INSTRUCTIONS:

1. Write your answers of the seven multiple choice questions into the table on the last page. Show all your work on the questions and you may use the back of the test pages as scratch paper if needed.

2. After you have finished the exam, hand in your test booklet to your instructor.

3. NO CALCULATORS, BOOKS, NOTES, PHONES OR CAMERAS ARE ALLOWED on this exam. Turn off or put away all electronic devices.

4. Please remain seated during the last 10 minutes of the exam. When time is called, all students must put down their writing instruments immediately. You may remain in your seat while your instructor will collect the exam booklets.

5. Anyone who violates these instructions will have committed an act of academic dishonesty. Penalties for such behavior can be severe and may include an automatic F on the course. All cases of academic dishonesty will be reported to the Office of the Dean of Students.

I have read and understand the above instructions regarding academic dishonesty:

STUDENT NAME

STUDENT SIGNATURE

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SECTION NUMBER
1. (10 points) Which of the following statements is not always true?

   A. If \( \mathbf{u}, \mathbf{v}, \) and \( \mathbf{w} \) are linearly independent vectors in a vector space, then \( \mathbf{u} + \mathbf{v}, \mathbf{v} + \mathbf{w}, \) and \( \mathbf{w} \) are also linearly independent vectors in the vector space.

   B. Every linearly independent set of vectors in \( \mathbb{R}^n \) consists of at most \( n \) vectors.

   C. Every spanning set of \( \mathbb{R}^n \) contains a basis of \( \mathbb{R}^n \).

   D. If the nullity of a matrix \( A \) is zero, then linear system \( A \mathbf{x} = \mathbf{b} \) has a unique solution for every \( \mathbf{b} \).

   E. Any integer between 0 and 4, inclusive, can be the rank of a \( 6 \times 4 \) matrix.

2. (10 points) Let \( M_{3\times3} \) be the vector space of all \( 3 \times 3 \) matrices, and let \( H \) be its subspace consisting of all \( A \) satisfying

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 3
\end{bmatrix} \ A = \begin{bmatrix}
1 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 3
\end{bmatrix}
\]

What is the dimension of \( H \)?

   A. 1

   B. 2

   C. 3

   D. 4

   E. 5
3. (10 points) For which number(s) \( a \) does the matrix

\[
A = \begin{bmatrix}
4 & 0 & 0 & 0 \\
-2 & -1 & 0 & 0 \\
10 & -9 & 6 & a \\
1 & 5 & a & 3
\end{bmatrix}
\]

have 2 as an eigenvalue?

A. \( a = 3 \) only
B. \( a = 3 \) and \( a = -3 \) only
C. \( a = 2 \) only
D. \( a = 2 \) and \( a = 3 \) only
E. \( a = 2 \) and \( a = -2 \) only

4. (10 points) Let \( A \) and \( B \) be similar \( n \times n \) matrices with real entries. Which of the following statements must be TRUE?

(i) \( A \) and \( B \) have the same characteristic polynomial.

(ii) If the columns of \( A \) are linearly independent, then 0 is an eigenvalue of \( A \).

(iii) If \( A \) is diagonalizable, then all the eigenvalues of \( A \) must be nonzero.

(iv) If \(-\lambda\) is an eigenvalue of \( A \), then \( \lambda^4 \) is an eigenvalue of \( B^4 \).

(v) If \( A \) is diagonalizable, then \( B \) is diagonalizable.

A. (i), (ii), (v)
B. (i), (iii), (iv), (v)
C. (iv), (v)
D. (i), (iv), (v)
E. (i), (iii), (iv)
5. (10 points) Let \( P_3 \) denote the vector space of all polynomials of degree at most 3, which of the following subsets are subspaces of either \( \mathbb{R}^3 \) or \( P_3 \)?

(i) The set of all vectors \( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \) in \( \mathbb{R}^3 \) such that \( x + 2y + 3z = 1 \).

(ii) The set of all vectors \( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \) in \( \mathbb{R}^3 \) such that \( 10x - 2y = z \).

(iii) The set of all polynomials \( p(t) \) in \( P_3 \) such that the degree of \( p(t) \) is 3.

(iv) The set of all polynomials \( p(t) \) in \( P_3 \) satisfying \( p(2) = 0 \).

A. (ii) and (iv) only
B. (i), (ii) and (iv) only
C. (ii) and (iii) only
D. (ii), (iii) and (iv) only
E. (iii) and (iv) only

6. (10 points) Consider the differential equation

\[
\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}.
\]

Then the origin is

A. an attractor
B. a repeller
C. a saddle point
D. a spiral point
E. none of the above
7. (10 points) A real 2 × 2 matrix $A$ has an eigenvalue $\lambda_1 = 1 + i$ with corresponding eigenvector $v_1 = \begin{bmatrix} 1 - 2i \\ 3 + 4i \end{bmatrix}$. Which of the following is the general REAL solution to the system of differential equations $x'(t) = Ax(t)$?

A. $c_1 e^t \begin{bmatrix} \cos t + 2 \sin t \\ 3 \cos t - 4 \sin t \end{bmatrix} + c_2 e^t \begin{bmatrix} \sin t - 2 \cos t \\ 3 \sin t + 4 \cos t \end{bmatrix}$

B. $c_1 e^t \begin{bmatrix} \cos t - 2 \sin t \\ 3 \cos t + 4 \sin t \end{bmatrix} + c_2 e^t \begin{bmatrix} \sin t + 2 \cos t \\ 3 \sin t - 4 \cos t \end{bmatrix}$

C. $c_1 e^t \begin{bmatrix} \cos t + 2 \sin t \\ 3 \cos t + 4 \sin t \end{bmatrix} + c_2 e^t \begin{bmatrix} \sin t - 2 \cos t \\ 3 \sin t - 4 \cos t \end{bmatrix}$

D. $c_1 e^t \begin{bmatrix} -\cos t + 2 \sin t \\ -3 \cos t - 4 \sin t \end{bmatrix} + c_2 e^t \begin{bmatrix} \sin t - 2 \cos t \\ 3 \sin t + 4 \cos t \end{bmatrix}$

E. $c_1 e^t \begin{bmatrix} \cos t + 2 \sin t \\ 3 \cos t - 4 \sin t \end{bmatrix} + c_2 e^t \begin{bmatrix} -\sin t - 2 \cos t \\ -3 \sin t + 4 \cos t \end{bmatrix}$
8. Let $\mathbb{P}_2$ denote the vector space of all polynomials of degree at most 2 in the variable $t$, and let $\mathbb{M}_{2 \times 2}$ denote the vector space of all $2 \times 2$ matrices. Consider a linear transformation:

$$T: \mathbb{P}_2 \rightarrow \mathbb{M}_{2 \times 2} \text{ given by } T(p(t)) = \begin{bmatrix} p(0) & p'(0) \\ p(1) & p'(1) \end{bmatrix}.$$ 

(1) (3 points) Find $T(at^2 + bt + c)$.

(2) (3 points) Find a polynomial $p(t)$ in $\mathbb{P}_2$ such that $T(p(t)) = \begin{bmatrix} 1 & 2 \\ 4 & 4 \end{bmatrix}$.

(3) (4 points) Find a basis for the range of $T$. 
9. (6 points) (1) Find all the eigenvalues of matrix $A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 5 & 1 \\ -1 & -3 & 1 \end{bmatrix}$, and find a basis for the eigenspace corresponding to each of the eigenvalues.

(4 points) (2) Find an invertible matrix $P$ and a diagonal matrix $D$ such that

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 5 & 1 \\ -1 & -3 & 1 \end{bmatrix} = PDP^{-1}.$$
10. (4 points) (1) Find the eigenvalues and corresponding eigenvectors of the matrix

\[ A = \begin{bmatrix} 9 & 5 \\ -6 & -2 \end{bmatrix}. \]

(2 points) (2) Find a general solution to the system of differential equations

\[
\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 9 & 5 \\ -6 & -2 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}.
\]

(4 points) (3) Let \( \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \) be a particular solution to the initial value problem

\[
\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 9 & 5 \\ -6 & -2 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \quad \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.
\]

Find \( x(1) + y(1) \).
Please write your answers of the 7 multiple choice questions in the following table.

<table>
<thead>
<tr>
<th>Question</th>
<th>Answer</th>
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Total Points: _____________________