. Finally, functoriality of the desingularization follows from Corollary 8.2.6.

9. EXTENSION OF MAIN THEOREMS TO OTHER CATEGORIES

In this section we will use functoriality to extend Theorems 1.2.3, 1.2.8 and 1.2.10 to other settings.

fpsec

9.1. Morphisms of finite presentation. Our first goal is to desingularize schemes over general valuation rings, but our argument applies to general non-noetherian bases as well.

9.1.1. Approximation. We briefly recall some facts, the main reference is [Gro67, IV₃, §8]. By [TT90, C.9] any quasi-compact quasi-separated (shortly qcqs) scheme B is a projective filtered limit of a family $\{B_i, f_{ij}\}$ with B_i of finite type over \mathbb{Z} and f_{ij} affine. If B is integral of characteristic zero, we can take all B_i to be integral of characteristic zero. Furthermore, any B -scheme X of finite presentation is the base change of a B_i -scheme X_i of finite type for a large enough i , and any two choices of such an X_i become isomorphic already after base change to some B_j . A similar theory exists for morphisms, coherent sheaves, etc.

The above theory easily extends to log schemes: a quasi-coherent log structure $M \rightarrow \mathcal{O}_B$ is a direct colimit of its fine log substructures $M_\alpha \hookrightarrow M$, and each M_j is

obtained by pullback of a fine log structure $M_{i\alpha}$ on some X_i , hence (X, M_α) is the filtered limit of fine log scheme $(X_j, f_{ij}^* M_{i\alpha})$, and varying α we obtain that (X, M) is the limit of fine log scheme of finite type over \mathbb{Z} . Moreover, if M is saturated we can take these log schemes to be saturated. In the same way one approximates morphisms of log schemes, etc.

9.1.2. *Proof of Theorem 1.2.13.* In the noetherian setting, log smooth morphisms possess abundance of derivations by Remark 2.5.13(ii), and any morphism of finite type is locally embeddable into a log smooth one. In particular, the functors $\mathcal{F}, \mathcal{R}, \mathcal{R}'$ from Theorems 1.2.3, 1.2.8 and 1.2.10 are defined for morphisms of finite type and satisfy the existence property (i). Therefore the approximation theory implies that they extend to morphisms of finite presentation between qcqs schemes, and then functoriality properties (ii) and (iii) imply that they extend to morphisms of locally finite presentation between log DM stacks. In particular, we even drop the quasi-compactness restriction on f .

9.2. Analytic spaces and formal schemes.

APPENDIX A.

regularapp

A.1. Regular morphisms.

A.1.1. *The definition.* Recall that a morphism $f: Y \rightarrow Z$ of noetherian schemes is *regular* if it is flat and has geometrically regular fibers. This notion is smooth-local and flat-local on the base, in particular, it extends to morphisms between algebraic stacks. If f is of finite type, then f is regular if and only if it is smooth. Thus, regularity is a natural extension of smoothness to arbitrary morphisms. In fact, a famous theorem of Popescu states that any regular morphism is a filtered limit of smooth ones.

A.1.2. *Parameters.* Assume that $f: Y \rightarrow Z$ is a regular morphism of schemes, $y \in Y$ is a point, and $S = Y \times_Z \text{Spec}(k(z))$ is the fiber over $z = f(y)$. If the extension $k(y)/k(z)$ is separable then we say that y is a *simple regular* point of f . If Y is a scheme but Z is a stack, we say that y is *simple* if for any base change $f': Y \times'_Z Z \rightarrow Z'$ with Z' a scheme any preimage of y is a simple point of f' . By a *family of regular parameters* at a simple point y we mean any family $t_1, \dots, t_n \in \mathcal{O}_y$ whose image is a family of regular parameters of the regular ring $\mathcal{O}_{S,y}$. Note that the images of dt_1, \dots, dt_n in $\Omega_{Y/Z,y} \otimes k(y)$ are linearly independent thanks to the assumption that y is a simple point.

regularlem

Lemma A.1.3. *Assume that $f: Y \rightarrow Z$ is a morphism with Y a scheme and Z a stack, $y \in Y$ a simple regular point of f , and t_1, \dots, t_l global functions on Y that form a subfamily of a regular family of parameters at y . Then,*

- (i) *The morphism $Y \rightarrow Z \times \mathbb{A}^l$ induced by f and t_1, \dots, t_l is regular at y .*
- (ii) *The closed subscheme $X \hookrightarrow Y$ given by the vanishing of t_1, \dots, t_l is regular over Z at x .*

The lemma is very easy in the classical case of a smooth f , and the general case immediately follows by Popescu's theorem. Since this is a little bit overkill, we provide another argument which is still far from being elementary. It seems

probable that any proof should involve a non-trivial technical part, similarly to the situation with Serre’s theorem that regularity is preserved by localizations.

Proof. The claim is flat-local on the base and local at y , hence we can assume that the schemes are local with closed points y and $z = f(y)$. Set $A = \mathcal{O}_z$ and $C = \mathcal{O}_y$. It suffices to prove that the homomorphism $\phi: A' = A[x_1, \dots, x_l] \rightarrow C$ taking x_i to t_i is regular. Indeed, this covers (i), and (ii) will follow since tensoring ϕ with $A'/(x_1, \dots, x_n)$ one obtains that $A \rightarrow C' = C/(t_1, \dots, t_n)$ is regular too.

Recall that by a theorem of André-Quillen a homomorphism of local noetherian rings $R \rightarrow S$ is regular if and only if the cotangent complex $\mathbb{L}_{S/R}$ has only zero-dimensional homology, which is a flat S -module. In the exact transitivity triangle $\mathbb{L}_{A'/A} \otimes_{A'} C \rightarrow \mathbb{L}_{C/A} \rightarrow \mathbb{L}_{C/A'}$ the first two complexes correspond to regular homomorphisms, hence up to a quasi-isomorphism the triangle reduces to the first fundamental sequence $0 \rightarrow \Omega_{A'/A} \otimes_{A'} C \rightarrow \Omega_{C/A} \rightarrow \Omega_{C/A'} \rightarrow 0$ and we should only prove that $\Omega_{C/A'}$ is flat. In fact, one even has that the sequence splits because $\Omega_{A'/A}$ is a free module with basis dx_1, \dots, dx_n and its image dt_1, \dots, dt_l in $\Omega_{C/A}$ remains linearly independent modulo m_C by our assumption on t_1, \dots, t_l . \clubsuit

Remark A.1.4. If y is not simple, then the images of dt_i in $\Omega_{Y/Z, y} \otimes k(y)$ may vanish, and both assertions of the lemma fail. This happens already for non-simple points on $Y = \mathbb{A}_k^1$ over $Z = \text{Spec}(k)$ for a non-perfect field k .

A.2. Special charts. Instead of working with the morphisms $Y \rightarrow \mathbf{Log}(Z)$ or $Y \rightarrow \mathbf{Log}(Z)_Y$ we will consider charts with as few units as possible. Then the orbits $O_x \subset Z[u^{Q/P}]$ are small and the morphism $Y \rightarrow Z[u^{Q/P}]$ can serve as an adequate replacement.

A.2.1. Good news and bad news. On the positive side, it is well known that étale-locally at a point z any fs log scheme Z possesses a chart $Z \rightarrow \mathbf{A}_P$, with $P = \overline{M}_z$ minimal possible (see Corollary A.2.6(i) below). However, in the case of general (non-integral) morphisms one has to consider non-sharp monoids, as can be seen from a simple example with blowing up at the origin z of the plane $Z = \text{Spec}(k[u, v])$ with the log structure given by u, v :

specialexam

Example A.2.2. We have that $P = \overline{M}_z = \mathbb{N}^2$ with basis $u = (1, 0)$ and $v = (0, 1)$. The monoid of the generic point of the preimage of z is $Q = \mathbb{N} \cdot (1, 0) \oplus \mathbb{Z} \cdot (1, -1)$. Thus $\overline{Q} = \mathbb{N}$ and the sharpening $\overline{P} \rightarrow \overline{Q}$ is not injective, but its kernel $Q^\times = \mathbb{Z} \cdot (1, -1)$ has trivial intersection with P .

A.2.3. Special homomorphisms. Thus, in order to construct minimal charts of morphisms one might only hope to restrict the units of Q to those that come from P^{gp} , and this leads to the following definition: a homomorphism $\phi: P \rightarrow Q$ of monoids will be called *special* if P is sharp and Q^\times is torsion free and finite over $Q^\times \cap \phi^{\text{gp}}(P^{\text{gp}})$.

speciallem

Lemma A.2.4. (i) If P is an fs monoid then there exists a (non-canonical) splitting $P = \overline{P} \oplus P^\times$.

(ii) If $\phi: P \rightarrow Q$ is an embedding of fs monoids and P is sharp, then there exist a Kummer embedding $Q \hookrightarrow R$ and a splitting $R = R' \oplus L$ such that $\overline{Q} = \overline{R} = \overline{R}'$ and $\phi(P) \subseteq R'$ giving rise to a special homomorphism $P \rightarrow R'$.

Note that in (ii) automatically L is a group and $R = Q \oplus_{Q^\times} R^\times$ is obtained by a finite extension of units of Q .

Proof. (i) Since P is fs, it is easy to see that P^\times splits off P^{gp} , say $P^{\text{gp}} = P^\times \oplus L$. It follows that the projection of P onto L is isomorphic to \bar{P} giving rise to the desired splitting.

(ii) Note that P^{gp} is torsion free and consider it as a sublattice of Q^{gp} . Then there exists a splitting $Q^{\text{gp}} = T \oplus L$, where T is torsion and L is torsion free and contains P . Hence $Q = T \oplus Q'$, where Q' is the projection onto L , and replacing Q by Q' we can assume that Q^{gp} is a lattice.

Let S be the saturation of P^{gp} in Q^{gp} . Then $Q_P^\times = Q^\times \cap S$ is saturated in Q^\times , hence there is a splitting $Q^\times = Q_P^\times \oplus L_0$. Since $L_0 \cap S = 0$, there exists a saturated lattice S' in Q^{gp} such that $L_0 \cap S' = 0$ and $S' \oplus L_0$ is of finite index in Q^{gp} . It follows easily that there exists an embedding of finite index $L_0 \hookrightarrow L$ such that $Q^{\text{gp}} \oplus_{L_0} L$ splits as $S' \oplus L$. Take $R = Q \oplus_{L_0} L$ and note that the projection R' of R onto S' yields a splitting $R = R' \oplus L$, and the homomorphism $P \rightarrow R'$ is special. ♣

22→ A.2.5. *Special charts.* A chart $Z \rightarrow \mathbf{A}_P, Y \rightarrow \mathbf{A}_Q$ of a morphism of log schemes $Y \rightarrow Z$ will be called *special at* $y \in Y$ ²² if $P = \bar{M}_{f(y)}, \bar{Q} = \bar{M}_y$ and $P \rightarrow Q$ is a special homomorphism of monoids.

specialcor

Corollary A.2.6. *Let $f: Y \rightarrow Z$ be a morphism of fs log schemes, $y \in Y$ a geometric point, $z = f(y)$, and $P = \bar{M}_z$. Then,*

(i) *Étale-locally at z there exists a chart $Z \rightarrow \mathbf{A}_P$.*

(ii) *Assume that the characteristic is zero. Then étale-locally at y any chart $Z \rightarrow \mathbf{A}_P$ can be extended to a special chart $Y \rightarrow \mathbf{A}_Q \rightarrow \mathbf{A}_P$ of f .*

Proof. (i) Start with any chart $Z \rightarrow \mathbf{A}_M$ at z . Since $\bar{M} = P$, the splitting $M = \bar{M} \oplus M^\times$ induces a sharp chart $Z \rightarrow \mathbf{A}_M \rightarrow \mathbf{A}_P$.

(ii) Lift a chart from (i) to any chart $Y \rightarrow \mathbf{A}_M \rightarrow \mathbf{A}_P$ at y . By Lemma A.2.4(ii), there exists a Kummer extension $M \hookrightarrow Q' = Q \oplus L$ such that $\bar{M} = \bar{Q}$. Note that Q' is obtained from M by a finite extension of units and, by our assumption on characteristic, we can extract in \mathcal{O}_y arbitrary roots from elements of $u^{P^\times} \subset \mathcal{O}_y^\times$. Therefore étale-locally we can lift the chart to a chart $Y \rightarrow \mathbf{A}_{Q'}$, and the latter induces a special chart $Y \rightarrow \mathbf{A}_Q$ of f at y . ♣

A.2.7. *Criterion of log regularity.* It turns out that any special chart can be used to describe log fibers and detect log regularity of $f: Y \rightarrow Z$.

logfiblem

Lemma A.2.8. *Assume that the characteristic is zero, $Y \rightarrow \mathbf{A}_Q, Z \rightarrow \mathbf{A}_P$ is a global chart of f which is special at a point $y \in Y$. Then*

(i) *Locally at y the log fiber S_y coincides with the fiber of $h: Y \rightarrow Z[u^{Q/P}]$ through y .*

(ii) *f is log regular at y if and only if h is regular at y .*

²²(Michael) An alternative is to call it almost minimal or something like that.

Proof. The first claim follows from the observation that the image of Q^\times in $Q^{\text{gp}}/P^{\text{gp}}$ is finite. Therefore the orbit of $T_{Q^{\text{gp}}/P^{\text{gp}}}$ through $x = h(y)$ is a finite group, which is étale by the assumption on the characteristic.²³

←23



Remark A.2.9. Lemma A.2.8 fails in positive characteristic p . For example, if $Q^{\text{gp}}/P^{\text{gp}}$ is p -torsion then the fibers of $Z[u^{Q/P}] \rightarrow \mathbf{Log}(Z)$ may be non-reduced, resulting in different non-reduced structure on the log fibers and fibers of h .

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²³(Michael) Complete later. Since this is really technical, can use $\mathbf{Log}(Y)$ and anything else.

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