

EQUISINGULAR RESOLUTION WITH SNC FIBERS AND COMBINATORIAL TYPE OF VARIETIES

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ABSTRACT. We introduce the notion of combinatorial type of varieties X which generalizes the concept of the dual complex of SNC divisors. It is a unique, up to homotopy, finite simplicial complex $\Sigma(X)$ which is functorial with respect to morphisms of varieties. Its cohomology $H^i(\Sigma(X), Q)$ for complex projective varieties coincide with weight zero part of the Deligne filtration $W_0(H^i(X, Q))$. The notion can be understood as a topological measure of the singularities of algebraic schemes of finite type.

We also prove that any variety in characteristic zero admits the Hironaka desingularization with all fibers having SNC. Moreover the dual complexes of the fibers are isomorphic on strata. Also for any morphism $f : X \rightarrow Y$ there exists a similar desingularization $\tilde{X} \rightarrow X$ for which the induce morphism $\tilde{X} \rightarrow Y$ has SNC fibers.

One of the consequence is that for any projective morphism $f : X \rightarrow Y$ the combinatorial type of the fiber is a constructible function. In particular $\dim(W_0 H^i(f^{-1}(y)))$ is constructible.

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0. INTRODUCTION

One of the purpose of the paper is to extend the Hironaka desingularization theorems in order to control the fibers of resolutions and more generally morphisms between varieties. In particular if D is a SNC divisor associated to a resolution of an isolated singularity then it was observed independently by Kontsevich, Soibelman [KS, A.4], Stepanov [Stp1], and others that the homotopy type of the dual complex associated with D is an invariant for the singularity ([KS]). More generally Thuillier [T] and Payne [P] proved homotopy invariance results for boundary divisors. In characteristic zero, all these results can be derived from the weak factorization theorem of Włodarczyk [Wlo], and Abramovich-Karu-Matsuki-Włodarczyk [AKMW]. A refinement of factorization theorem gives a more general version of the homotopy invariance for varieties with SNC, and allows to study lower dimensional fibers of morphisms. In particular, in the paper [ABW] we show that the invariants of the dual complexes of the fibers of the resolution can be defined for arbitrary varieties. Moreover they can be computed directly when using fibers with SNC crossings of smaller dimension. This raises a question of existence of resolution with SNC fibers. In this paper we show that such a resolution can be constructed. Moreover one can associate with it a stratification, with points in the strata defining isomorphic dual complexes. In this sense it can be considered as a version of equisingular resolution. On the other hand if one considers an arbitrary morphism $f : X \rightarrow Y$ the method allows to desingularize the variety X and the fibers of the morphism transforming them into SNC varieties. Since, in general, we cannot

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eliminate SNC singularities of the fibers the theorem can be considered as the desingularization of the fibers of the morphism. The proofs rely on Hironaka desingularization theorems and, introduced here, notion of SNC morphism generalizing smooth morphisms.

The considerations of the dual complexes associated with SNC fibers of desingularizations naturally lead to the idea of extension of the concept of the dual complex and placing it in a more general context. We introduce here the notion of *combinatorial type* which allows to approach the theory of singularities from the topological perspective. The combinatorial type $\Sigma(X)$ is a roughly a homotopy type of topological space (simplicial complex) which is assigned to an algebraic scheme X and which reflects the singularities of the scheme. Moreover the notion, in fact defines, a functor from the category of (quasiprojective) algebraic schemes of finite type to the homotopy category of topological spaces. In the particular case of SNC divisors the combinatorial type is the homotopy type of its dual complex. On the other hand, in the case of complex projective varieties, the rational cohomology $H^i(\Sigma(X), \mathbb{Q})$ of the combinatorial type coincides with weight zero part of the Deligne filtration $W_0 H^i(X, \mathbb{Q})$.

The paper is organized as follows. In the first section we briefly formulate the Hironaka desingularization theorems in their stronger form which are used frequently in the remaining part of the paper. In Section 2 we briefly discuss the language of varieties with SNC and associated simplicial complexes. One of the main tool used in the considerations is, introduced here, operation of cone extension. It generalizes the operation of star subdivision and can be conveniently used for natural modification of dual complexes associated with (embedded) blow-ups of SNC varieties at compatible SNC centers. (One should mention that the case of varieties with SNC differs quite significantly from the case of SNC divisors.) In Section 3 we introduce the notion of combinatorial type of varieties and associated morphisms between them, and study their basic properties. In Section 4 we introduce the notion of SNC morphism and prove the analog of generic smoothness theorem for morphisms of SNC varieties. In section 5 we study étale trivialization of SNC morphisms. In section 6 we prove the desingularization theorems with SNC fibers, and some of their consequences. In particular we show that the combinatorial type of the fibers of a projective morphism is constant on strata. We note that in this paper the term “scheme” refers to a scheme of finite type over a ground field K of characteristic zero. A variety is a reduced scheme.

1. FORMULATION OF THE HIRONAKA RESOLUTION THEOREMS

In the paper we are going to use and generalize the following Hironaka desingularization theorems in the strongest form. (see [H],[BM2],[Vi],[Wlo3]). Observe that in the actual Hironaka algorithm we have no control on the fibers of the morphisms and it seems rather difficult to alter the steps of the algorithm to ensure the SNC condition on the fibers. Throughout the paper we shall work over a ground field K of characteristic zero.

(1) Strong Hironaka Resolution of Singularities

Theorem 1.0.1. *Let Y be an algebraic variety over a field of characteristic zero.*

There exists a canonical desingularization of Y that is a smooth variety \tilde{Y} together with a projective birational morphism $\text{res}_Y : \tilde{Y} \rightarrow Y$ such that

- (a) *res_Y is a composition of blow ups $Y = Y_0 \leftarrow Y_1 \leftarrow \dots \leftarrow \tilde{Y}$ with smooth centers disjoint from the set of regular points $\text{Reg}(Y)$ of Y .*
- (b) *res_Y is functorial with respect to smooth morphisms. For any smooth morphism $\phi : Y' \rightarrow Y$ there is a natural lifting $\tilde{\phi} : \tilde{Y}' \rightarrow \tilde{Y}$ which is a smooth morphism.*

Moreover the centers of blow ups are defined by the liftings of the centers of blow-ups

- (c) *$\text{res}_Y : \tilde{Y} \rightarrow Y$ is an isomorphism over the nonsingular part of Y . Moreover res_Y is equivariant with respect to any group action not necessarily preserving the ground field.*

(2) Strong Hironaka Embedded Desingularization

Theorem 1.0.2. *Let Y be a subvariety of a smooth variety X over a field of characteristic zero. There exists a sequence*

$$X_0 = X \xleftarrow{\sigma_1} X_1 \xleftarrow{\sigma_2} X_2 \leftarrow \dots \leftarrow X_i \leftarrow \dots \leftarrow X_r = \tilde{X}$$

of blow-ups $\sigma_i : X_{i-1} \leftarrow X_i$ of smooth centers $C_{i-1} \subset X_{i-1}$ such that

- (a) *The exceptional divisor E_i of the induced morphism $\sigma^i = \sigma_1 \circ \dots \circ \sigma_i : X_i \rightarrow X$ has only simple normal crossings and C_i has simple normal crossings with E_i .*

- (b) Let $Y_i \subset X_i$ be the strict transform of Y . All centers C_i are contained in Y_i , and are disjoint from the regular point set $\text{Reg}(Y) \subset Y_i$ of points where Y_i is smooth.
- (c) The strict transform $\tilde{Y} := Y_r$ of Y is smooth and has only simple normal crossings with the exceptional divisor E_r .
- (d) The morphism $(X, Y) \leftarrow (\tilde{X}, \tilde{Y})$ defined by the embedded desingularization commutes with smooth morphisms and embeddings of ambient varieties. It is equivariant with respect to any group action not necessarily preserving the ground K .

(3) Canonical Principalization

Theorem 1.0.3. Let \mathcal{I} be a sheaf of ideals on a smooth algebraic variety X , and $Y \subset X$ be any subvariety of X . There exists a principalization of \mathcal{I} that is, a sequence

$$X = X_0 \xleftarrow{\sigma_1} X_1 \xleftarrow{\sigma_2} X_2 \leftarrow \dots \leftarrow X_i \leftarrow \dots \leftarrow X_r = \tilde{X}$$

of blow-ups $\sigma_i : X_{i-1} \leftarrow X_i$ of smooth centers $C_{i-1} \subset X_{i-1}$ such that

- (a) The exceptional divisor E_i of the induced morphism $\sigma^i = \sigma_1 \circ \dots \circ \sigma_i : X_i \rightarrow X$ has only simple normal crossings and C_i has simple normal crossings with E_i .
- (b) The total transform $\sigma^{r*}(\mathcal{I})$ is the ideal of a simple normal crossing divisor \tilde{E} which is a natural combination of the irreducible components of the divisor E_r .

The morphism $(\tilde{X}, \tilde{\mathcal{I}}) \rightarrow (X, \mathcal{I})$ defined by the above principalization commutes with smooth morphisms and embeddings of ambient varieties. It is equivariant with respect to any group action not necessarily preserving the ground field K .

(4) Extended Hironaka Embedded Desingularization

In the constructions used in this paper it will be convenient to use the following modification of the strong Hironaka Embedded desingularization.

Definition 1.0.4. Let Y be a subvariety of a smooth variety X over a field of characteristic zero.

By the *extended Hironaka desingularization* we shall mean a sequence of blow-ups as in Hironaka embedded desingularization followed by the blow-up $\bar{X} \rightarrow \tilde{X}$ of the smooth center of the strict transform $C = \tilde{Y}$ on \tilde{X} .

Observe that extended Hironaka desingularization creates SNC divisor D on \bar{X} . It is functorial, commutes with embeddings of ambient varieties. Moreover all the centers of the blow-ups are contained in the strict transforms of Y , and in particular are of dimension $\leq \dim(Y)$.

2. DUAL COMPLEXES OF SNC VARIETIES AND CONE EXTENSIONS

Definition 2.0.1. By an *abstract polyhedral complex* Σ we mean a finite partially ordered \leq collection of polytopes σ together with inclusion maps $i_{\tau\sigma} : \tau \rightarrow \sigma$ for any $\tau \leq \sigma$ that

- (1) For any $\tau \leq \sigma$, the subset $i_{\tau\sigma}(\tau)$ is a face of σ
- (2) For any face σ' of a polytope $\sigma \in \Sigma$, there exists a $\tau \in \Sigma$ such that $i_{\tau\sigma}(\tau) = \sigma'$.
- (3) For any $\tau, \tau' \leq \sigma$ such that $i_{\tau\sigma}(\tau) = i_{\tau'\sigma}(\tau')$ we have that $\tau = \tau'$.

If all faces are simplices then the complex will be called *simplicial*. A subset Σ' of a polyhedral complex which is a polyhedral complex itself will be called a subcomplex. If $\tau \leq \sigma$ then we shall also call τ a *face* of σ slightly abusing terminology.

By the *geometric realization* $|\Sigma|$ of Σ we mean the topological space

$$\coprod_{\sigma \in \Sigma} \sigma / \sim,$$

when \sim is the equivalence relation generated by the inclusion maps $i_{\tau\sigma}$

Note that we do not require that polytopes intersect along faces but rather subcomplexes.

The faces of polytopes σ define unique elements of Σ , and thus $|\Sigma|$ is obtained by gluing maximal polytopes along their faces.

Definition 2.0.2. The map of polyhedral complexes $\phi : \Sigma \rightarrow \Sigma'$ is a continuous map of the geometric realizations $|\phi| : |\Sigma| \rightarrow |\Sigma'|$, and the compatible affine maps of the faces $\phi_\sigma : \sigma \rightarrow \sigma'$, where σ' is the smallest face $\sigma' \in \Sigma'$ such that $|\phi|(\sigma) \subset \sigma'$, and the following diagram commutes:

$$\begin{array}{ccc} \sigma & \xrightarrow{\phi_\sigma} & \sigma' \\ \cap & & \cap \\ |\Sigma| & \xrightarrow{|\phi|} & |\Sigma'| \end{array}$$

If $|\phi|$ is a homeomorphism then we shall call Σ a *subdivision* of Σ' . If, moreover Σ is simplicial then it will be called *triangulation* of Σ' .

In the paper we shall consider mostly simplicial complexes referring to them as *complexes*. The polyhedral complexes occur only when introducing cartesian products.

Definition 2.0.3. A *variety with simple normal crossings (SNC)* is a reduced scheme X of finite type with all maximal components X_i smooth over K , and having SNC crossings. That is for any $p \in X$ there is a neighborhood U of p in X , and an étale morphism of $U \rightarrow Z$, where $Z \subset \mathbf{A}^n$ is a union of coordinate subspaces of possibly different dimension. If X is an SNC variety on a smooth ambient variety Z , and C is a smooth subvariety then we say that C has SNC with X if for any $p \in X$ there is a neighborhood U of p in Z , and an étale morphism of $\phi : U \rightarrow \mathbf{A}^n$, where C is the preimage of a coordinate subspace and X is the preimage of a union of coordinate subspaces of possibly different dimension.

Recall that given an SNC divisor or more generally a variety with SNC $D = \bigcup D_i$ with maximal components D_i one can associate with it the dual complex. More precisely we define the dual complex $\Delta(D)$ to be the simplicial complex whose vertices correspond to the maximal components D_i , and simplices $\Delta_{j_0 \dots j_p}^s$ or shortly Δ_α , where $\alpha := (j_0 \dots j_p : s)$ are in correspondence to the irreducible D_α components of $(p+1)$ -fold nonempty intersections $D_{j_0} \cap \dots \cap D_{j_p}$. Note that the components D_α need not to be distinct for different α .

The dual complexes are often constructed from SNC divisors which are obtained by successive blow-ups of smooth centers having SNC with the exceptional divisors. It is well known fact that when blowing up a center which is an intersection component of the divisor we form a new complex which is the star subdivision at the simplex corresponding to the component. In general one needs to extend the notion of star subdivision to the, introduced here more universal transformation of *cone extension* (Definition 2.1.3, Proposition 2.1.6).

For any two faces σ and τ in a complex Σ we write $\sigma < \tau$ if σ is a proper face of τ .

For any complex Σ and its subset $\Delta \subset \Sigma$ denote by *closure* $\overline{\Delta} = \{\tau \mid \tau \leq \sigma, \sigma \in \Delta\}$.

Definition 2.0.4. Let Σ be a complex and $\tau \in \Sigma$ be its face. The *star* of the cone τ , the *closed star*, and the *link* of τ are defined as follows:

$$\begin{aligned} \text{Star}(\tau, \Sigma) &:= \{\sigma \in \Sigma \mid \tau \leq \sigma\}, \\ \overline{\text{Star}}(\tau, \Sigma) &:= \{\sigma \in \Sigma \mid \sigma' \leq \sigma \text{ for some } \sigma' \in \text{Star}(\tau, \Sigma)\}, \\ \text{Link}(\tau, \Sigma) &:= \overline{\text{Star}}(\tau, \Sigma) \setminus \text{Star}(\tau, \Sigma) \end{aligned}$$

We say that a subset Δ of a complex Σ is *star-closed* iff whenever $\tau \leq \sigma$ and $\tau \in \Delta$, and $\sigma \in \Sigma$ we have that $\sigma \in \Delta$.

By the *support* of a subset Δ of a complex Σ we mean the union of the relative interior of all its faces, $|\Delta| = \bigcup_{\tau \in \Delta} \text{int}(\tau) \subset |\Sigma|$. Observe that the support of the complex Σ coincides with its geometric realization $|\Sigma|$. That is why we use the same notation for both notions. In general $|\Delta|$ is a locally closed subset of $|\Sigma|$.

The definition of the support gives a convenient description of topology on $|\Sigma|$.

Lemma 2.1. *Let Δ be a subset of Σ . Then*

- (1) *The subset Δ is star closed in Σ iff its complement $\Sigma \setminus \Delta$ is a subcomplex of Σ .*
- (2) *The subset Δ is star closed in Σ iff $|\Delta|$ is open in $|\Sigma|$*
- (3) *Δ is a subcomplex of Σ iff $|\Delta|$ is closed in $|\Sigma|$.*
- (4) *$|\overline{\Delta}|$ is the closure of $|\Delta|$.*

Remark. Note that the components D_α may correspond to different simplices. For any such a component $C = D_\alpha$ there exists a unique maximal face σ_C corresponding to all the divisors D_i containing C . Then all other simplices τ with the property $D_\tau = C$ correspond to certain faces of σ_C .

Remark. If $\tau \leq \tau'$ then $D_{\tau'} \subset D_\tau$. On the other hand if $D_{\tau'} \subset D_\tau$ then $D_{\tau'} \cap D_\tau \subset D_\tau$ which means that there exists a maximal simplex τ'_{max} such that $D_{\tau'} = D_{\tau'_{max}}$ and for which $\tau' \leq \tau'_{max}$, and $\tau \leq \tau'_{max}$. Thus for $\tau' = \tau'_{max}$, we get that $D_\tau \supseteq D_{\tau'_{max}}$ iff $\tau \leq \tau'_{max}$.

Definition 2.1.1. By the *interior* of a subset Δ in the complex Σ we mean the maximal star closed subset $\text{int}(\Delta)$ of Δ

If v is independent of cones $\sigma \in \Delta$ then by $v * \Delta(D)$ we shall denote the complex consisting of the cones $\text{conv}(v, \sigma)$ over $\sigma \in \Delta$ with vertex v . We also put $v * \emptyset = \{v\}$.

Definition 2.1.2. Let Σ be a complex and v be a point in the relative interior of $\tau \in \Sigma$. Then the *star subdivision* of Σ with respect to τ is defined to be

$$v \cdot \Sigma = (\Sigma \setminus \text{Star}(\tau, \Sigma)) \cup v * (\overline{\text{Star}(\tau, \Sigma)} \setminus \text{Star}(\tau, \Sigma)).$$

Note that in the definition v can be chosen as any vertex independent of the faces in the $\text{Link}(\tau, \Sigma)$. The assumption that v is in the relative interior of τ is to visualize the operation and can be dropped.

Definition 2.1.3. Given a complex Σ , and its subset Δ^0 which is star closed and contained in the subcomplex Δ of Σ we define a *cone extension* of Σ at (Δ, Δ^0) to be the complex

$$(\Delta, \Delta^0) \cdot \Sigma := \Sigma \setminus \Delta^0 \cup v * (\Delta \setminus \Delta^0),$$

where v is any new vertex which is independent from faces in $\Delta \setminus \Delta^0$. By the *pure cone extension* of Σ we mean the complex $(\Delta, \emptyset) \cdot \Sigma$, where Δ is a subcomplex of Σ

Note that $\Delta^1 := \Delta \setminus \Delta^0$ is necessarily a subcomplex of Σ and Δ . On the other hand $\Delta^0 \subset \text{int}(\Delta)$.

Example 2.1.4. Let $\tau \in \Sigma$ be arbitrary face. Consider the subsets

$$\Delta^0 = \text{Star}(\sigma, \Sigma), \quad \Delta = \overline{\text{Star}(\sigma, \Sigma)}.$$

In that case

$$\Delta^1 = \overline{\text{Star}(\sigma, \Sigma)} \setminus \text{Star}(\sigma, \Sigma) = \text{Link}(\sigma, \Sigma),$$

and the cone extension of Σ at $(\Delta, \Delta^0) = (\overline{\text{Star}(\sigma, \Sigma)}, \text{Star}(\sigma, \Sigma))$ is the star subdivision of Σ at σ .

Example 2.1.5. The cone extension of Σ at (\emptyset, \emptyset) is the disjoint union $\Sigma \cup \{v\}$, and thus the operation changes the homotopy type of $|\Sigma|$.

For any variety D with SNC denote by D_τ , where $\tau \in \Delta(D)$, the stratum which is the intersection of divisors corresponding to vertices of τ , and by v_C

Proposition 2.1.6. Let C be a center having SNC with a variety with SNC D on a smooth variety X . Set

$$\Delta_C^0 := \{\tau \in \Delta(D) \mid D_\tau \subset C\}$$

$$\Delta_C := \{\tau \in \Delta(D) \mid D_\tau \cap C \neq \emptyset\}$$

Then Δ_C^0 is star closed, and the new complex $\Delta(D')$ arising from D after blow-up at C is given by the cone extension of $\Delta(D)$ at (Δ_C, Δ_C^0) .

$$\Delta(D') = (\Delta_C, \Delta_C^0) \cdot \Delta(D)$$

Moreover

- (1) The new vertex $v = v_C$ corresponds to the exceptional divisor E .
- (2) The components which are eliminated after blow-up are those corresponding to Δ_C^0 .
- (3) The components created in the blow-up correspond to faces in $(v_C * \Delta_C^1) \setminus \Delta_C^1$.
- (4) If the center C is the intersection component and corresponds to the face σ_C then

$$\text{Star}(\sigma_C, \Delta(D)) \subseteq \Delta_C^0 \subseteq \Delta_C = \overline{\text{Star}(\sigma_C, \Delta(D))}.$$

In this case $|\Delta(D)|$ and $|\Delta(D')|$ are homotopy equivalent.

(5) If the center C is properly contained in the smallest intersection component containing it then

$$\Delta_C \subseteq \overline{\text{Star}(\sigma_C, \Delta(D))}$$

with the maximal simplices of Δ_C in $\text{Star}(\sigma_C, \Delta(D))$. In this case Δ_C is contractible and $|\Delta(D')|$ retracts homotopically to $|\Delta(D)|$.

Proof. (1), (2), and (3) follow from the definition and properties of blow-ups.

(4) Assume that $C = D_{\sigma_C}$ be the minimal component describing C . Then $\tau \in \Delta_C$ then $D_\tau \cap C \neq \emptyset$, and $D_\tau \cap D_{\sigma_C} \neq \emptyset$. In other words $\text{Star}(\tau, \Delta(D)) \cap \text{Star}(\sigma_C, \Delta(D)) \neq \emptyset$, and the maximal simplices of Δ_C are in $\text{Star}(\sigma_C, \Sigma_D)$. Also if $\tau \in \text{Star}(\sigma_C, \Delta(D))$ then $D_\tau \subset D_{\sigma_C} = C$, which means that $\tau \in \Delta_C$. Thus $\text{Star}(\sigma_C, \Delta(D)) \subset \Delta_C$.

Consider the pure cone extension $(\Delta(D))'' = \Delta(D) \cup v * (\Delta_C)$ of $\Delta(D)$ at $\Delta_C = \overline{\text{Star}(\sigma_C, \Delta(D))}$. Let v_C be the barycenter of σ_C . Any maximal face τ in Δ_C^0 is star closed in $\Delta(D)$. Consider its barycenter v_τ . Then moving v_τ to v defines the retraction eliminating face τ . Then we eliminate the next maximal face τ_2 in $\Delta_C^0 \setminus \{\tau\}$. The process will continue upon eliminating all the faces in Δ_C^0 and transforming homotopically $|(\Delta(D))''|$ into $|(\Delta(D))'| = |\Delta(D) \setminus \Delta_C^0 \cup v * (\Delta_C \setminus \Delta_C^0)|$.

Note also that both $\Delta_C = \overline{\text{Star}(\sigma_C, \Delta(D))}$, and $v * (\Delta_C)$ are contractible. Moving v to v_τ defines retraction of $(\Delta(D))'' = \Delta(D) \cup v * (\Delta_C)$ to its subcomplex $\Delta(D)$.

(5) Assume C is properly contained in D_{σ_C} . Then as in (4) if $\tau \in \Delta_C$ then $D_\tau \cap C \neq \emptyset$ and $D_\tau \cap D_{\sigma_C} \neq \emptyset$. As before $\text{Star}(\tau, \Delta(D)) \cap \text{Star}(\sigma_C, \Delta(D)) \neq \emptyset$, and the maximal simplices of Δ_C are in $\text{Star}(\sigma_C, \Sigma_D)$.

Write $\Delta(D') = \Delta(D) \setminus \Delta_C^0 \cup v * (\Delta_C \setminus \Delta_C^0)$. Since $\text{Star}(\sigma_C, \Delta(D)) \subset \Delta_C^0$ the complex $\Delta(D')$ this is a subcomplex of the star subdivision $\Delta(D) \setminus \text{Star}(\sigma_C, \Delta(D)) \cup v * (\Delta_C \setminus \text{Star}(\sigma_C, \Delta(D)))$. As before we eliminate one by one maximal faces τ in Δ_C^0 in $\Delta(D) \setminus \text{Star}(\sigma_C, \Delta(D))$ by moving their barycenter v_τ to v defines the retraction eliminating face τ . The process will transform $\sigma \cdot \Delta(D)$ to $\Delta(D')$. □

Remark. Assume C is the component of D that corresponds to certain maximal face $\sigma_C \in \Delta(D)$ with the property $C = D_{\sigma_C}$. Then the set $\Sigma_C := \{\sigma \in \Delta(D) \mid D_\sigma = C\}$ consists σ_C and some of its faces. Then if $D_\tau \subset C$ then $\sigma_C \leq \tau_{max}$ which means $\tau \in \text{Star}(\sigma, \Delta(D))$ for $\sigma \in \Sigma_C$. The component D_τ is contained in $C = D_\tau \subset \overline{D_{\sigma_C}}$ iff $\tau \in \text{Star}(\sigma_C, \Delta(D))$.

Since the set $\Delta_C^0 = \{\tau \in \Delta(D) \mid D_\tau \subset C\}$ is contained in $\overline{\text{Star}(\sigma_C, \Delta(D))}$, it is equal to

$$\Delta_C^0 = \bigcup_{\sigma \in \Sigma_C} \text{Star}(\sigma, \overline{\text{Star}(\sigma_C, \Delta(D))})$$

Remark. Consider a union of the coordinate axes in \mathbf{A}^3 . It is an SNC variety E with maximal components $E_i, i = 0, 1, 2$ given by the axes. The dual complex $\Delta(E)$ consists of the simplex $\Delta(e_0, e_1, e_2)$ and their faces. The cone $\Delta(e_0, e_1, e_2)$ and its all 1-dimensional faces correspond the origin O , and they form the subset Δ_C^0 for $C = \{0\}$. The set Δ_C coincides with $\Delta(E)$. The cone extension of $\Delta(E)$ at (Δ_C, Δ_C^0) corresponding to the blow-up of $C = \{0\}$ is given by $v * (\Delta(E) \setminus \Delta_C^0)$, and thus consists of three one dimensional simplices $\Delta(v, e_i)$ intersecting at v . It corresponds to the exceptional divisors meeting the three disjoint strict transforms of the axes.

The particular case of SNC divisor is somewhat simpler and was first studied by Stepanov [Stp1] (Cases (1),(2), and (4))

Proposition 2.1.7. *Let C be center having SNC with a SNC divisor D on a smooth variety X .*

Then the following possibilities hold

- (1) *If the center C is the intersection component and corresponds to the face σ_C then $\Delta_C^0 = \text{Star}(\sigma_C, \Delta(D))$, and $\Delta_C = \overline{\text{Star}(\sigma_C, \Delta(D))}$. Consequently the new complex $\Delta(D')$ is given by the star subdivision of $\Delta(D)$ at σ_C .*
- (2) *If the center C is properly contained in the smallest intersection component corresponding to σ_C then $\Delta_C^0 = \emptyset$, $\Delta_C \subset \overline{\text{Star}(\sigma_C, \Delta(D))}$. Moreover the maximal simplices of Δ_C are in $\text{Star}(\sigma_C, \Delta(D))$. In this case $\Delta(D')$ is a pure cone extension at $(\overline{\text{Star}(\sigma_C, \Delta(D))}, \emptyset)$, and $\Delta(D')$ is homotopy equivalent to $\Delta(D)$.*
- (3) *If C is not contained in D , and D is an SNC divisor then D contains no components of D and thus $\Delta_C^0 = \emptyset$, and $\Delta(D')$ is a pure cone extension of $\Delta(D)$ at (Δ_C, \emptyset) .*

(4) In the case (3) if D'' is the inverse image of D then $\Sigma(D) = \Sigma(D')$.

Proof. Suppose C contains a component D_α which is locally described by the compatible coordinates $u_1 = \dots = u_k = 0$, where u_i describe the divisorial components. Then the center must have a form $u_1 = \dots = u_r = 0$, for $r \leq k$, and appropriate coordinate rearrangements. This means that it is a component of D . Then $\Delta_C^0 = \emptyset$ in case (2) and (3). The rest of the assertions follows from the previous Proposition. \square

Lemma 2.2. *Consider any embedding $Z \subset T$ of smooth varieties. Let $\bar{Z} \rightarrow Z$ be a sequence of blow ups of nonsingular centers having SNC with exceptional divisors and let $\bar{T} \rightarrow T$ be the induced morphism. Denote the exceptional (SNC) divisor of $\bar{Z} \rightarrow Z$ by D^Z , and by D^T be the exceptional divisor of $\bar{T} \rightarrow T$. Then*

- (1) $D^Z = D^T \cap \bar{Z}$
- (2) There is an inclusion of complexes $\Delta_Z := \Delta(D^Z) \subset \Delta_T := \Delta(D^T)$.
- (3) Each intersection component D_α^Z is contained in a unique minimal component D_α^T . Moreover the components D_α^Z are in bijective correspondence with the components of D_α^T intersecting Z .
- (4) Δ_Z is a deformation retract of its subcomplex Δ_T .

Proof. We use induction on the number of blow-ups k . If $k = 0$ then both statements are obvious. We can assume by the induction that $\Delta_Z \subset \Delta_T$ and Δ_Z corresponds to those of intersection components of T which intersect \bar{Z} .

By the proof of the previous theorem each component D_α^T is created exactly after the same blow-up as the component D_α^Z , with $D_\alpha^Z = D_\alpha^T \cap Z$. Moreover the component D_α^T (or D_α^Z) will vanish if the center of the blow-up C contains D_α^T (or D_α^Z). Thus if the component D_α^T vanishes then D_α^Z also does. If C contains D_α^Z but not D_α^T then the component D_α^Z will vanish while D_α^T will become disjoint from the strict transform of Z . Let $C \subset Z$ be a smooth center having SNC with ZS . The new complexes $\Delta_{Z'}$, and $\Delta_{T'}$ are obtained from Δ_Z and Δ_T by the cone extensions at $(\Delta_{Z,C}, \Delta_{Z,C}^0)$ and $(\Delta_{T,C}, \Delta_{T,C}^0)$ respectively:

$$\begin{aligned}\Delta_{Z'} &= \Delta_Z \setminus \Delta_{Z,C}^0 \cup v_C * (\Delta_{Z,C} \setminus \Delta_{Z,C}^0), \\ \Delta_{T'} &= \Delta_T \setminus \Delta_{T,C}^0 \cup v_C * (\Delta_{T,C} \setminus \Delta_{T,C}^0)\end{aligned}$$

Note that

$$\Delta_{Z,C} = \{\tau \in \Delta_Z \mid D_\tau^Z \cap C \neq \emptyset\} = \{\tau \in \Delta_T \mid D_\tau^T \cap C \neq \emptyset\} = \Delta_{T,C}$$

On the other hand

$$\Delta_{Z,C}^0 = \{\tau \in \Delta_Z \mid D_\tau^Z \subset C\} \supseteq \{\tau \in \Delta_T \mid D_\tau^T \subset C\} = \Delta_{T,C}^0$$

This implies immediately that $\Delta_{Z'}$ is a subcomplex of $\Delta_{T'}$.

Moreover one can retract homotopically

$$|\Delta_{T'}| = |\Delta_T \setminus \Delta_{T,C}^0 \cup v_C * (\Delta_{T,C} \setminus \Delta_{T,C}^0)|$$

to

$$|\Delta_T \setminus \Delta_{Z,C}^0 \cup v_C * (\Delta_{T,C} \setminus \Delta_{Z,C}^0)|$$

Consider a simplex τ of maximal dimension in $\Delta_{Z,C}^0 \setminus \Delta_{T,C}^0$. Let v_τ be its barycenter. The face τ can be eliminated and $|\Delta_T|$ retracted to $|(\Delta_T)' \setminus \{\tau\}|$ by putting $v_\tau \mapsto (1-t)v_\tau + tv_C$, and mapping all vertices identically. Note that any simplex $\tau' \in \Delta_T$ such that $\tau \leq \tau'$ is necessarily in $\Delta_{Z,C}^0 \subset \Delta_{T,C}$, and by maximality is in $\Delta_{T,C}^0$. Thus τ is a maximal face in $\Delta_{T'}$ and the retraction affects only the internal points of τ collapsing them to v_C . We transform $\Delta_{T'}$ to $\Delta_T \setminus (\Delta_{T,C}^0 \setminus \{\tau\}) \cup v_C * (\Delta_{T,C} \setminus (\Delta_{T,C}^0 \setminus \{\tau\}))$ By the repeating this procedure one by one we collapse, or eliminate all simplices in $\Delta_{Z,C}^0 \setminus \Delta_{T,C}^0$, thus transforming $\Delta_{T'}$ into $\Delta_T \setminus \Delta_{Z,C}^0 \cup v_C * (\Delta_{T,C} \setminus \Delta_{Z,C}^0)$.

Note that $\Delta_{Z,C}^0$ is star closed in Δ_T . If $\tau \in \Delta_{Z,C}^0$ and $\tau \leq \tau'$ for $\tau' \in \Sigma_T$. Then $\tau' \in \Delta_{T,C} = \Delta_{Z,C} \subset \Delta_Z$. But $\Delta_{Z,C}^0$ is star closed in Δ_Z , and thus $\tau' \in \Delta_{Z,C}^0$.

This implies that $|\Delta_{Z,C}^0|$ is open in $|\Sigma_T|$. The retraction $|\Delta_T|$ to $|\Delta_Z|$ transforms an open $|\Delta_{Z,C}^0|$ identically. Thus it extends to the retraction of $|\Delta_T \setminus \Delta_{Z,C}^0|$ to $|\Delta_Z \setminus \Delta_{Z,C}^0|$ and the retraction of $|\Delta_T \setminus \Delta_{Z,C}^0 \cup v_C * (\Delta_{Z,C} \setminus \Delta_{Z,C}^0)|$ to $|(\Delta_Z)'| = |\Delta_Z \setminus \Delta_{Z,C}^0 \cup v_C * (\Delta_{Z,C} \setminus \Delta_{Z,C}^0)|$. \square

Corollary 2.2.1. *Consider any embedding $Z \subset T$ of smooth varieties, and let $\bar{Z} \rightarrow Z$ be the sequence of blow-ups of centers having SNC with exceptional divisors, and $\bar{T} \rightarrow T$ be the induced sequence of blow-ups on T . Denote the exceptional (SNC) divisor of $\bar{Z} \rightarrow Z$ by D^Z , and by D^T be the exceptional divisor of $\bar{T} \rightarrow T$. Let $D_1^Z \subset \dots \subset D_k^Z \subset D^Z$ and $D_1^T \subset \dots \subset D_k^T \subset D^T$ denote filtrations of divisors such that*

- (1) $D_i^Z = D_i^T \cap \bar{Z}$
- (2) *There is an inclusion of complexes $\Delta_i^Z := \Delta(D^Z) \subset \Delta_T := \Delta(D^T)$.*
- (3) *Each intersection component D_α^Z is contained in a unique minimal component D_α^T . Moreover the components D_α^Z are in bijective correspondence with the components of D_α^T intersecting Z .*
- (4) Δ^Z *is a deformation retract of its subcomplex Δ^T . Moreover the deformation transforms each Δ_i^T into Δ_i^Z*

Let $C \subset \bar{Z}$ be a center having SNC with D^Z (and thus D^T). Denote by $D^{T'}$ and $D^{Z'}$ the full transformation of the divisors D^T and D^Z . Consider the natural filtration of $D^{T'}$ (or $D^{Z'}$) by $D_i^{T'}$ and $D_i^{Z'}$, where $D_i^{Z'}$ is the inverse image of D_i^Z , and $D_i^{T'}$. Then the above conditions are satisfied after the blow-up of C for

$$D_1^{Z'} \subset \dots \subset D_k^{Z'} \subset D^{Z'} \text{ and } D_1^{T'} \subset \dots \subset D_k^{T'} \subset D^{T'}.$$

Proof. The blow-up of the center C defines cone extensions $\Delta_{Z'}$ and $\Delta_{T'}$ of Δ^Z and Δ^T respectively at $(\Delta_{Z,C}, \Delta_{Z,C}^0)$ and $(\Delta_{T,C}, \Delta_{T,C}^0)$. Moreover by the previous proof we know that $\Delta_{Z,C} = \Delta_{T,C}$ and $\Delta_{Z,C}^0$ contains $\Delta_{T,C}^0$ and is a star closed subset of both Δ^Z and Δ^T . Correspondingly $\Delta_{Z,C,i} = \Delta_{Z,C} \cap \Delta_i^Z$, and $\Delta_{Z,C,i}^0 = \Delta_{Z,C}^0 \cap \Delta_i^Z$.

Consider 3 cases.

Case 1. Assume the center C is not a component of D_Z . Then it is also not a component of D_T . The complexes $\Delta_{Z'}$ and $\Delta_{T'}$ are obtained from Δ_Z and Δ_T by pure cone extension at $\Delta_{Z,C} = \Delta_{T,C}$. Then the retraction of $|\Delta_{T'}|$ to $|\Delta_{Z'}|$ is identical on $|\Delta_{Z,C}| = |\Delta_{T,C}|$ and extends identically to the retraction of $|\Delta_{T'}|$ to $|\Delta_{Z'}|$. This is compatible with the retraction of filtration $|\Delta_{iT'}|$ to $|\Delta_{iZ'}|$.

Case 2. Assume the center C is a component of D^Z corresponding to $\sigma_C \in \Delta_Z$ but is not a component of D_T . Then as before $\Delta_{Z,C} = \Delta_{T,C}$, and $\Delta_{Z,C}^0 = \text{Star}(\sigma_C, \Delta_Z)$, with $\Delta_{T,C}^0 = \emptyset$.

Then $|\Delta_{T'}|$ retracts to $|\Delta_T|$ which further retracts by the inductive assumption to $|\Delta_{Z'}| = |\Delta^Z|$ as $\Delta_{Z'}$ is a star subdivision of Δ_Z . The subcomplexes $\Delta_{T'i}$ are retracted to $\Delta_{T,i}$ (by moving v to v_σ) and then further to $\Delta_{Z'i}$ with $|\Delta_{Z'i}| = |\Delta_{Z'i}|$.

Case 3. Assume the center C is a component of both D^Z and D^T then

$$\Delta_{Z,C} = \Delta_{T,C} = \text{Star}(\sigma_C, \Delta_Z) = \text{Star}(\sigma_C, \Delta_T),$$

and complexes $\Delta_{Z'}$ and $\Delta_{T'}$ are obtained from Δ_Z and Δ_T , by the star subdivision at σ_C . The retraction of $|\Delta_{T',C}|$ to $|\Delta_{Z',C}|$ is identical on $|\text{Star}(\sigma_C, \Delta_Z)|$ and defines the retraction of $|\Delta_{T',C}|$ to $|\Delta_{Z',C}|$ compatible with filtration $(\Delta_{T'i})$. □

For any simplicial complex Σ and its face σ let $\bar{\sigma}$ be the set of all faces of σ . In particular the set $\bar{\sigma}$ is subcomplex of Σ . If $\tau \leq \sigma$ then the face map $i_{\tau,\sigma} : \tau \rightarrow \sigma$ extends to a map (inclusion) between complexes $\bar{i}_{\tau,\sigma} : \bar{\tau} \rightarrow \bar{\sigma}$. Moreover Σ is a union of the subcomplexes $\bar{\sigma}$ for $\sigma \in \Sigma$, with the natural identifications of simplices determined by the maps $\bar{i}_{\tau,\sigma}$.

Definition 2.2.2. The *simplicial product* of simplices $\sigma_1 = \sigma_1(e_0, \dots, e_k)$ and $\sigma_2 = \sigma_2(e'_0, \dots, e'_n)$ with vertices respectively e_0, \dots, e_k , and e'_0, \dots, e'_n , is the simplex

$$\sigma_1 \otimes \sigma_2 = \sigma_1 \otimes \sigma_2(e_{00}, \dots, e_{10}, \dots, e_{kn}),$$

with the set of vertices $\{e_{00}, \dots, e_{10}, \dots, e_{kn}\}$ corresponding to the product $\{e_0, \dots, e_k\} \times \{e'_0, \dots, e'_n\}$.

If $\tau_1 \leq \sigma_1$, and $\tau_2 \leq \sigma_2$ are defined by the inclusion of vertices then there is a face map

$$i_{\tau_1 \otimes \tau_2, \sigma_1 \otimes \sigma_2} : \tau_1 \otimes \tau_2 \rightarrow \sigma_1 \otimes \sigma_2$$

which is also defined by the inclusion of vertices.

Definition 2.2.3. The *simplicial product* of simplicial complexes Σ_1 and Σ_2 is the simplicial complex $\Sigma_1 \otimes \Sigma_2$ which is a union of all the complexes $\overline{\sigma_1 \otimes \sigma_2}$ where $\sigma_i \in \Sigma_i, i = 1, 2$, with inclusion maps $\bar{i}_{\tau_1 \otimes \tau_2, \sigma_1 \otimes \sigma_2} : \tau_1 \otimes \tau_2 \rightarrow \sigma_1 \otimes \sigma_2$ for $\tau_1 \leq \sigma_1$, and $\tau_2 \leq \sigma_2$

The definition is, essentially equivalent to the definition of the product in the category of simplicial sets.

This notion is closely related to the *cartesian product* of simplicial complexes which is a polyhedral complex:

$$\Sigma_1 \times \Sigma_2 := \{\sigma_1 \times \sigma_2 \mid \sigma_i \in \Sigma_i, i = 1, 2\}$$

The face maps in $\Sigma_1 \times \Sigma_2$ are given by the products of the face maps in Σ_i . As before $\Sigma_1 \times \Sigma_2$ can be represented as the union of the complexes $\overline{\sigma_1 \times \sigma_2} = \overline{\sigma_1} \times \overline{\sigma_2}$.

Lemma 2.3. *If D_1 and D_2 are SNC divisors then $D_1 \times D_2$ is a variety with SNC. Moreover*

$$\Delta(D_1 \times D_2) = \Delta(D_1) \otimes \Delta(D_2).$$

Proof. If $D_1 = \sum D_{1i}$ and $D_2 = \sum D_{2i}$ then $D_1 \times D_2 = \bigcup D_{1j} \times D_{2i}$ which defines the dual complex $\Delta(D_1) \otimes \Delta(D_2)$. □

Observe that the star subdivision of a polyhedral complex Σ at any of its vertices v creates a new complex

$$\Sigma' = \Sigma \setminus \text{Star}(v, \Sigma) \cup v * \text{Link}(v, \Sigma),$$

with the same set of vertices and all the faces containing v are of the form $v * \sigma$, with v independent of σ . Thus applying the star subdivisions to all the vertices of Σ creates its triangulation, that is a simplicial complex Σ^{simp} with the same geometric realization.

Let Σ_1, Σ_2 be two simplicial complexes. For any faces $\sigma_1 \in \Sigma_1$ and $\sigma_2 \in \Sigma_2$ the bijective correspondence between the vertices defines a natural projection $\phi_{\sigma_1, \sigma_2} : \sigma_1 \otimes \sigma_2 \rightarrow \sigma_1 \times \sigma_2$ which agrees on the faces and extends to the map

$$\phi_{\sigma_1, \sigma_2} : \overline{\sigma_1 \otimes \sigma_2} \rightarrow \overline{\sigma_1 \times \sigma_2}$$

and which globalizes to

$$\phi : \Sigma_1 \otimes \Sigma_2 \rightarrow \Sigma_1 \times \Sigma_2,$$

and the map of its geometric realizations:

$$|\phi| : |\Sigma_1 \otimes \Sigma_2| \rightarrow |\Sigma_1 \times \Sigma_2|.$$

Note that the map ϕ is the product of two natural projections $\pi_1 : \Sigma_1 \otimes \Sigma_2 \rightarrow \Sigma_1$ and $\pi_2 : \Sigma_1 \otimes \Sigma_2 \rightarrow \Sigma_2$, induced by the projections of the vertices. On the other hand the inclusion maps defined on vertices induces a unique inclusion map

$$i_{\sigma_1, \sigma_2} : \overline{(\sigma_1 \times \sigma_2)}^{\text{simp}} \rightarrow \overline{\sigma_1 \otimes \sigma_2}$$

extending to

$$i : (\Sigma_1 \times \Sigma_2)^{\text{simp}} \rightarrow \Sigma_1 \otimes \Sigma_2.$$

and the closed embedding of the geometric realizations:

$$|i| : |(\Sigma_1 \times \Sigma_2)^{\text{simp}}| = |(\Sigma_1 \times \Sigma_2)^{\text{simp}}| \rightarrow |\Sigma_1 \otimes \Sigma_2|.$$

Lemma 2.4. *The maps $|i|$ and $|\phi|$ defines a homotopy equivalence between the geometric realizations of both complexes. That is*

$$|\Sigma_1 \otimes \Sigma_2| \simeq |(\Sigma_1 \times \Sigma_2)^{\text{simp}}| = |(\Sigma_1 \times \Sigma_2)|$$

Proof. Indeed $\phi i = id_{(\Sigma_1 \times \Sigma_2)^{\text{simp}}}$. On the other hand consider the homotopy which takes $y \in |\Sigma_1 \otimes \Sigma_2|$ to $y_t = (1-t)y + t \cdot i(\phi(y))$ which defines homotopical equivalence of $id_{|\Sigma_1 \otimes \Sigma_2|}$ and $i\phi : |\Sigma_1 \otimes \Sigma_2| \rightarrow |\Sigma_1 \otimes \Sigma_2|$. □

3. COMBINATORIAL TYPE OF VARIETIES

The following theorem gives us a functorial correspondence between quasiprojective varieties and their *combinatorial types*, that is the homotopy types of certain associated complexes. It is a generalization of the correspondence between SNC divisors and their dual complexes. While the notion of the combinatorial type is defined for any schemes of finite type, the induced maps between combinatorial types are considered only for quasiprojective varieties.

Theorem 3.0.1. *Let K be a field of characteristic zero. One can associate with any scheme X of finite type over K its **combinatorial type** $\Sigma(X)$ that is a unique canonical homotopy type of a finite complex such that*

- (1) If X is any variety with SNC crossings then $\Sigma(X)$ is the homotopy type of its dual complex $\Delta(X)$.
- (2) For any morphism $\phi : X \rightarrow Y$ of the quasiprojective schemes of the finite type there is a canonical topological map $\Sigma(f) : |\Sigma(X)| \rightarrow |\Sigma(Y)|$ defined uniquely up to homotopy.
- (3) For any morphisms $Z \xrightarrow{\phi} Y \xrightarrow{\psi} X$ of quasiprojective varieties the maps of the topological spaces:

$$\Sigma(\psi\phi) : |\Sigma(Z)| \rightarrow |\Sigma(X)| \quad \text{and} \quad \Sigma(\psi)\Sigma(\phi) : |\Sigma(Z)| \rightarrow |\Sigma(X)|$$

are homotopy equivalent

- (4) $\Sigma(X \times Y) = \Sigma(X) \otimes \Sigma(Y) \simeq |\Sigma(X) \times \Sigma(Y)|$, and for the projection $\pi_Y : X \times Y \rightarrow Y$, the induced map $\Sigma(\pi_Y) : |\Sigma(X)| \times |\Sigma(Y)| \rightarrow |\Sigma(Y)|$ is homotopy equivalent to the projection $\pi_{|\Sigma(Y)|} : |\Sigma(X)| \times |\Sigma(Y)| \rightarrow |\Sigma(Y)|$.
- (5) If $\phi : Z_1 \dashrightarrow Z_2$ is a proper birational map of smooth varieties which transforms closed subvariety $X_1 \subset Z_1$ to $X_2 \subset Z_2$ then $\Sigma(X_1) = \Sigma(X_2)$.
- (6) If $X = X_1 \cup X_2$ is a union of the closed subvarieties X_1 and X_2 then $\Sigma(X)$ is a push out of $\Sigma(X_1) \leftarrow \Sigma(X_1 \cap X_2) \rightarrow \Sigma(X_2)$.
- (7) If X is a complex projective variety then $W_0(H^i(X, \mathbb{C})) \simeq H^i(\Sigma(X), \mathbb{C})$.

The theorem will be proven in a few steps. First we introduce the definition of the combinatorial type for quasiprojective varieties.

Step 1. The combinatorial type of quasiprojective varieties.

Given a quasiprojective scheme of finite type X one can associate with it its combinatorial type. Embed X into a smooth quasiprojective variety Z , and consider the extended Hironaka desingularization $\bar{Z} \rightarrow Z$ of $X \subset Z$. It defines an SNC divisor $D \subset Z$. We define the *combinatorial type* of X , denoted $\Sigma(X)$ to be the homotopy type of the dual complex $\Delta(D)$.

We will show that the combinatorial type $\Sigma(X)$ of X does not depend upon the embedding and desingularization or principalization which can be used instead of desingularization.

Proposition 3.0.2. *Let s be the number of blow-ups used in the extended canonical desingularization of X , and $n = \dim(X)$ be its dimension. For any extended canonical desingularization of $X \subset Z$, where $m > s + n$ the dual complex $\Delta(D_X^Z)$ of the exceptional divisor D_X^Z is completely determined by the intersections of the centers of blow-ups on the strict transform of X with exceptional divisors. Thus it depends only on the strict transforms and it is independent of the embedding.*

More precisely denote by D^i for $i = 1, \dots, s$ the exceptional divisors on the varieties Z^i obtained by consecutive blow-ups of Z with $D^s = D$. Then $\Delta^{can}(X) = \Delta(D_X^Z)$ is obtained by the sequence of the pure cone extensions of $\Delta^i = \Delta(D^i)$ at the canonical subcomplexes Δ_C^i , where

$$\Delta_C^i = \{\tau \in \Delta^i \mid D_\tau \cap C \neq \emptyset\}$$

is independent of the ambient varieties Z^i (and depends only on the intersections of D^i with the strict transforms of X).

Proof. In fact we can assume that the exceptional divisors D_1, \dots, D_k are ordered by the sequence of blow-ups. The component D_{j_1, \dots, j_k}^s is created when the component $D_{j_1, \dots, j_{k-1}}^{s'}$ intersects the center C_{j_k} on the strict transform X_{j_k} of X . Moreover it becomes the exceptional divisor of the component of the intersection $C_{j_k} \cap D_{j_1, \dots, j_{k-1}}^{s'}$ on $D_{j_1, \dots, j_{k-1}}^{s'}$. Once the component is created it will define a nonsingular subvariety of codimension $\leq s$ so of dimension $> n$. The consecutive centers are contained in the strict transforms of X , and are of dimension $\leq n$ so they won't affect the created components D_{j_1, \dots, j_k}^s . Thus the component D_{j_1, \dots, j_k}^s depends entirely on its creation on the strict transform of X and is independent on the embedding. Similarly the intersection component $D_{j_1, \dots, j_k}^s \cap \bar{X}$ depends on the strict transform \bar{X} of X , and the restrictions of the exceptional divisors to \bar{X} . □

Proposition 3.0.3. *For any quasiprojective variety X there exists a canonical dual complex $\Delta^{can}(X)$ which is functorial with respect to smooth morphisms such that*

- (1) For any embedding $X \hookrightarrow Z$ into a smooth variety Z of sufficiently large dimension $\Delta^{can}(X) = \Delta(D_X^Z)$
- (2) Any smooth morphism $\phi : Y \rightarrow X$ defines the map of complexes $\Delta^{can}(\phi) : \Delta^{can}(Y) \rightarrow \Delta^{can}(X)$

(3) For any smooth morphisms $Z \rightarrow Y \rightarrow X$ we have that

$$\Delta^{can}(\psi\phi) = \Delta^{can}(\psi)\Delta^{can}(\phi)$$

(4) If $\phi : U \rightarrow X$ is an open inclusion then $\Delta^{can}(\phi) : \Delta^{can}(U) \subset \Delta^{can}(X)$ is a canonical inclusion of complexes.

Proof. Follows immediately from the canonicity of the Hironaka (extended) desingularization. Note that any simplex $\Delta_{j_1, \dots, j_k}^s$ in $\Delta^{can}(Y)$ can be identified canonically with the subset $D_{j_1, \dots, j_{k-1}}^s \cap C_k$ on the strict transform X_k of X . This gives functoriality of $\Delta(\phi)$ defined for the smooth maps $\phi : X \rightarrow X'$, since such a map necessarily extends to $X_k \rightarrow X'_k$, defining the unique maps between components transforming divisor into divisors. \square

By the proposition one can construct the canonical dual complex for any scheme X of finite type. Consider an open affine cover (X_i) of X and construct $\Delta^{can}(X)$ by gluing the complexes $\Delta^{can}(X_i)$ along $\Delta^{can}(X_i \cap X_j)$.

Corollary 3.0.4. *If $X = X_1 \cup X_2$ then there exist canonical inclusion $\Delta^{can}(X_1 \cap X_2) \subset \Delta^{can}(X_i)$. Then $\Delta^{can}(X)$ is obtained by the canonical glueing of $\Delta^{can}(X_i)$ along $\Delta^{can}(X_1 \cap X_2)$.*

Theorem 3.0.5. *If X is a variety with SNC then $\Sigma(X)$ is a homotopy type of the dual complex $\Delta(X)$.*

Proof. The complex $\Delta^{can}(X)$ is obtained from $\Delta(X)$ by the sequence of the canonical cone extensions of the form (4) and (5) from Proposition 2.1.6 corresponding to the blow-ups in the extended Hironaka desingularization. These operations do not change the homotopy type. \square

Proposition 3.0.6. *Let $X \subset Z$ be any closed embedding of X as a closed subscheme of a smooth variety Z . Let $Z_0 \rightarrow Z$ be any proper birational map from smooth variety Z_0 , such that the proper transform of X is a variety $D_X^{Z_0}$ with SNC on Z_0 . Then the combinatorial type $\Sigma(X)$ is the homotopy type the dual complex $\Delta(D_X^{Z_0})$.*

Proof. Consider a further closed embedding $Z \subset T =: Z \times \mathbb{P}^n$ into a closed embedding of sufficiently large dimension n . Then the extended canonical embedded desingularization $\bar{Z} \rightarrow Z$ of $X \subset Z$ defines an extended canonical embedded desingularization $\bar{T} \rightarrow T$ of $X \subset T$. It transforms X into $D_X^{\bar{Z}}$ on \bar{Z} and $D_X^{\bar{T}}$ on \bar{T} . Moreover the dual complex $\Delta(D_X^{\bar{T}}) = \Delta^{can}(X)$ on \bar{T} is independent of the embedding by Proposition 3.0.2. On the other hand, by Lemma 2.2, the dual complex $\Delta(D_X^{\bar{Z}})$ is a deformation retract of $\Delta(D_X^{\bar{T}})$. To finish the proof we need the following theorem

Theorem 3.0.7. ([ABW], see also [Stp2],[P]) *Suppose that X is a smooth variety, and $D \subset X$ be its closed subscheme which is a variety with SNC. Then the homotopy type of the dual complex of D depends only the complement $X - D$, and in fact only on its proper birational class.*

By Theorem 3.0.7, $\Delta(D_X^{Z_0})$ is homotopy equivalent to $\Delta(D_X^{\bar{Z}})$, and thus to $\Delta^{can}(X) = \Delta(D_X^{\bar{T}})$. \square

Step 2. Maps of complexes associated with closed embeddings.

Let $i : Y \subset X$ be closed embedding of quasiprojective varieties. Consider the embeddings $Y \subset X \subset Z$ into smooth variety Z of a sufficiently large dimension and the canonical extended desingularization $Z_1 \rightarrow Z$ of $Y \subset Z$. It produces a divisor $D_Y^{Z_1}$ on Z_1 . Next apply the canonical extended desingularization $\bar{Z} \rightarrow Z_1$ of the proper transform $X_1 \subset Z_1$ with SNC to $D_Y^{Z_1}$.

This defines a pair of SNC divisors $D_Y^{\bar{Z}} \subset D_X^{\bar{Z}}$ and the inclusion $\Delta(D_Y^{\bar{Z}}) \hookrightarrow \Delta(D_X^{\bar{Z}})$. As in Theorem 3.0.2 the process is canonical and produces a unique pair of complexes

$$\Delta^{can}(Y; (Y, X)) := D_Y^{\bar{Z}} \subset \Delta^{can}(X; (Y, X)) := \Delta(D_X^{\bar{Z}}).$$

Observe that, by Proposition 3.0.6, $\Sigma(Y)$ and $\Sigma(X)$ are the homotopy type of $\Delta^{can}(Y; (Y, X))$ and $\Delta^{can}(X; (Y, X))$. We define the map $\Sigma(i) : \Sigma(X) \rightarrow \Sigma(Y)$ of the combinatorial types associated with the embedding i to be the homotopy type of the inclusion

$$\Delta^{can}(Y; (Y, X)) \subset \Delta^{can}(X; (Y, X))$$

The construction can be extended to a sequence closed embeddings $X_1 \xrightarrow{i_1} X_2 \xrightarrow{i_2} \dots \xrightarrow{i_{k-1}} X_k \subset Z$ where Z is an ambient smooth field of sufficiently large dimension. By applying the canonical embedded

desingularization of X_1 followed by the canonical embedded desingularization of X_2 and etc, up X_k we produce the sequence of SNC divisors $D_{X_1}^Z \subset \dots \subset D_{X_k}^Z$. As before (Theorem 3.0.2) the process is canonical and produces a unique sequence of complexes

$$\Delta^{can}(X_1; (X_1, \dots, X_k)) := \Delta(D_{X_1}^Z) \subset \dots \subset \Delta^{can}(X_k; (X_1, \dots, X_k)) := \Delta(D_{X_k}^Z) \quad (*)$$

Note that the by Proposition 3.0.6, the homotopy type of $\Delta^{can}(X_k; (X_1, \dots, X_k))$ is equal to $\Sigma(X_i)$. Recall the following result (A simple consequence of the Weak factorization Theorem):

Lemma 3.1. ([P],[ABW]) *Let $X_1 \subset \dots \subset X_k \subset Z$ be embeddings of X_i into smooth Z , then for any proper birational transformation Z' of Z into SNC divisors (or varieties with SNC) $X'_1 \subset \dots \subset X'_k \subset Z'$ on the smooth Z' , the induced embeddings of the dual complexes $\Delta(X'_1) \subset \dots \subset \Delta(X'_k)$ are homotopy equivalent.*

By the the lemma, the sequence (*) defines the sequence of the induced maps of the cominatorial types

$$\Sigma(X_1) \xrightarrow{\Sigma(i_1)} \Sigma(X_2) \xrightarrow{\Sigma(i_2)} \dots \xrightarrow{\Sigma(i_{k-1})} \Sigma(X_k) \quad (**)$$

This observation can be further extended

Lemma 3.2. *Let $X_1 \xrightarrow{i_1} X_2 \xrightarrow{i_2} \dots \xrightarrow{i_{k-1}} X_k \subset Z$ be embeddings of X_i into smooth Z , then for any proper birational transformation of Z taking X_i into SNC divisors (or varieties with SNC) $X'_1 \subset \dots \subset X'_k \subset Z'$ on the smooth Z' , the induced embeddings of the dual complexes*

$$\Delta(X'_1) \subset \dots \subset \Delta(X'_k) \quad (***)$$

define the induced sequence of combinatorial types

$$\Sigma(X_1) \xrightarrow{\Sigma(i_1)} \Sigma(X_2) \xrightarrow{\Sigma(i_2)} \dots \xrightarrow{\Sigma(i_{k-1})} \Sigma(X_k)$$

Proof. The embeddding of X_i into Z can be further extended by embedding $Z \rightarrow T \times \mathbb{P}^n$ into a smooth ambient variety T of sufficiently large dimension. Applying the extended Hironaka desingularization to X_i on T gives a sequence of the dual complexes

$$\Delta^{can}(X_1; (X_1, \dots, X_k)) \subset \dots \subset \Delta^{can}(X_k; (X_1, \dots, X_k))$$

which is homotopy equivalent to the sequence (**). Now by Corollary 2.2.1 the sequece is homotopy equivalent to the sequence of the embedding of the induced dual complexes on the subvariety Z . The latter, in view of Lemma 3.1, is homotopy equivalent to the sequence (***) \square

Step 3. Maps associated with projections

Let X and Y be quasiprojective varieties and $X \subset T$ and $Y \subset V$ be closed embeddings into a smooth ambient varieties of sufficiently large dimension. Consider the extended Hironaka desingularization of $X \subset T$ and $Y \subset V$ producing SNC divisors D_X , and D_Y . This defines a variety with SNC $D_X \times D_Y$ corresponding to the dual complex $\Delta(D_X) \otimes \Delta(D_Y)$ whose homotopy type is equivalent to $|\Delta(D_X)| \times |\Delta(D_Y)|$ and thus of $\Sigma(X) \times \Sigma(Y)$. In the case of arbitrary schemes of finite types we consider affine covers (X_i) of X , and Y_i of Y . Then $\Delta^{can}(X \times Y)$ is obtained by the canonical glueing of $\Delta^{can}(X_i \times Y_i) := \Delta^{can}(X_i) \otimes \Delta^{can}(Y_i)$. This allows to construct

$$\Delta^{can}(X \times Y) = \Delta^{can}(X_i) \otimes \Delta^{can}(Y_i),$$

and, in view of Lemma 2.4, leads to a homotopy equivalence:

$$\Sigma(X \times Y) \simeq \Sigma(X) \times \Sigma(Y).$$

In the particular case, when Y is smooth we get the isomorphism $\Sigma(X \times Y) = \Sigma(X) \times \Sigma(Y) \simeq \Sigma(X)$. We shall describe this isomorphism in a functorial way.

Lemma 3.3. *Let D be an SNC divisor on a smooth variety Z and let E be its closed subscheme which is avriety with SNC, and let $i : E \subset D$ be the closed embedding with the induced map of combinatorial types $\Sigma(i) : \Sigma(E) \rightarrow \Sigma(D)$ (introduced in the previous Step).*

Assume that each maximal irreducible components E_i of E is contained in a unique maximal component D_i of D . Then any component E_α^s is contained in the relevant component $D_\alpha^{s'}$ for a unique s' . This induces a map on vertices of the dual complex $\Delta(E)$ which extends uniquely to the map $\Delta(E) \rightarrow \Delta(D)$ and its geometric realization $\alpha : |\Delta(E)| \rightarrow |\Delta(D)|$ which represents the map between combinatorial types of $\Sigma(E)$, and $\Sigma(D)$. Then the map α is homotopy equivalent to $\Sigma(i)$.

Proof. By definition one can identify $\alpha(\Sigma(E))$ with a subcomplex of $\Sigma(D)$. Consider the blow-up of the maximal component E_i which is not divisorial. It defines a cone extension $\Sigma(E')$ of $\Sigma(E)$ at $(\Delta_{E_i, \Sigma(E)}, \Delta_{E_i, \Sigma(E)}^0)$ with $\Delta_{E_i, \Sigma(E)} = \overline{\text{Star}(v_i, \Sigma(E))} \Delta_{E_i, \Sigma(E)}^0 \supseteq \text{Star}(v_i, \Sigma(E))$ (see Lemma 2.1.6).

It replaces v_i and all the simplices in $\Delta_{E_i, \Sigma(E)}^0 = \text{Star}(v_i, \Sigma(E))$ with a new vertex v and the new simplices in $v * (\Delta_{E_i, \Sigma(E)} \setminus \Delta_{E_i, \Sigma(E)}^0)$. We obtain the complex $\Sigma(E')$ which can be identified with the subcomplex of $\Sigma(E)$ by putting $v \rightarrow v_i$, and which is homotopy equivalent to the latter one (see Lemma 2.1.6).

On the other hand, the center E_i defines the set $\Delta_{E_i, \Sigma(D)}$ which is contained in $\overline{\text{Star}(\alpha(v_i), \Sigma(D))}$. Moreover, by the definition, $\alpha(\Delta_{E_i, \Sigma(E)}) \subset \Delta_{E_i, \Sigma(D)}$. The new complex $\Sigma(D')$ is obtained from $\Sigma(D)$ by the cone extension at $(\Delta_{C, \Sigma(D)}, \emptyset)$. The vertex v corresponds to the exceptional divisor \overline{E}_i . The inclusion $E' \subset D'$ defines a new map $\alpha' : |\Sigma(E')| \rightarrow |\Sigma(D')|$ can be described by glueing two maps $\alpha'_0 : \Sigma(E) \setminus \Delta_{E_i, \Sigma(E)}^0 \rightarrow \Sigma(D)$ and $\alpha'_1 : v * (\Delta_{E_i, \Sigma(E)} \setminus \Delta_{E_i, \Sigma(E)}^0) \rightarrow v * \Delta_{C, \Sigma(D)}$.

The map is homotopy α' equivalent to α , as we can move v to $\alpha(v_i)$. By the blowing of all the nondivisorial maximal component we create a closed embedding of SNC divisors $\overline{E} \subset \overline{D}$. Then the induced map $|\Sigma(\overline{E})| \rightarrow |\Sigma(\overline{D})|$ is homotopy equivalent to α and $\Sigma(i)$. □

Lemma 3.4. *Let Z be a smooth variety with a fixed point $z \in Z$. Consider the closed embedding $j_z : X \simeq X \times \{z\} \hookrightarrow X \times Z$. Then the induced map $\Sigma(j_z) : \Sigma(X) \rightarrow \Sigma(X \times Z)$ is an isomorphism.*

Proof. Let $X \subset T_X$ be a closed embedding into smooth variety. Consider the canonical principalization $\overline{T}_X \rightarrow T_X$ of $X \subset T_X$. It induces the closed embedding $D_X \hookrightarrow D_X \times Z$ of varieties with SNC. That is, by Lemma 3.3, $\Sigma(D_X) \rightarrow \Sigma(D_X \times Z)$ is an isomorphism. □

Lemma 3.5. *Let $i : X \subset Y$ be a closed embedding of quasiprojective varieties and Z be a smooth variety with a fixed point $z \in Z$ consider the induced morphism $i_Z : X \times Z \subset Y \times Z$ extended by the identity on Z and closed embeddings $j_{X,z} : X \simeq X \times \{z\} \subset X \times Z$, and $j_{Y,z} : Y \simeq Y \times \{z\} \subset Y \times Z$. Then the commutative diagram*

$$\begin{array}{ccc} X & \hookrightarrow & Y \\ \downarrow & & \downarrow \\ X \times Z & \hookrightarrow & Y \times Z \end{array}$$

induces the commutative diagram of the maps of complexes

$$\begin{array}{ccc} \Sigma(X) & \rightarrow & \Sigma(Y) \\ \downarrow & & \downarrow \\ \Sigma(X \times Z) & \rightarrow & \Sigma(Y \times Z) \end{array}$$

with induced vertical maps given by isomorphism.

Proof. Let $X \subset Y \subset Z_X$ be a closed embedding into smooth variety Z_X . Consider the canonical principalization $\overline{Z}_X \rightarrow Z_X$ of $X \subset Y \subset Z_X$. It induces the closed embedding $D_X \subset D_Y \subset \overline{Z}_X$ and $D_X \times Z \subset D_Y \times Z \subset \overline{Z}_X \times Z$, and commutative diagram of SNC divisors

$$\begin{array}{ccc} D_X & \hookrightarrow & D_Y \\ \downarrow & & \downarrow \\ D_X \times Z & \hookrightarrow & D_Y \times Z \end{array}$$

and the corresponding diagram of the dual complexes:

$$\begin{array}{ccc} \Delta(D_X) & \hookrightarrow & \Delta(D_Y) \\ \downarrow j_{X,z} & & \downarrow j_{Y,z} \\ \Delta(D_X \times Z) & \hookrightarrow & \Delta(D_Y \times Z) \end{array}$$

with vertical maps given by isomorphisms, as in Lemma 3.3. □

Lemma 3.6. *Let X, Y be quasiprojective varieties. Let $X \subset Z_X$ be a closed embedding into a smooth variety Z_X . Consider the map $X \times Y \rightarrow Z_X \times Y$. Let $j : Y \subset Z_X \times Y$ denote an embedding. Then the composition of the maps*

$$\Sigma(X) \times \Sigma(Y) \simeq \Sigma(X \times Y) \rightarrow \Sigma(Z_X \times Y) \xrightarrow{(\Sigma(j))^{-1}} \Sigma(Y)$$

defines a projection $\pi_{\Sigma(Y)}$.

Proof. Consider the closed embedding $X \times Y \subset Z_X \times Z_Y$, and the induced $D_X \times D_Y \subset \bar{Z}_X \times D_Y \subset \bar{Z}_X$. It satisfies the condition from Lemma 3.3. The induced map α is nothing but the projection on $\Sigma(Z_X \times Y) \simeq \Sigma(Y)$. Thus, by Lemmas 3.3, and 3.4, the embedding $j : X \times Y \rightarrow Z_X \times Y$ induces the map $\Sigma(j) : \Sigma(X \times Y) \rightarrow \Sigma(Z_X \times Y)$ which is homotopy equivalent to the projection via the identification $\Sigma(Z_X \times Y) \rightarrow \Sigma(Y)$. \square

Step 4. Maps associated with general morphisms

Definition 3.6.1. Let $f : X \rightarrow Y$ be a morphism of quasiprojective varieties. Then the induced map $\Sigma(f) : \Sigma(X) \rightarrow \Sigma(Y)$ is defined as follows. Let $X \subset Z_X$ be a closed embedding into a smooth ambient variety Z_X , with a fixed point $x \in X \subset Z_X$. Consider the composition $X \xrightarrow{f_1} X \times Y \xrightarrow{f_2} Z_X \times Y$ of the graph inclusion f_1 followed by the induced inclusion f_2 . The composition morphism $X \rightarrow Z_X \times Y$ defines the map

$$\Sigma(f) : \Sigma(X) \xrightarrow{\Sigma(f_2 f_1)} \Sigma(Z_X \times Y) \xrightarrow{(\Sigma(j))^{-1}} \Sigma(Y)$$

Moreover the map $\Sigma(f)$ is a composition of two uniquely defined maps

$$\Sigma(f_1) : \Sigma(X) \rightarrow \Sigma(X \times Y) \simeq \Sigma(X) \times \Sigma(Y)$$

and the projection (see Lemma 3.6)

$$\pi_{\Sigma(Y)} = (\Sigma(j))^{-1} \Sigma(f_2) : \Sigma(X \times Y) \simeq \Sigma(X) \times \Sigma(Y) \rightarrow \Sigma(Y)$$

and thus it is uniquely defined up to homotopy type.

Lemma 3.7. *Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be morphisms of quasiprojective algebraic varieties and fix $x \in X$, $y = f(x) \in Y$ and $z = gf(x) \in Z$. Then $\Sigma(g)\Sigma(f) : \Sigma(X) \rightarrow \Sigma(Z)$ and $\Sigma(gf) : \Sigma(X) \rightarrow \Sigma(Z)$ are homotopy equivalent.*

Proof. Consider the composition of closed embeddings maps

$$\begin{array}{ccccccc} X \hookrightarrow & X \times Y \hookrightarrow & T_X \times Y & \hookrightarrow & T_X \times Y \times Z & \hookrightarrow & T_X \times T_Y \times Z \\ & & \uparrow & & \uparrow & & \uparrow \\ & & Y & \hookrightarrow & Y \times Z & \hookrightarrow & T_Y \times Z \\ & & & & & & \uparrow \\ & & & & & & Z \end{array}$$

induces, by Lemmas 3.2, 3.5, the commutative diagram of combinatorial types with vertical arrows defined by isomorphisms.

$$\begin{array}{ccccccc} \Sigma(X) \hookrightarrow & \Sigma(X \times Y) \hookrightarrow & \Sigma(T_X \times Y) & \hookrightarrow & \Sigma(T_X \times Y \times Z) & \hookrightarrow & \Sigma(T_X \times T_Y \times Z) \\ \uparrow id_{\Sigma(X)} & & \uparrow & & \uparrow & & \uparrow \\ \Sigma(X) & \xrightarrow{\Sigma(f)} & \Sigma(Y) & \hookrightarrow & \Sigma(Y \times Z) & \hookrightarrow & \Sigma(T_Y \times Z) \\ \uparrow id_{\Sigma(X)} & & \uparrow id_{\Sigma(Y)} & & & & \uparrow \\ \Sigma(X) & \xrightarrow{\Sigma(f)} & \Sigma(Y) & & \xrightarrow{\Sigma(g)} & & \Sigma(Z) \end{array}$$

It follows from commutativity of the diagram that $\Sigma(g)\Sigma(f)$ and $\Sigma(gf)$ are homotopy equivalent. \square

Step 5. Weight Deligne filtration and the combinatorial type of complex projective schemes of finite type.

Recall that for a (projective) simple normal crossing divisor D , or in general a SNC variety the weight zero part exactly coincides with the cohomology of the dual complex $\Delta(D)$:

Lemma 3.8 (Deligne). $W_0(H^i(D, \mathbb{C})) = H^i(\Delta(D), \mathbb{C})$

Thus W_0 can be understood as the ‘‘combinatorial part’’ of the cohomology, and as a generalization of the cohomology of the dual complexes of SNC varieties. Both notions can be compared via the relevant Hironaka resolutions. In fact we have

Proposition 3.8.1. *Let $\phi : X \dashrightarrow X'$ be a proper birational map between complete nonsingular smooth varieties, and suppose $Z \subset X$ and $Z' \subset X'$ are two closed subvarieties such that $\phi(Z) = Z'$. Then*

$$\dim W_0 H^i(Z) = \dim W_0 H^i(Z')$$

Proof. By the Weak Factorization theorem ([Wlo], [AKMW]) the varieties X and X' can be connected by a sequence of blow-ups with smooth centers. It suffices prove the proposition in the case when $\phi : X \rightarrow X'$ is the blow up along a smooth center C . Let $U = X \setminus Z$, $U' = X' \setminus Z'$. Consider the diagram

$$\begin{array}{ccccccc} \dots & W_0 H^{i-1}(Z) & \longrightarrow & W_0 H_c^i(U) & \longrightarrow & W_0 H^i(X) & \longrightarrow & W_0 H^i(Z) & \dots \\ & \downarrow & & \downarrow \simeq & & \downarrow \simeq & & \downarrow & \\ & W_0 H^{i-1}(Z') & \longrightarrow & W_0 H_c^i(U') & \longrightarrow & W_0 H_c^i(X') & \longrightarrow & W_0 H^i(Z') & \end{array}$$

It follows from lemma 5.4 [ABW] that $W_0 H_c^i(U) \rightarrow W_0 H_c^i(U')$ and $W_0 H_c^i(X) \rightarrow W_0 H_c^i(X')$ are isomorphisms. By the diagram and 5 lemma we get that

$$W_0 H^i(Z) \rightarrow W_0 H^i(Z')$$

is an isomorphism. □

Remark. In case Z and Z' are SNC varieties we get a somewhat weaker version (over \mathbb{Q}) of Theorem 7.9 ([ABW]). On the other hand Z and Z' are arbitrary dual complexes generalize the observation for to a more general situation.

The following theorem extends the Deligne Lemma

Theorem 3.8.2. *Let X be a projective complex variety, and $\Sigma(X)$ be its combinatorial type. Then*

$$H^i(\Sigma(X), \mathbb{C}) \simeq W_0(H_i(X, \mathbb{C}))$$

Proof. Consider any embedding $X \rightarrow \mathbb{P}^n$, and let $\bar{\sigma} : \bar{Z} \rightarrow \mathbb{P}^n$ be the canonical Hironaka principalization $\bar{\sigma} : \bar{Z} \rightarrow \mathbb{P}^n \times \mathbb{P}^m$ of the ideal of $X \rightarrow \mathbb{P}^n$ and D be the exceptional divisor. Then, by Proposition 3.8.1 and Deligne Lemma, we have

$$\dim(W_0(H_i(X, \mathbb{C})) = \dim(W_0(H_i(D, \mathbb{C})) = \dim(H_i(\Delta(D), \mathbb{C})) = \dim(H_i(\Sigma(X), \mathbb{C}))$$

□

4. GENERIC SNC

Our next goal is to study SNC morphisms which can be understood as an extensions of the notion of smooth morphisms.

Definition 4.0.1. Let Y be an algebraic scheme over a ground field K . An algebraic scheme D over Y together with the morphism $\psi : D \rightarrow Y$ will be called a *SNC-variety over Y of dimension n* if for any point $p \in D$ there exists an open neighborhood $U \subset D$ and the morphism $\phi : U \rightarrow D_A$, where $D_A \subset \mathbf{A}^{n+1}$ is a SNC divisor in \mathbf{A}^{n+1} defined by the equation $x_1 \cdots x_k = 0$, where $k \leq n$, and such that the induced morphism

$$\phi \times \psi : U \rightarrow D_A \times Y$$

is étale. Alternatively we say that the morphism $D \rightarrow Y$ is SNC of dimension n . An algebraic scheme D is an SNC variety of dimension n if it is an SNC variety of dimension n over $\text{Spec}K$.

(In other words the morphism $D \rightarrow Y$ is étale equivalent to the projection of $D_A \times Y \rightarrow Y$ along a SNC variety D_A .)

If a variety D is SNC over $\text{Spec}(K)$ then for any point $p \in D$ there exists local parameters on D that is the set of functions $u_1, \dots, u_k, u_{k+1}, \dots, u_n$ defining the étale morphism to $D_A \times \mathbf{A}^{n-k}$ as above. We shall

call the function $u_1 \dots u_k$ the local set of *fixed parameters*. Then, it follows from the definition that the completion of the local ring

$$\widehat{\mathcal{O}}_{D,p} \simeq K_p[[u_1, \dots, u_k, u_{k+1}, \dots, u_n]]/(u_1 \dots u_k),$$

where K_p is the residue field of D at p .

Remark. The notion of SNC morphism is a natural extension of the smoothness of the morphism.

We need the following extension of generic smoothness theorem [Har].

Proposition 4.0.2. Generic SNC theorem *Let $D = \sum D_i$ be a SNC variety of dimension n . Consider any morphism $\psi : D \rightarrow Y$. There exists an open subset $V \subset Y$ such that $\psi^{-1}(V) \rightarrow V$ is SNC.*

Proof. We can assume that Y is nonsingular and irreducible and ψ is dominating. Consider all the nonempty intersections $D_{i_1} \cap \dots \cap D_{i_k}$. They are nonsingular and thus by generic smoothness we can find a nonempty open affine $V \subset Y$ such that all the induced morphisms from intersections

$$\psi_{i_1 \dots i_k} : D_{i_1} \cap \dots \cap D_{i_k} \cap \psi^{-1}(V) \rightarrow V$$

are smooth of dimension $n + 1 - k$. We shall prove that $\psi^{-1}(V) \rightarrow V$ is SNC at any point $p \in \psi^{-1}(V)$. We can assume that $p \in \psi^{-1}(V) \cap D_1 \cap \dots \cap D_k$, and that it does not lie in the intersection component of the smaller dimension. Consider the local fixed parameters $u_1 \dots u_k$ at p describing the components $D_i \cap \psi^{-1}(V)$ through p . Let $u_{k+1} \dots u_r$ be the local parameters on V such that for the smooth morphism

$$\psi_{1 \dots k} : D_1 \cap \dots \cap D_k \cap \psi^{-1}(V) \rightarrow V$$

the induced morphism

$$\phi_0 := \psi_{1 \dots k} \times (u_{k+1} \dots u_r) : D_1 \cap \dots \cap D_k \cap \psi^{-1}(V) \rightarrow V \times \mathbf{A}^r$$

is étale. Denote by v_{r+1}, \dots, v_n the local parameters at $y := \psi_{1 \dots k}(p) \in Y$, and by

$$u_{r+1} = \psi^*(v_{r+1}), \dots, u_n := \psi^*(v_n)$$

the induced functions on $\psi^{-1}(V)$.

Then

$$u_1 \dots u_k, u_{k+1} \dots u_r, u_{r+1}, \dots, u_n$$

are local parameters at $p \in V$.

Denote by v_1, \dots, v_r the natural coordinates on the closed subschemes $D_A \times \mathbf{A}^{r-k} \subset \mathbf{A}^r$ defined by $v_1 \dots v_k = 0$. Consider the morphism

$$\phi := \psi \times (u_1 \dots u_k, u_{k+1} \dots u_r) : \psi^{-1}(V) \rightarrow Z := V \times D_A \times \mathbf{A}^r.$$

The induced morphism $\widehat{\psi}_p^* : \widehat{\mathcal{O}}_{Z,y} \rightarrow \widehat{\mathcal{O}}_{D,p}$ sends v_i to u_i . Moreover, by the construction

$$\widehat{\mathcal{O}}_{Z,y} \simeq K_y[[v_1, \dots, v_n]]/(v_1 \dots v_k),$$

where K_y is the residue field of Z at $y = \phi(p)$. Similarly

$$\widehat{\mathcal{O}}_{D,p} \simeq K_p[[u_1, \dots, u_n]]/(u_1 \dots u_k).$$

Moreover since ϕ_1 is a restriction of ϕ the residue field K_p is a finite (separable extension) of K_y . Thus $\widehat{\mathcal{O}}_{Z,y} \otimes_{K_y} K_p \simeq K_p[[v_1, \dots, v_n]]/(v_1 \dots v_k)$, and $\widehat{\psi}_p^*$ induces the isomorphism the complete local rings :

$$\widehat{\mathcal{O}}_{Z,y} \otimes_{K_y} K_p \rightarrow \widehat{\mathcal{O}}_{D,p},$$

that is

$$K_p[[v_1, \dots, v_n]]/(v_1 \dots v_k) \rightarrow K_p[[u_1, \dots, u_n]]/(u_1 \dots u_k), \quad v_i \mapsto u_i.$$

Then $\phi : \psi^{-1}(V) \rightarrow V \times D^A \times \mathbf{A}^r$ is étale and $\psi^{-1}(V) \rightarrow V$ is SNC in a neighborhood of p . □

5. ÉTALE TRIVIALIZATIONS OF STRICTLY SNC MORPHISMS

Recall a well known fact of trivialization of étale proper morphisms:

Lemma 5.1. *Let $\phi : X \rightarrow Y$ be a finite étale morphism of smooth varieties. Then there exists a finite étale morphism $\bar{Y} \rightarrow Y$ of smooth varieties such that the fiber product $\bar{X} := \bar{Y} \times_Y X$ is a finite disjoint union of the varieties \bar{Y}_i isomorphic to \bar{Y} , and the induced morphism*

$$\bar{X} = \bigcup \bar{Y}_i \rightarrow \bar{Y}$$

is an isomorphism on each component.

Proof. Let d be the degree of finite morphism. We shall give a proof by induction on d . Consider first the extension $Y_1 := X$ and induced morphism $\pi_1 : X_1 = Y_1 \times_Y X \rightarrow Y_1$. There is natural finite morphism $i : X \rightarrow X_1 = X \times_Y X$, such that $\pi_1 \circ i = id_X$. Thus $i(X)$ is a closed subvariety of X_1 of the same dimension, which means that $i(X)$ is an irreducible and thus connected component of a smooth subscheme. In other words X_1 is a disjoint union $X_1 = i(Y_1) \cup X'_1$, where degree $\pi'_1 : X'_1 \rightarrow Y_1$ is $\leq d - 1$. By the inductive assumption one can find the desired extension $\bar{Y} \rightarrow Y_1$ by successive repetition of this construction which then trivializes $\pi'_1 : X'_1 \rightarrow Y_1$ and consequently $\pi_1 : X_1 \rightarrow Y_1$. \square

Definition 5.1.1. A morphism $f : X \rightarrow Y$ will be called *strictly smooth* if it is proper and can be factored as $f : X \rightarrow X' \rightarrow Y$ where $g : X \rightarrow X'$ is smooth with connected fibers and $X' \rightarrow Y$ is étale. A morphism $D \rightarrow Y$ will be called a *strictly SNC* if it is SNC and each induced morphism $f : D_\alpha \rightarrow Y$ is strictly smooth.

Corollary 5.1.2. (1) *Let $\phi : X \rightarrow Y$ be a proper morphism of smooth varieties. Then there exists a (canonical) nonempty open subset $U \subset Y$ such that $\phi^{-1}(U) \rightarrow U$ is strictly smooth.*
 (2) *Let D be an SNC variety over ground field K and $f : D \rightarrow Y$ be a proper surjective morphism. Then there exists a (canonical) open subset $U \subset Y$ such that $f^{-1}(U) \rightarrow U$ is strictly SNC.*

Proof. (1) Consider Stein factorization $X \rightarrow Z \rightarrow Y$ of ϕ , where $\psi_1 : Z \rightarrow Y$ is finite and $\psi_2 : X \rightarrow Z$ has connected fibers. By generic smoothness there exists $U \subset Y$ such that $\psi_1^{-1}(U) \rightarrow U$ is étale and $\psi_2^{-1}\psi_1^{-1}(U) \rightarrow \psi_1^{-1}(U)$ is smooth. (2) Follows from (1) and from generic SNC. \square

Corollary 5.1.3. *Let $X \rightarrow Y$ be strictly smooth morphism. Then there exists an étale extension $\bar{Y} \rightarrow Y$ such that $\bar{X} := X \times_Y \bar{Y} \rightarrow \bar{Y}$ splits into a disjoint union of the irreducible components $\bar{X} = \bigcup \bar{X}_i$, where each morphism $\phi : \bar{X}_i \rightarrow \bar{Y}$ is smooth with connected fibers.*

Proof. Let $X \rightarrow Z \rightarrow Y$ be a factorization into a smooth morphism with connected fibers and étale morphism.

Consider the étale extension $\bar{Y} \rightarrow Y$ defined for $Z \rightarrow Y$ and inducing a trivial cover $\bar{Z} := Z \times_Y \bar{Y} \rightarrow \bar{Y}$. This implies that \bar{Z} is a union of irreducible components isomorphic to \bar{Y} . Thus the induced morphism $\bar{X} := X \times_Y \bar{Y} \rightarrow \bar{Z}$ is smooth and has connected fibers. Consequently \bar{X} is a union of disjoint components which are smooth over \bar{Y} . \square

Corollary 5.1.4. *Let $f : D = \sum D_i \rightarrow Y$ be a strictly SNC morphism of dimension n to a smooth variety Y . Then there exists a finite étale morphism $\bar{Y} \rightarrow Y$ such that*

- (1) *The induced morphism $\bar{f} : \bar{D} := D \times_Y \bar{Y} \rightarrow \bar{Y}$ is SNC with $\bar{D} = \bigcup \bar{D}_j$ being a union of irreducible components.*
- (2) *For any irreducible component of $\bar{D}_\alpha = D_{i_1, \dots, i_k}^s$, where $\alpha := \{i_1, \dots, i_k; s\}$ of the intersection of \bar{D}_i the restriction $\bar{f}|_{\bar{D}_\alpha} : \bar{D}_\alpha \rightarrow \bar{Y}$ is smooth proper with irreducible fibers $\bar{D}_{\alpha x} := \bar{f}|_{\bar{D}_\alpha}^{-1}(x)$.*
- (3) *Any fiber $\bar{D}_x := \bar{f}^{-1}(x)$ of $x \in \bar{Y}$ under \bar{f} is a SNC variety with maximal components \bar{D}_{jx} which are intersections of the maximal components \bar{D}_j and $\bar{f}^{-1}(x)$.*
- (4) *There exists a bijective correspondence between the intersection components \bar{D}_α of \bar{D} and the intersection components of $\bar{D}_{\alpha x} = \bar{D}_\alpha \cap \bar{f}^{-1}(x)$. In particular $\Delta(\bar{D}) = \Delta(\bar{D}_x)$.*
- (5) *For any $x, y \in Y$, $\Delta(D_x) = \Delta(D_y)$.*

Proof. Consider a strictly smooth morphism from an intersection component $f_\alpha : D_\alpha \rightarrow Y$ induced by f and its factorization $D_\alpha \xrightarrow{g_\alpha} \mathbb{Z}_\alpha \xrightarrow{h_\alpha} Y$ into a smooth g_α and étale morphism $Z_\alpha \rightarrow Y$. Let $Z \rightarrow Y$ be the component which dominates Y in the product $\prod_U \tilde{Z}_\alpha$ over U . Then the morphism $Z \rightarrow Y$ is finite and étale. Consider the extension $\bar{Y} \rightarrow Y$ for the morphism $Z \rightarrow U$ such that $\bar{Z} := Z \times_Y \bar{Y} \rightarrow \bar{Y}$ is a trivial cover.

The factorization $Z \rightarrow Z_\alpha \rightarrow Y$ determines the factorization of the trivial cover $\bar{Z} \rightarrow \bar{Z}_\alpha := \tilde{Z}_\alpha \times_Y \bar{Y} \rightarrow \bar{Y}$. Thus all $\bar{Z}_\alpha = \bigcup \bar{Z}_{\alpha i} \rightarrow \bar{Y}$ are trivial covers over \bar{Y} . Since the morphism $\bar{D}_\alpha := D_\alpha \times_Y \bar{Y} \rightarrow \bar{Z}_\alpha$ is smooth with connected fibers, the variety \bar{D}_α splits into components $D_{\alpha i}$ which are smooth over $\bar{Y}_{\alpha i}$ with connected fibers. In other words all the connected components of \bar{D}_α are smooth and have connected, and in fact, irreducible fibers over \bar{Y} .

Each fiber $f^{-1}(x)$ is an SNC variety. It is a union of varieties which are intersections of the fiber and irreducible divisor $D_{ix} = D_i \cap f^{-1}(x)$. The components intersections of these $D_{\alpha x}$ are exactly the intersections of components of D_α with $f^{-1}(x)$. This shows that there is a bijective correspondence between the D_α and $D_{\alpha x}$ as well as their intersections proving that the dual complexes of both are the same. In particular, the dual complexes of the fibers are the same. □

6. DESINGULARIZATION OF MORPHISMS AND EQUISINGULAR HIRONAKA DESINGULARIZATION

Theorem 6.0.1. Desingularization of the fibers of morphism *Let $\pi : X \rightarrow Y$ be any morphism over a field K , of characteristic zero. There exists a canonical resolution, that is a nonsingular variety \bar{X} and a projective morphism $\phi : \bar{X} \rightarrow X$ such that the composition morphism $\bar{X} \rightarrow Y$ has all SNC fibers. Moreover*

- (1) *Let $U \subset Y$ be a maximal open nonsingular subset such that $V := \pi^{-1}(U) \rightarrow U$ is smooth. Then ϕ is an isomorphism over V .*
- (2) *The exceptional locus of ϕ is a SNC divisor D .*
- (3) *The fibers of ψ are SNC varieties and they have SNC with D .*
- (4) *There exist a nonsingular locally closed (smooth) stratification S of Y with the generic stratum U and such that for any nongeneric stratum $s \in S$ the set $\phi^{-1}(s)$ is a SNC divisor contained in D and the morphism $\phi^{-1}(s) \rightarrow s$ is SNC.*
- (5) *If ϕ is proper then the combinatorial type of the fibers $\Delta(f^{-1}(x))$ is the same for all points in the stratum $x \in s$.*
- (6) *For each stratum there is an étale morphism $\bar{s} \rightarrow s$, such that the combinatorial type Δ_s of the SNC variety $D_s = D_s \times_s \bar{s}$ is equal to $\Delta(f^{-1}(x))$.*

Proof. Let $\phi_0 : X_0 \rightarrow X$ be the Hironaka desingularization of X with SNC divisor D_0 and denote by $\pi_0 : X_0 \rightarrow Y$ the induced morphism. By the assumption $\phi_0 : X_0 \rightarrow X$ is smooth over $\pi^{-1}(U)$. By generic smoothness there is a maximal open subset $U_1 \subset Y$ (containing U) such that $\pi_0^{-1}(U_1) \rightarrow U_1$ is smooth. Consider a canonical Hironaka principalization $\phi_1 : X_1 \rightarrow X_0$ of the ideal of the complement $Z_1 := X_0 \setminus \pi_0^{-1}(U_1)$ on (X_0, D_0) .

Then $D_1 := \phi_1^{-1}(Z_1)$ is a SNC divisor on X_1 with the induced morphism $\pi_1 : X_1 \rightarrow Y$. Set $Y_1 := Y \setminus U_1$ and consider the induced morphism $D_1 \rightarrow Y_1$. Find a maximal nonsingular open subset $U_1 \subset Y_1$ such that the morphism $\pi_1^{-1}(U_1) \rightarrow U_1$ is SNC. Set $Y_2 := Y_1 \setminus U_1$, and take Hironaka principalization of $Z_2 := \pi_1^{-1}(Y_2)$ on $(X_1, D_0 \cup D_1)$.

We continue this process until $Y_{k+1} = \emptyset$. We obtain successive varieties X_i with the exceptional SNC divisors $D_0 \cup \dots \cup D_i$ projections $\phi_i : X_i \rightarrow Y$, defining the restrictions $\pi_i : D_i \rightarrow Y_i$ and open subsets $U_i \subset Y_i$ for which $\pi_i^{-1}(U_i)$ is SNC over U_i . We define stratification S on Y by taking irreducible components of the locally closed subsets $U_i \subset Y_i \subset Y$. □

Applying the theorem to the identical morphism yields the corollary

Theorem 6.0.2. Hironaka's equisingular desingularization with SNC fibers *Let X be any algebraic variety over a field K , of characteristic zero. There exists a canonical resolution, that is a nonsingular variety \bar{X} such that $\bar{X} \rightarrow X$ has all SNC fibers. Moreover*

- (1) *Let $U \subset X$ be the open subset of nonsingular points on X .*

- (2) The exceptional locus of ϕ is a SNC divisor D .
- (3) The fibers of $\bar{\psi}$ are SNC varieties and they have SNC with D .
- (4) There exist a nonsingular locally closed smooth stratification S of Y such that the generic stratum is U and for any nongeneric stratum $s \in S$ the set $\phi^{-1}(s)$ is a SNC divisor contained in D and the morphism $\phi^{-1}(s) \rightarrow s$ is SNC.
- (5) The combinatorial type of the fibers is the same for all points in the stratum .

□

Theorem 6.0.3. Hironaka's Canonical Principalization with SNC fibers *Let \mathcal{I} be a sheaf of ideals on a smooth algebraic variety X , and $X \rightarrow Y$ be a proper morphism of varieties. There exists a principalization of \mathcal{I} that is, a sequence*

$$X = X_0 \xleftarrow{\sigma_1} X_1 \xleftarrow{\sigma_2} X_2 \leftarrow \dots \leftarrow X_i \leftarrow \dots \leftarrow X_r = \tilde{X}$$

of blow-ups $\sigma_i : X_{i-1} \leftarrow X_i$ of smooth centers $C_{i-1} \subset X_{i-1}$ such that

- (1) The exceptional divisor E_i of the induced morphism $\sigma^i = \sigma_1 \circ \dots \circ \sigma_i : X_i \rightarrow X$ has only simple normal crossings and C_i has simple normal crossings with E_i .
- (2) The total transform $\sigma^{r*}(\mathcal{I})$ is the ideal of a simple normal crossing divisor \tilde{E} which is a natural combination of the irreducible components of the divisor E_r .
- (3) The fibers of $\bar{\psi}$ are SNC varieties and they have SNC with D .
- (4) There exist a nonsingular locally closed stratification S of Y such that the generic stratum is U and for any nongeneric stratum $s \in S$ the set $\phi^{-1}(s)$ is a SNC divisor contained in \tilde{E} and the morphism $\phi^{-1}(s) \rightarrow s$ is SNC.
- (5) The combinatorial type of the fibers is the same for all points in the stratum.

The morphism $(\tilde{X}, \tilde{\mathcal{I}}) \rightarrow (X, \mathcal{I})$ defined by the above principalization commutes with smooth morphisms and embeddings of ambient varieties. It is equivariant with respect to any group action not necessarily preserving the ground field K .

Proof. Consider the canonical Hironaka principalization $\sigma_1 : X_1 \rightarrow X$ of the ideal of \mathcal{I} . We create a SNC divisor D_1 on Z_1 with induced morphism $\pi_1 : D_1 \rightarrow Y$. Then find a maximal open subset U_1 of Y such that $\pi_1^{-1}(U_1) \rightarrow U_1$ is strictly SNC Corollary 5.1.4. Set $Y_2 = Y \setminus U_1$, and consider the canonical Hironaka principalization $\sigma_2 : Z_2 \rightarrow Z_1$ of \mathcal{I}_{Y_2} on (Z_1, D_1) . Then we create the exceptional divisor $D_2 = \sigma_2^{-1}(Y_2)$ having SNC crossing with the strict transform of D_1 and defining the morphism $\pi_2 : D_2 \rightarrow Y_2$, which is SNC over an open nonsingular $U_2 \subset Y_2$. Put $Y_3 := Y_2 \setminus U_2$ principalize Y_3 on $(Z_2, D_1 \cup D_2)$, and continue the process until $Y_{k+1} = \emptyset$. Then put $\bar{Z} := Z_k$ and let $\bar{\pi} : \bar{D} := D_1 \cup \dots \cup D_k \rightarrow Y$ $\bar{\sigma} : \bar{X} \rightarrow X$ be the induced projections.

Consider the stratification S is determined by an irreducible components of the locally closed subsets $U_i \subset Y_i \subset Y$. Then $\bar{\pi}^{-1}(s) \rightarrow s$ is SNC and the combinatorial type $\Delta(\bar{\pi}^{-1}(x))$ is the same for all $x \in s$.

□

Theorem 6.0.4. Constructibility of the combinatorial type of fibers *Let $f : X \rightarrow Y$ be a projective morphism of complex projective varieties. Then there is a locally closed stratification $S = \{s\}$ of Y such that for any stratum s the combinatorial type of fibers $\Sigma(f^{-1}(x))$ is constant for all points $x \in s$. In particular the combinatorial type of fibers is a constructible invariant.*

Proof. X, Y can be embedded as the closed subvarieties of \mathbb{P}^n , and \mathbb{P}^m respectively. Then the graph of $f : X \rightarrow Y$ can be embedded into $\mathbb{P}^n \times \mathbb{P}^m$ so that we have the embedding $X \rightarrow \mathbb{P}^n \times \mathbb{P}^m$ defined by the graph and the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ \mathbb{P}^n \times \mathbb{P}^m & \xrightarrow{\pi} & \mathbb{P}^m \end{array}$$

Consider the canonical Hironaka principalization $\bar{\sigma} : \bar{Z} \rightarrow \mathbb{P}^n \times \mathbb{P}^m$ of the ideal of $X \rightarrow \mathbb{P}^n \times \mathbb{P}^m$ with respect to the morphism $\pi : \mathbb{P}^n \times \mathbb{P}^m \rightarrow \mathbb{P}^m$. Then the exceptional divisor D maps to Y .

For any $x \in X$ the fiber $f^{-1}(x) \subset X \subset \mathbb{P}^n \times \mathbb{P}^m$ we have that

$$\bar{\sigma}^{-1}(f^{-1}(x)) = \bar{\pi}^{-1}(x),$$

which implies

$$\Sigma(f^{-1}(x)) = \Sigma(\bar{\sigma}^{-1}(f^{-1}(x))) = \Sigma(\bar{\pi}^{-1}(x)) = (\Delta(\bar{\pi}^{-1}(x)))$$

and is the same for all points in the same stratum s .

□

Corollary 6.0.5. Constructibility of zero Weight Deligne filtration of fibers *Let $f : X \rightarrow Y$ be a projective morphism of complex projective varieties. Then there is a locally closed stratification $S = \{s\}$ of Y such that for any stratum s the zero weight Deligne filtration $\dim W_0(H^i(f^{-1}(x)))$ of the cohomology of the fiber $f^{-1}(x)$, is constant for all $x \in s$. In particular the invariant $\dim W_0 H^i(f^{-1}(x))$ is constructible.*

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