MA262 — FINAL EXAM — FALL 2022 — DECEMBER 12, 2022
TEST NUMBER 01

INSTRUCTIONS:

1. **DO NOT OPEN THIS BOOKLET UNTIL INSTRUCTED TO DO SO**
2. Before you open the booklet fill in the information below and use a # 2 pencil to fill in the required information on the scantron.
3. **MAKE SURE YOU WRITE YOUR 10 DIGIT ID # AND YOUR TEST NUMBER ON YOUR SCANTRON. THIS IS TEST NUMBER 01.**
4. Once you are allowed to open the exam, make sure you have a complete test. **There are 14 different test pages including this cover page.**
5. Do any necessary work for each problem on the space provided or on the back of the pages of this test booklet. No extra paper is allowed. Circle your answers on this test booklet.
6. There are 20 problems, each problem is worth 10 points. The maximum possible score is 200 points. No partial credit.
7. After you finish the exam, hand in your scantron and your test booklet to your professor, your TA or one of the proctors.

RULES REGARDING ACADEMIC DISHONESTY:

1. Do not seek or obtain any kind of help from anyone to answer questions on this exam. If you have questions, consult only your instructor.
2. Do not look at the exam of another student. You may not compare answers with anyone else or consult another student until after you have finished your exam, handed it in to your instructor and left the room.
3. Do not consult notes, books, calculators.
4. Do not handle phones or cameras, or any electronic device until after you have finished your exam, handed it in to your instructor and left the room.
5. After time is called, the students have to put down all writing instruments and remain in their seats, while the proctors will collect the scantrons and the exams.
6. Anyone who violates these instructions will have committed an act of academic dishonesty. Penalties for academic dishonesty may include an F in the course. All cases of academic dishonesty will be reported to the Office of the Dean of Students.

I have read and understand the above statements regarding academic dishonesty:

STUDENT NAME: ____________________ ID NUMBER: ____________________

SIGNATURE: ____

RECITATION SEC. NUMBER ________________ TA's NAME: ________________

1. Let $y(x)$ be the solution of the following initial value problem

$$y'(x) = y\cos x, \quad y(0) = 1.$$

Then $y(\frac{\pi}{2})$ is equal to

A. $y(\frac{\pi}{2}) = 1$

B. $y(\frac{\pi}{2}) = e^2$

C. $y(\frac{\pi}{2}) = e$

D. $y(\frac{\pi}{2}) = 2e$

E. $y(\frac{\pi}{2}) = 3e$

2. Let $y(t)$ be the solution to the initial value problem

$$\frac{dy}{dt} - 2y = 3e^{-t}, \quad y(0) = 7$$

Find T such that $y(T) = 0$.

A. $T = \frac{1}{2}\ln 3$

B. $T = -\frac{1}{2}\ln 3$

C. $T = \frac{1}{3}\ln 8$

D. $T = -\frac{1}{3}\ln 8$

E. $T = \frac{1}{2}\ln 2$

3. Let $y(x)$ satisfy the following exact equation with initial condition

$$(2x^2e^{2y} - 3x\cos y)dy + (-3\sin y + 2xe^{2y} + 2)dx = 0, \quad y(1) = 0.$$

Which of the following give an implicit formula for $y(x)$?

A. $2y - x^2e^{2y} + 3x \sin y = -1$

B. $2y + x^2e^{2y} + 3x \sin y = 1$

C. $x^2 + x^2e^{2y} + 3x \sin y = 2$

D. $2x + x^2e^{2y} - 3x \sin y = 3$

E. $2x - x^2e^{2y} - 3x \sin y = 1$

4. Let A be a 3×3 matrix and let E be the reduced row-echelon form of A. If the second row of E is $[0 \quad 1 \quad -1]$, consider the following statements:

I. A is not row-equivalent to the 3×3 identity matrix.
II. It is possible that the rank of A is equal to one.
III. The rank of A is equal to two.

A. I is the only true statement

B. II is the only true statement

C. I and II are the only true statements

D. II and III are the only true statements

E. I and III are the only true statements

5. If the system

$$\begin{aligned} x - y - 2z &= a \\ 2x + 2y - 3z &= b \\ 3x + y - 5z &= c \end{aligned}$$

is consistent, what can we conclude about a, b, and c?

A. $c = a$

B. $c = a - b$

C. $c = a + b$

D. $c = -a - b$

E. a, b, and c can be any numbers.

6. Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$. If Det $A = 10$ and if $B = \begin{bmatrix} 2a & 3b & (c+5a) \\ 2d & 3e & (f+5d) \\ 2g & 3h & (i+5g) \end{bmatrix}$, then

A. Det $B = 10$

B. Det $B = 20$

C. Det $B = 40$

D. Det $B = 60$

E. Det $B = 80$

7. A certain matrix A is such that the only solution of the linear system $Ax = 0$ is $x = 0$. Which of the following statements about the matrix A must be true?

A. A must be a square matrix and $\det A \neq 0$.

B. A must be a square matrix and $\det A = 0$.

C. The rank of A equals the number of columns of A.

D. The rank of A equals the number of rows of A.

E. The reduced row-echelon form of A has at most one row of zeros.

8. Find all the values of a such that

$$\begin{bmatrix} 1 & (2-a) & 0 \\ 1 & 0 & 0 \\ 0 & 2 & (1-a) \end{bmatrix}$$

is invertible.

A. $a \neq 1$

B. $a \neq 1$ and $a \neq 2$

C. $a \neq 2$

D. $a \neq 0$

E. It is invertible for all values of a.

9. What must a and b satisfy such that the set of vectors

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ a \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ b \\ 1 \end{bmatrix}$$

is a basis for the vector space $\mathbb{R}^3$?

A. $a - b - 1 \neq 0$

B. $a - b - 1 = 0$

C. $a \neq b$

D. $a = b$

E. $a \neq 0$ and $b \neq 1$

10. $\lambda = 1$ is an eigenvalue of multiplicity 3 of the matrix $\begin{bmatrix} -2 & -9 & 0 \\ 1 & 4 & 0 \\ 1 & 3 & 1 \end{bmatrix}$. The defect of the eigenvalue $\lambda = 1$ is equal to

A. 0

B. 1

C. 2

D. 3

E. 4

11. Let S be the subspace of $\mathbb{R}^3$ consisting of all the vectors x of the form $x = \begin{bmatrix} r+s+t \\ r-s \\ 2r+2s+2t \end{bmatrix}$,

where r and s are real numbers. Which of the following is basis of S?

A. $\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \right\}$ B. $\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\}$

C. $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} \right\}$ D. $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\}$

E. $\left\{ \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \right\}$

12. The general solution of the linear homogeneous equation

$$(D^2 - 9)^2 \, (D^2 + 9)^2 \, y = 0,$$

(this means that its characteristic polynomial is $(r^2 - 9)^2(r^2 + 9)^2$) is given by

A. $y = C_1 e^{3x} + C_2 e^{-3x} + C_4 \cos 3x + C_5 \sin 3x$

B. $y = (C_1 + C_2 x)e^{3x} + (C_3 + C_4 x)e^{-3x} + C_5 \cos 3x + C_6 \sin 3x$

C. $y = x(C_1 + C_2 x)e^{3x} + x(C_3 + C_4 x)e^{-3x} + (C_5 + C_6 x) \cos 3x + (C_7 + xC_8) \sin 3x$

D. $y = (C_1 + C_2 \, x) \, e^{3x} + (C_3 + C_4 \, x) \, e^{-3x} + (C_5 + C_6 \, x) \cos 3x + (C_7 + C_8 \, x) \sin 3x$

E. $y = x(C_1 + C_2 x)e^{3x} + x(C_3 + C_4 x)e^{-x} + x(C_5 + C_6 x) \cos 3x + x(C_7 + C_8 x) \sin 3x$

13. Given that the general solution of the homogeneous equation

$$y^{(4)} + 3y^{(3)} + 3y'' + y' = 0 \text{ is } y_h(x) = C_1 + C_2e^{-x} + C_3xe^{-x} + C_4x^2e^{-x}$$

the general solution to the corresponding nonhomogeneous equation

$$y^{(4)} + 3y^{(3)} + 3y'' + y' = 6x\cos x + 6xe^{-x}$$

looks like:

A. $y(x) = y_h(x) + (Ax + B)\cos x + (Cx + D)\sin x + x^3(Ex + F)e^{-x}$

B. $y(x) = y_h(x) + (Ax + B)\cos x + (Cx + D)\sin x + x^2(Ex + F)e^{-x}$

C. $y(x) = y_h(x) + (Ax + B)\cos x + (Cx + D)\sin x + x(Ex + F)e^{-x}$

D. $y(x) = y_h(x) + x(Ax + B)\cos x + x(Dx + E)\sin x + x^4(Fx + G)e^{-x}$

E. $y(x) = y_h(x) + x(Ax + B)\cos x + x(Cx + D)\sin x + x^3(Ex + F)e^{-x}$

14. Using the method of variation of parameters, we know that a particular solution to the differential equation

$$y'' + 4y = 2\sec x$$

is of the form

$$y_p(x) = u_1(x)\cos 2x + u_2(x)\sin 2x.$$

Then

A. $u_1'(x) = \sin 2x \sec x, \quad u_2'(x) = \cos 2x \sec x$

B. $u_1'(x) = -\sin 2x \sec x, \quad u_2'(x) = \cos 2x \sec x$

C. $u_1'(x) = -2\sin 2x \sec x, \quad u_2'(x) = -2\cos 2x \sec x$

D. $u_1'(x) = 2\sin 2x \sec x, \quad u_2'(x) = -2\cos 2x \sec x$

E. $u_1'(x) = -2\sin 2x \sec x, \quad u_2'(x) = 2\cos 2x \sec x$

15. Let A be a 2×2 matrix whose entries are real numbers. If $\lambda = 2 + 3i$ is a complex eigenvalue of A with corresponding complex eigenvector $\mathbf{w} = \begin{bmatrix} 1-i \\ 4 \end{bmatrix}$, then the general solution to $\mathbf{x}' = A\mathbf{x}$ is:

A. $\mathbf{x} = C_1 e^{2t} \begin{bmatrix} \cos 3t + \sin 3t \\ 4\cos 3t \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} \sin 3t - \cos 3t \\ 4\sin 3t \end{bmatrix}$

B. $\mathbf{x} = C_1 e^{2t} \begin{bmatrix} \cos 3t - \sin 3t \\ 4\cos 3t \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} \sin 3t - \cos 3t \\ 4\sin 3t \end{bmatrix}$

C. $\mathbf{x} = C_1 e^{3t} \begin{bmatrix} \cos 2t + \sin 2t \\ 4\cos 2t \end{bmatrix} + C_2 e^{3t} \begin{bmatrix} \sin 2t - \cos 2t \\ 4\sin 2t \end{bmatrix}$

D. $\mathbf{x} = C_1 e^{2t} \begin{bmatrix} \cos 3t + \sin 3t \\ 4\cos 3t \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} \sin 3t + \cos 3t \\ 4\sin 3t \end{bmatrix}$

E. $\mathbf{x} = C_1 e^{2t} \begin{bmatrix} \cos 3t + \sin 3t \\ 4\cos 3t \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} \sin 3t - \cos 3t \\ -4\sin 3t \end{bmatrix}$

16. Given that $\lambda = 1$ is a defective eigenvalue of the matrix $\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$, which of the following is the solution of the initial value problem:

$$\mathbf{x}' = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{x} \quad \mathbf{x}(0) = \begin{bmatrix} 4 \\ 2 \end{bmatrix}?$$

A. $\mathbf{x}(t) = 2e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^t \left\{ t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right\}$

B. $\mathbf{x}(t) = 2e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2te^t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

C. $\mathbf{x}(t) = 2e^t \begin{bmatrix} 1 \\ -1 \end{bmatrix} + 2e^t \left\{ t \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$

D. $\mathbf{x}(t) = 2e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2e^t \left\{ t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

E. $\mathbf{x}(t) = e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^t \left\{ t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$

17. Let $\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ be the solution of the system of differential equations

$$\mathbf{x}' = A\mathbf{x}, \quad A = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix}.$$

with the initial data $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$. Given that the eigenvalues of A and their corresponding eigenvectors of A are $\lambda_1 = -5$, $V_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\lambda_2 = 2$, $V_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, find $x_1(1)$.

A. $3e^2$

B. $-2e^{-5}$

C. e^{-5}

D. $-3e^{-5}$

E. $e^{-5} + 3e^2$

18. Find all constants a such that the origin is a saddle point for the system

$$X'(t) = \begin{bmatrix} a & 2 \\ 2 & a \end{bmatrix} X(t), \quad a \text{ in } \mathbb{R},$$

A. $a < 2$

B. $a > -2$

C. $a < -3$ or $a > 3$

D. $a < -2$ or $a > 2$

E. $-2 < a < 2$

19. Find all constants b such that the origin is a spiral source of the system

$$X'(t) = \begin{bmatrix} 3 & b \\ 1 & 4 \end{bmatrix} X(t), \quad b \text{ in } \mathbb{R}$$

are

A. $b < -\frac{1}{3}$

B. $b > -\frac{1}{3}$

C. $-\frac{1}{4} < b < -\frac{1}{4}$

D. $b > -\frac{1}{4}$

E. $b < -\frac{1}{4}$

20. The general solution of the system

$$\mathbf{x}' = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \mathbf{x}$$

is given by

A. $\mathbf{x}(t) = ae^{t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + be^{t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + ce^{2t} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

B. $\mathbf{x}(t) = ae^{t} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + be^{-t} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + ce^{2t} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

C. $\mathbf{x}(t) = ae^{t} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + be^{t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + ce^{2t} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

D. $\mathbf{x}(t) = ae^{t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + be^{t} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + ce^{2t} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$

E. $\mathbf{x}(t) = ae^{2t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + be^{2t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + ce^{t} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$