MA26500 FINAL EXAM Fall 2022

GREEN - Test Version 01

NAME ______________________ INSTRUCTOR ______________________

1. NO CALCULATORS, BOOKS, NOTES, PHONES OR CAMERAS ARE ALLOWED on this exam. Use the back of the test pages for scratch paper. **PLEASE PUT YOUR SCANTRON UNDERNEATH YOUR QUESTION SHEETS WHEN YOU ARE NOT FILLING IN THE SCANTRON SHEET.**

2. Fill in your name and your instructor's name on the question sheets (above).

3. You must use a **#2 pencil** on the mark–sense sheet (answer sheet). Fill in the instructor's name and the course number(**MA265**), fill in the correct TEST/QUIZ NUMBER (**GREEN** is 01), your name, your section number (see below if you are not sure), and your 10-digit PUID(BE SURE TO INCLUDE THE TWO LEADING ZEROS of your PUID.) and shade them in the appropriate spaces. Sign the mark–sense sheet.

101	MWF	10:30AM	Ying Zhang
102	MWF	9:30AM	Ying Zhang
153	MWF	11:30AM	Ying Zhang
154	MWF	11:30AM	Daniel Tuan-Dan Le
205	TR	1:30PM	Oleksandr Tsymbaliuk
206	TR	3:00PM	Oleksandr Tsymbaliuk
357	MWF	1:30PM	Yiran Wang
410	TR	1:30PM	Arun Debray
451	TR	10:30AM	Arun Debray
501	TR	12:00PM	Vaibhav Pandey
502	MWF	10:30AM	Ayan Maiti
600	MWF	1:30PM	Seongjun Choi
601	MWF	1:30PM	Ayan Maiti
650	MWF	10:30AM	Yevgeniya Tarasova
651	MWF	9:30AM	Yevgeniya Tarasova
701	MWF	3:30PM	Seongjun Choi
702	MWF	11:30AM	Yiran Wang
703	MWF	12:30PM	Ke Wu
704	MWF	1:30PM	Ke Wu
705	MWF	12:30PM	Seongjun Choi
706	TR	1:30PM	Vaibhav Pandey
707	MWF	3:30PM	Siamak Yassemi
708	MWF	2:30PM	Siamak Yassemi

4. There are 25 questions, each is worth 8 points. Show your work on the question sheets. also **CIRCLE** your answer choice for each problem on the question sheets in case your scantron is lost. Although no partial credit will be given, any disputes about grades or grading will be settled by examining your written work on the question sheets.

5. **Please remain seated during the last 10 minutes of the exam.** When time is called, all students must put down their writing instruments immediately.

6. **Turn in both the mark-sense sheets and the question sheets to your instructor when you are finished**.

1. Let A be a 4×4 matrix. Which of the following statements is always TRUE?

A. The reduced echelon form of A has at least 1 pivot.

B. If $A^2 = 0$ then the system $A\mathbf{x} = 0$ has only the trivial solution.

C. If $A^2 = A$ then the reduced echelon form of A is I_4.

D. If $A^2 = I_4$ then the reduced echelon form of A is I_4.

E. A is diagonalizable over the complex numbers.

2. For which of the following five values of the parameter a is the set $\left\{ \begin{bmatrix} a \\ a \\ a \end{bmatrix}, \begin{bmatrix} 1 \\ a \\ a \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ a \end{bmatrix} \right\}$ linearly independent?

(i) $a = 0$

(ii) $a = 1$

(iii) $a = 2$

(iv) $a = 3$

(v) $a = 4$

A. (iv) and (v) only

B. (iii) and (v) only

C. (i), (ii), and (iv) only

D. (ii), (iii), and (iv) only

E. (i), (ii), (iii), and (iv) only

3. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and $T(\mathbf{x}) = A\mathbf{x}$, where A is the standard matrix for T. Which of the following statements must be TRUE?

(i) A is an $n \times m$ matrix.

(ii) If the columns of A are linearly independent, then T is one-to-one.

(iii) If $m > n$, then T is onto.

(iv) If $n > m$, then T is one-to-one.

(v) If the columns of A span $\mathbb{R}^m$, then T is onto.

A. (i), (ii), (iii), (iv), and (v)

B. (i), (ii), and (iii) only

C. (ii) and (v) only

D. (ii), (iii), and (iv) only

E. (iii) and (iv) only

4. Consider the system of linear equations

$$\begin{aligned} x + 3y - z &= 5 \\ 2x + 5y + az &= 9 \\ x + y + a^2 z &= a \end{aligned}$$

Under which conditions does this system have infinitely many solutions?

A. $a \neq -1$

B. $a \neq 3$

C. $a = -1$

D. $a = 3$

E. $a \neq -1$ and $a \neq 3$

5. Which of the following statements is always TRUE?

A. Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^3$. If $\mathbf{z}$ is not a multiple of $\mathbf{y}$, and $\mathbf{y}$ is not a multiple of $\mathbf{x}$, then $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is linearly independent.

B. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a one-to-one linear transformation. If $\{\mathbf{x}, \mathbf{y}\}$ is a linearly independent set in $\mathbb{R}^n$, then so is $\{T(\mathbf{x}), T(\mathbf{y})\}$ in $\mathbb{R}^m$.

C. If A is an $m \times n$ matrix and $m \geq n$, then A has linearly independent columns.

D. If A is an $m \times n$ matrix and $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b} \in \mathbb{R}^m$, then the columns of A are linearly independent.

E. If the columns of a matrix A are linearly independent, then the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$.

6. Let $L : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation such that

$$L\left(\begin{bmatrix} -1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \quad L\left(\begin{bmatrix} 2 \\ -3 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}, \text{ and } \quad L\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}.$$

Then $a + b =$

A. 5

B. -5

C. 1

D. -1

E. 9

7. Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -1 & 0 & 3 & 4 & 2 \\ 2 & 2 & 2 & 2 & 3 \end{bmatrix}$. Which of the following statements is FALSE?

A. $\left\{ \begin{bmatrix} 3 \\ -4 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -5 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis for $\text{Nul}(A)$.

B. $A\mathbf{x} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$ is consistent.

C. $\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$ is a basis for $\text{Col}(A)$.

D. $\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$ is a basis for $\text{Col}(A)$.

E. $\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} \right\}$ is a basis for $\text{Col}(A)$.

8. Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 2 \\ -1 & 3 & 1 \end{bmatrix}$ and $B = A^{-1}$. Find b_{12}, the $(1,2)$-entry of B.

A. 0

B. 2

C. -2

D. -1

E. 1

9. Which of the following sets of vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is a subspace of $\mathbb{R}^3$?

(i) The set of all vectors satisfying $x + y + z = 0$.

(ii) The set of all vectors satisfying $xy - z = 0$.

(iii) The set of all vectors satisfying $xyz = 0$.

(iv) The set of all vectors satisfying $x + y - z^2 = 0$.

(v) The set of all vectors satisfying $x + y - z \geq 0$.

A. (i) only

B. (i), (iii), (iv), and (v) only

C. (i) and (ii) only

D. (i) and (v) only

E. None of them is a subspace of $\mathbb{R}^3$

10. The points $(1, 2)$, $(2, 4)$, $(4, 5)$, and $(5, 7)$ are the vertices of a parallelogram on the coordinate plane. What is the area of this parallelogram?

A. -3

B. 3

C. 6

D. 0

E. 1

11. If

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix} = -5,$$

find

$$\begin{vmatrix} 2d-3f & e & f \\ 2a-3c & b & c \\ 2(a+g)-3(c+k) & b+h & c+k \end{vmatrix}.$$

A. 5

B. -5

C. 10

D. -10

E. 30

12. Let $A = \begin{bmatrix} 1 & -1 & 0 & 2 \\ 2 & 0 & 1 & 5 \\ -1 & 1 & 0 & 3 \end{bmatrix}$. Which of the following statements is always TRUE?

A. $Col(A)$ is a subspace of $\mathbb{R}^4$.

B. $Nul(A)$ is a subspace of $\mathbb{R}^3$.

C. The equation $A\mathbf{x} = 0$ has a unique solution.

D. The dimension of the null space of A is 1.

E. The rank of the matrix A is 2.

13. If the characteristic polynomial of A is $9\lambda - \lambda^3$, then

A. A is invertible and diagonalizable.

B. A is diagonalizable, but not invertible.

C. A is invertible, but not diagonalizable.

D. A is neither invertible nor diagonalizable.

E. A is invertible, but may or may not be diagonalizable.

14. The mapping $T : \mathbb{P}_2 \to \mathbb{P}_2$ defined by $T(a_0 + a_1t + a_2t^2) = a_1 + 2a_2t$ is a linear transformation. Find $[T]_{\mathcal{B}}$, the $\mathcal{B}$-matrix for T when $\mathcal{B}$ is the basis $\{1, t, t^2\}$ for $\mathbb{P}_2$.

A. $[T]_{\mathcal{B}} = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

B. $[T]_{\mathcal{B}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

C. $[T]_{\mathcal{B}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

D. $[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

E. $[T]_{\mathcal{B}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

15. Let the transformation $T : \mathbf{x} \to A\mathbf{x}$ be the composition of a rotation by angle $\theta \in (-\pi, \pi]$, followed by a scaling by the factor $r > 0$. If $A = \begin{bmatrix} -9\sqrt{3} & 9 \\ -9 & -9\sqrt{3} \end{bmatrix}$, find θ and r.

A. $\theta = -5\pi/6, \quad r = 18$

B. $\theta = \pi/3, \quad r = 9$

C. $\theta = 5\pi/6, \quad r = 18$

D. $\theta = \pi/6, \quad r = 18$

E. $\theta = -\pi/6, \quad r = 18$

16. Consider the following system of differential equations

$$x'(t) = 4x(t) + 2y(t)$$
$$y'(t) = 2x(t) + 4y(t)$$

with the initial condition $x(0) = 8, y(0) = 2$. What is the value of $x(1) + y(1)$?

A. $10e^6$

B. $10e^{-6}$

C. $6e^6$

D. $10e^6 + 6e^2$

E. $6e^6 + 10e^2$

17. If $\mathbf{v}$ is an eigenvector of A with eigenvalue 2 and also an eigenvector of B with eigenvalue 3, then $\mathbf{v}$ is an eigenvector of $A^2 - BA + 3I$ with eigenvalue

A. 5

B. -5

C. 1

D. -1

E. -2

18. Consider the real solution to the initial value problem

$$\begin{cases} \mathbf{x}'(t) = A\mathbf{x}(t), \\ \mathbf{x}(0) = \mathbf{x}_0, \end{cases}$$

where $\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$, $A = \begin{bmatrix} 0 & 2 \\ -8 & 0 \end{bmatrix}$, and $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Then $x_1(\frac{\pi}{4})$ is

A. 2

B. 0

C. 1

D. -1

E. -2

19. Which of the following statements is FALSE?

A. If rows of an $m \times n$ matrix A are orthonormal, then the linear transformation $\mathbf{x} \to A\mathbf{x}$ preserves lengths.

B. If columns of a $k \times m$ matrix A are orthonormal and columns of an $m \times n$ matrix B are orthonormal, then the columns of their product AB are also orthonormal.

C. If A is an orthogonal $n \times n$ matrix and B is an $n \times n$ matrix obtained from A by interchanging some rows, then B is also orthogonal.

D. If columns of an $m \times n$ matrix A are orthonormal, then the linear transformation $\mathbf{x} \to A\mathbf{x}$ preserves lengths.

E. If A and B are two orthogonal $n \times n$ matrices, then AB^T is also orthogonal.

20. Let $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} 9 \\ 2 \\ -2 \end{bmatrix}$. Then the closest point $\hat{\mathbf{y}}$ in $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$ to $\mathbf{y}$ is:

A. $\hat{\mathbf{y}} = \begin{bmatrix} 4 \\ -1 \\ -1 \end{bmatrix}$

B. $\hat{\mathbf{y}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

C. $\hat{\mathbf{y}} = \begin{bmatrix} 9 \\ 0 \\ 0 \end{bmatrix}$

D. $\hat{\mathbf{y}} = \begin{bmatrix} 28/3 \\ 11/3 \\ 11/3 \end{bmatrix}$

E. $\hat{\mathbf{y}} = \begin{bmatrix} 9 \\ -2 \\ 2 \end{bmatrix}$

21. Which of the following statements is FALSE?

A. If the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is orthonormal, then the set $\{\mathbf{u}+\mathbf{v}, \mathbf{u}-\mathbf{v}, \mathbf{w}\}$ is orthogonal.

B. If a matrix A has orthogonal columns, then A^TA is a diagonal matrix.

C. Orthogonal matrices are invertible.

D. For vectors $\mathbf{u}$ and $\mathbf{v}$, if $\|\mathbf{u}-\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$, then $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

E. Performing the Gram-Schmidt process on the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ and the set $\{\mathbf{v}_2, \mathbf{v}_1\}$ always gives the identical orthonormal basis for the Span$\{\mathbf{v}_1, \mathbf{v}_2\}$.

22. The least-squares solution of an inconsistent system

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

is

A. $\hat{\mathbf{x}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

B. $\hat{\mathbf{x}} = \frac{1}{2}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

C. $\hat{\mathbf{x}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

D. $\hat{\mathbf{x}} = \frac{1}{3}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

E. $\hat{\mathbf{x}} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

23. If one orthogonally diagonalizes $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} = PDP^T$, then P and D can be chosen as

A. $P = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \quad D = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$

B. $P = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \quad D = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$

C. $P = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \quad D = \begin{bmatrix} -2 & 0 \\ 0 & -4 \end{bmatrix}$

D. $P = \begin{bmatrix} 5/\sqrt{26} & 7/\sqrt{50} \\ 1/\sqrt{26} & 1/\sqrt{50} \end{bmatrix} \quad D = \begin{bmatrix} -2 & 0 \\ 0 & -4 \end{bmatrix}$

E. $P = \begin{bmatrix} -5/\sqrt{26} & 7/\sqrt{50} \\ 1/\sqrt{26} & -1/\sqrt{50} \end{bmatrix} \quad D = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$

24. Let $C[-3,3]$ be the vector space of all continuous real-valued functions defined on $[-3,3]$. Define the inner product on $C[-3,3]$ by

$$\langle f, g \rangle = \int_{-3}^{3} f(t)g(t)dt.$$

Let W be the subspace spanned by the polynomials 1 and t. Find the orthogonal projection of $t^2 + t$ onto W with respect to the above inner product on $C[-3,3]$.

A. $3 + t$

B. $3 - t$

C. $t^2 + t$

D. $3 + 2t$

E. $3 - 2t$

25. Consider the basis S for $\mathbb{R}^3$ given by

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

If we apply the Gram–Schmidt process to S to obtain an orthonormal basis, we obtain

A. $\left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

B. $\left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{3\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}, \frac{1}{3} \begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix} \right\}$

C. $\left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{3\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}, \frac{1}{3} \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \right\}$

D. $\left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \right\}$

E. $\left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$