INSTRUCTIONS:

1. Write your answers of the seven multiple choice questions into the table on the last page. Show all your work on the questions and you may use the back of the test pages as scratch paper if needed.

2. After you have finished the exam, hand in your test booklet to your instructor.

3. NO CALCULATORS, BOOKS, NOTES, PHONES OR CAMERAS ARE ALLOWED on this exam. Turn off or put away all electronic devices.

4. Please remain seated during the last 10 minutes of the exam. When time is called, all students must put down their writing instruments immediately.

5. Anyone who violates these instructions will have committed an act of academic dishonesty. Penalties for such behavior can be severe and may include an automatic F on the course. All cases of academic dishonesty will be reported to the Office of the Dean of Students.

I have read and understand the above instructions regarding academic dishonesty:

STUDENT NAME ________________________________

STUDENT SIGNATURE ________________________________

STUDENT PUID ________________________________

SECTION NUMBER ________________________________
1. (10 points) Let $A$ be a $5 \times 7$ real matrix such that $\text{rank}(A) = 5$. Which of the following statements is true?

A. The dimension of the null space of $A$ is equal to 0.
B. The columns of $A$ are linearly independent.
C. The rank of $A^T$ is equal to 7.
D. The rows of $A$ are linearly independent.
E. The dimension of the row space of $A$ is 2.

2. (10 points) Find the dimension of the subspace

$$H = \left\{ \begin{bmatrix} a - 2b + 9c + 5d + 4e \\ a - b + 6c + 5d - 3e \\ b - 3c + d - 9e \\ 5d - 10e \end{bmatrix} : a, b, c, d, e \in \mathbb{R} \right\}.$$

A. 1
B. 2
C. 3
D. 4
E. 5
3. (10 points) Consider the differential equation
\[
\begin{bmatrix}
x'(t) \\
y'(t)
\end{bmatrix} = \begin{bmatrix} 2 & 2 \\
3 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\
y(t) \end{bmatrix}.
\]
Then the origin is
A. a saddle point
B. a spiral point
C. an attractor
D. a repeller
E. none of the above

4. (10 points) Which of the following statements are TRUE?

(i) If \( n \times n \) matrices \( A \) and \( B \) have the same characteristic polynomial, then \( A \) must be similar to \( B \).

(ii) If \( A \) is an \( n \times n \) diagonalizable matrix, then \( A \) must have \( n \) distinct eigenvalues.

(iii) \( n \times n \) matrix \( A \) and its transpose \( A^T \) must have the same eigenvalues.

(iv) There exists an \( n \times n \) matrix whose eigenvectors span \( \mathbb{R}^n \).

(v) If an \( n \times n \) matrix \( A \) is invertible, then none of the eigenvalues of \( A \) can be zero.

A. (iii) and (v) only
B. (i), (ii) and (v) only
C. (ii) and (iii) only
D. (ii), (iv) and (v) only
E. (iii), (iv) and (v) only
5. (10 points) Which of the following sets describe a subspace of the vector space $V$?

A. The set of all vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ in $V = \mathbb{R}^2$ such that $\begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -6 \end{bmatrix}$.

B. The set of all vectors $v$ in $V = \mathbb{R}^2$ such that $Av = 3v$ where $A = \begin{bmatrix} 3 & 0 \\ 2 & 3 \end{bmatrix}$.

C. The set of all vectors $v$ in $V = \mathbb{R}^4$ such that $v = \begin{bmatrix} x + 1 \\ 1 - x \\ 1 \\ 2x \end{bmatrix}$, $x \in \mathbb{R}$.

D. The set of all polynomials $p(t)$ in $V = \mathbb{P}_2 = \{\text{polynomials of degree at most 2}\}$ such that $p(1)p(2) = 0$.

E. The set of all $n \times n$ invertible matrices in $V = M_{n \times n}(\mathbb{R}) = \{\text{all the } n \times n \text{ real matrices}\}$.

6. (10 points) Which of the following matrices are diagonalizable over the real numbers?

(i) $\begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$ (ii) $\begin{bmatrix} -3 & 1 \\ 0 & -3 \end{bmatrix}$ (iii) $\begin{bmatrix} 2 & 3 & 5 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$ (iv) $\begin{bmatrix} 5 & -9 & 6 \\ 0 & 4 & -3 \\ 0 & 0 & 2 \end{bmatrix}$

A. (i) and (iv) only

B. (iii) and (iv) only

C. (i), (ii) and (iii) only

D. (i), (ii) and (iv) only

E. (i), (iii) and (iv) only
7. (10 points) Consider $\mathbf{x}' = A\mathbf{x}(t)$ with $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. Which of the following is the general real solution?

A. $c_1 \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} e^t + c_2 \begin{bmatrix} \sin t \\ -\cos t \end{bmatrix} e^t$ with $c_1, c_2$ real numbers

B. $c_1 \begin{bmatrix} \sin t \\ \cos t \end{bmatrix} e^t + c_2 \begin{bmatrix} -\cos t \\ \sin t \end{bmatrix} e^t$ with $c_1, c_2$ real numbers

C. $c_1 \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} e^t$ with $c_1, c_2$ real numbers

D. $c_1 \begin{bmatrix} \sin t \\ \cos t \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} -\cos t \\ \sin t \end{bmatrix} e^t$ with $c_1, c_2$ real numbers

E. $c_1 \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} e^t + c_2 \begin{bmatrix} \sin t \\ \cos t \end{bmatrix} e^t$ with $c_1, c_2$ real numbers
8. Let $\mathbb{P}_2$ be the vector space consisting of all polynomials of degree at most 2. Note that $\mathbb{P}_2$ contains the zero polynomial. Consider the linear transformation $T : \mathbb{P}_2 \rightarrow \mathbb{P}_2$ defined by

$$T(p(t)) = p(0) + p(1)t + p(2)t^2.$$ 

(4 points) (1) Find the image of $p(t) = t^2 - 1$ under the linear transformation $T$.

(6 points) (2) Find the matrix $[T]_B$ for $T$ relative to the ordered basis $B = \{1, t, t^2\}$. 
9. (6 points) (1) Find all the eigenvalues of matrix \( A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & -1 \\ 0 & -2 & 2 \end{bmatrix} \), and find a basis for the eigenspace corresponding to each of the eigenvalues.

(4 points) (2) Find an invertible matrix \( P \) and a diagonal matrix \( D \) such that

\[
\begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & -1 \\ 0 & -2 & 2 \end{bmatrix} = PDP^{-1}.
\]
10. (4 points) (1) Find the eigenvalues and corresponding eigenvectors of the matrix

\[ A = \begin{bmatrix} -4 & -5 \\ 2 & 3 \end{bmatrix}. \]

(2 points) (2) Find a general solution to the system of differential equations

\[
\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -4 & -5 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}.
\]

(4 points) (3) Let \( \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \) be a particular solution to the initial value problem

\[
\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -4 & -5 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \quad \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \end{bmatrix}.
\]

Find \( x(1) + y(1) \).
Please write your answers of the 7 multiple choice questions in the following table.

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<tr>
<th>Question</th>
<th>Answer</th>
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Total Points: _____________________