MA26500 FINAL EXAM Fall 2024

GREEN - Test Version 01

NAME ______________________ INSTRUCTOR ______________________

1. NO CALCULATORS, BOOKS, NOTES, PHONES OR CAMERAS ARE ALLOWED on this exam. Use the back of the test pages or the last two sheets of blank paper for scratch paper. **PLEASE PUT YOUR SCANTRON UNDERNEATH YOUR QUESTION SHEETS WHEN YOU ARE NOT FILLING IN THE SCANTRON SHEET.**

2. Fill in your name and your instructor's name on the question sheets (above).

3. You must use a **#2 pencil** on the mark–sense sheet (answer sheet). Fill in the instructor's name and the course number(**MA265**), fill in the correct TEST/QUIZ NUMBER (**GREEN is 01**), your name, your section number (see below if you are not sure), and your 10-digit PUID(BE SURE TO INCLUDE THE TWO LEADING ZEROS of your PUID.) and shade them in the appropriate spaces. Sign the mark–sense sheet.

101	MWF	11:30AM	Oleksandr Tsymbaliuk	600	MWF	8:30AM	Youssef Hakiki
102	MWF	12:30PM	Oleksandr Tsymbaliuk	650	MWF	8:30AM	Yuxi Han
153	TR	1:30PM	Yilong Zhang	701	MWF	12:30PM	Mathew George
154	TR	10:30AM	Yilong Zhang	702	MWF	12:30PM	Nour Khoudari
205	MWF	3:30PM	Soheil Memariansorkhabi	704	MWF	4:30PM	Michael Monaco
206	MWF	11:30AM	Yuxi Han	705	MWF	2:30PM	Kuan-Ting Yeh
357	MWF	11:30AM	Vaibhav Pandey	706	MWF	3:30PM	Michael Monaco
410	MWF	2:30PM	Vaibhav Pandey	707	MWF	1:30PM	Soheil Memariansorkhabi
451	MWF	10:30AM	Youssef Hakiki	708	MWF	11:30AM	Nour Khoudari
501	MWF	11:30AM	Mathew George	709	MWF	11:30AM	Ying Zhang
502	MWF	3:30PM	Kuan-Ting Yeh	710	MWF	12:30PM	Ying Zhang

4. There are 25 questions, each is worth 8 points. Show your work on the question sheets. also **CIRCLE** your answer choice for each problem on the question sheets in case your scantron is lost. Although no partial credit will be given, any disputes about grades or grading will be settled by examining your written work on the question sheets.

5. **Please remain seated during the last 10 minutes of the exam.** When time is called, all students must put down their writing instruments immediately.

6. **Turn in both the mark-sense sheets and the question sheets to your instructor when you are finished**.

1. Consider the following linear system

$$\begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 5 \\ 0 & 1 & 2 \end{bmatrix} \mathbf{x} = \begin{bmatrix} -2 \\ 4 \\ b \end{bmatrix}.$$

Which of the following statements is always **TRUE**?

A. There is no solution for $b = 0$.

B. There is no solution for $b = 1$.

C. If $b = 2$ then $\mathbf{x} = \begin{bmatrix} -1 \\ 4 \\ -1 \end{bmatrix}$.

D. If $b = 3$ then $\mathbf{x} = \begin{bmatrix} 1 \\ 7 \\ -2 \end{bmatrix}$.

E. The system is inconsistent for some choice of b.

2. Let A and B be two $n \times n$ matrices such that $AB = BA$ and A is invertible. Which of the following statements is **FALSE?**

A. $A^{-1}B = BA^{-1}$

B. $A^2B = BA^2$

C. $B = A^{-1}BA$

D. $(AB)^T = A^TB^T$

E. A and B are similar matrices

3. Let $\mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}$. Which of the following is **FALSE**?

A. $\{\mathbf{x}, \mathbf{y}\}$ spans a plane through the origin.

B. $\begin{bmatrix} -1 \\ -11 \\ 9 \end{bmatrix}$ lies in the span of $\mathbf{x}$ and $\mathbf{y}$.

C. There is a parallelogram with vertices given by $\mathbf{0}$, $\mathbf{x}$, $\mathbf{y}$, and $\begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$.

D. $\left\{\mathbf{x}, \mathbf{y}, \begin{bmatrix} 0 \\ 6 \\ -6 \end{bmatrix}\right\}$ is linearly independent.

E. $\mathbf{x}$, $\mathbf{y}$ and $\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$ form a basis of $\mathbb{R}^3$.

4. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation defined by $T(\mathbf{e_1}) = \begin{bmatrix} a \\ 1 \\ 2 \end{bmatrix}$, $T(\mathbf{e_2}) = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}$, $T(\mathbf{e_3}) = \begin{bmatrix} 4 \\ a \\ 4 \end{bmatrix}$, where a is a real number. Which of the following is **TRUE?**

A. The linear transformation T is one-to-one for $a \in \{2, 8\}$ only.

B. The standard matrix of T has rank 3 for $a \notin \{2, 8\}$.

C. The linear transformation T is onto for $a \in \{2, 8\}$ only.

D. The columns of the standard matrix of T form a basis for $\mathbb{R}^3$ for $a \in \{2, 8\}$.

E. The standard matrix of T is singular for $a \notin \{2, 8\}$.

5. Which of the following is a subspace of $\mathbb{R}^3$?

A. $H = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3 : 3a + 4b - 5c = 0 \right\}$

B. $H = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3 : a^2 - 3b^2 + 2c^2 = 1 \right\}$

C. $H = \left\{ \begin{bmatrix} 0 \\ t \\ t^2 \end{bmatrix} \in \mathbb{R}^3 : t \in \mathbb{R} \right\}$

D. $H = \text{Nul } A$ where A is a 3×4 real matrix

E. $H = \text{Col } A$ where A is a 4×3 real matrix

6. Suppose A is a 3×3 matrix with $A^3 = PBQ$, where

$$QP = B = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 3 \end{bmatrix}.$$

Compute $k = \det(A^2)$.

A. $k = 3$

B. $k = 3^2$

C. $k = 3^3$

D. $k = 3^4$

E. $k = 3^5$

7. Let A be the standard matrix of the linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ which first rotates by the angle $3\pi/4$ counterclockwise around the origin and then reflects in the line $x = y$. The inverse matrix A^{-1} is equal to:

A. $\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ -1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$

B. $\begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$

C. $\begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

D. $\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

E. $\begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

8. Let S be the parallelogram on $\mathbb{R}^2$ with vertices $(-1,-1), (2,1), (3,5), (0,3)$ and $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation with the standard matrix $\begin{bmatrix} 2 & 1 \\ 7 & 5 \end{bmatrix}$. Then the area of $T(S)$ equals:

A. 3

B. 10

C. 20

D. 30

E. 40

9. Let $L : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation such that

$$L\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad L\left(\begin{bmatrix} 3 \\ -2 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \text{ and } \quad L\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} 5 \\ 4 \\ 3 \end{bmatrix}.$$

Then $a + b =$

A. 2

B. -2

C. 1

D. -1

E. 0

10. Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 0 & 3 \\ -3 & 2 & 1 \end{bmatrix}$ and $B = A^{-1}$. Find b_{12}, the $(1, 2)$-entry of B.

A. 0

B. 2

C. -2

D. -1

E. 1

11. Let V be the subspace of polynomials $p(t)$ in $\mathbb{P}_3$ satisfying $p(0) = p'(0) = 0$. Which of the following sets is a basis for V?

A. $\{1 - t, t - t^2\}$

B. $\{t - t^3, t + t^3\}$

C. $\{t^2 - t^3, t^2 + t^3\}$

D. $\{1, t + 2, t^2 - 1\}$

E. $\{1, t + t^2, t^2 - t^3\}$

12. Let A be a 3×3 matrix with eigenvalues -1 and 0. If the dimension of the eigenspace associated with the eigenvalue -1 is 2. Which of the following statements is/are always **TRUE**?

(i) A is diagonalizable

(ii) A is invertible

(iii) $\det A = 1$

A. (i) only

B. (ii) only

C. (i) and (ii)only

D. (i) and (iii) only

E. (i), (ii) and (iii)

13. Suppose $\begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ is an eigenvector of the matrix $A = \begin{bmatrix} 2 & 0 & -2 & 0 \\ a & -1 & 1 & -4 \\ 2 & 5 & 0 & b \\ 1 & 0 & -5 & 0 \end{bmatrix}$. Find the product of a and b.

A. 4

B. 2

C. -2

D. -4

E. -8

14. Let $A = \begin{bmatrix} -1 & 3 \\ 0 & 2 \end{bmatrix}$. For which matrix P is it true that $P^{-1}AP = D$, where D is a diagonal matrix?

A. $P = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

B. $P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

C. $P = \begin{bmatrix} -1 & 1 \\ 0 & 4 \end{bmatrix}$

D. $P = \begin{bmatrix} 1 & -4 \\ 0 & 3 \end{bmatrix}$

E. $P = \begin{bmatrix} 1 & 1 \\ 0 & -3 \end{bmatrix}$

15. Let $T : M_{2\times 2} \to M_{2\times 2}$ be the linear transformation given by $T(A) = A + 2A^T$. Consider the ordered basis $B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ for $M_{2\times 2}$. Which of the following matrices is $[T]_B$?

A. $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix}$

B. $\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

C. $\begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}$

D. $\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

E. $\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

16. For the system of differential equations $\mathbf{x}'(t) = A\mathbf{x}(t)$ with $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$, the origin is

A. a repeller

B. an attractor

C. a saddle point

D. a spiral point

E. none of the above

17. Consider the differential equation

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

with the initial condition

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}.$$

Then, the sum $x_1(1) + x_2(1)$ is equal to:

A. $2e^2(-\cos(2) + 3\sin(2))$

B. $2e^2(3\cos(2) - \sin(2))$

C. $2e^2(3\cos(2) + \sin(2))$

D. $2e^2(\cos(2) + 3\sin(2))$

E. none of the above

18. Let A be a 2×2 matrix with real entries. Suppose that A is similar to the matrix $B = \begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix}$. Which of the following statements is **FALSE**?

A. If λ is a complex eigenvalue of A, then $|\lambda| = 5$.

B. A is similar to some diagonal matrix with real entries.

C. The origin is a spiral point of the system of differential equation $\mathbf{x}' = A\mathbf{x}$.

D. The linear transformation $\mathbf{x} \to B\mathbf{x}$ is a composition of a scaling and a rotation.

E. If $\mathbf{v}$ is an eigenvector of A corresponding to the complex eigenvalue λ, then $\bar{\mathbf{v}}$ is an eigenvector of A corresponding to the complex eigenvalue $\bar{\lambda}$.

19. Which of the following statements is always **TRUE**?

A. If $\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n\}$ is an orthogonal set in $\mathbb{R}^n$, then $\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n\}$ is linearly independent.

B. If $\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n\}$ is a basis of $\mathbb{R}^n$ and $\mathbf{x} \cdot \mathbf{u}_j = 0$ for $j = 1, \cdots, n$, then $\mathbf{x} = \mathbf{0}$.

C. Suppose an $m \times n$ matrix U has orthogonal columns. Then $\|U\mathbf{x}\| = \|\mathbf{x}\|$ for any $\mathbf{x}$ in $\mathbb{R}^n$.

D. Let A be an $m \times n$ matrix. Then $(\text{Col } A)^{\perp}$=Nul A.

E. Let L be a line spanned by a nonzero vector $\mathbf{u}$ in $\mathbb{R}^n$. Then for any vector $\mathbf{y}$ in $\mathbb{R}^n$, $\text{proj}_L\mathbf{y}$ (the projection of $\mathbf{y}$ onto L) must have the same direction as $\mathbf{u}$.

20. Which of the following matrices is/are diagonalizable over real numbers?

(i) $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

(ii) $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$

(iii) $\begin{bmatrix} 0 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 2 & 3 \end{bmatrix}$

(iv) $\begin{bmatrix} 2 & 3 & 5 \\ 3 & 8 & 9 \\ 5 & 9 & 8 \end{bmatrix}$

A. (iii) only

B. (iv) only

C. (i) and (iii) only

D. (iii) and (iv) only

E. (i), (iii) and (iv) only

21. Consider the following linear system

$$\begin{bmatrix} 1 & -1 & a \\ 0 & 2 & b \\ -2 & 1 & c \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

If the system has no solution, then which of the following is **TRUE** about the vector $\mathbf{z} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ in $\mathbb{R}^3$?

A. The set of all possible vectors $\mathbf{z}$ is a subspace.

B. The orthogonal complement of the subspace spanned by the vectors $\mathbf{z}$ is a plane.

C. $\mathbf{z}$ can only be the zero vector.

D. $\mathbf{z}$ is orthogonal to $\begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix}$.

E. none of the above

22. Let $\mathbf{y} = \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}$, $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$, and $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$. Then the distance from $\mathbf{y}$ to W is

A. $\dfrac{\sqrt{3}}{3}$

B. $\dfrac{2\sqrt{3}}{3}$

C. $\dfrac{4\sqrt{3}}{3}$

D. $\dfrac{5\sqrt{3}}{3}$

E. $\dfrac{7\sqrt{3}}{3}$

23. Find a least-squares solution of an inconsistent system $A\mathbf{x} = \mathbf{b}$ where $A = \begin{bmatrix} 2 & 0 \\ -1 & 1 \\ 0 & 2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 3 \\ 0 \\ -3 \end{bmatrix}$.

A. $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

B. $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

C. $\begin{bmatrix} -3 \\ 3 \end{bmatrix}$

D. $\begin{bmatrix} 3 \\ -3 \end{bmatrix}$

E. $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$

24. Let $C[-1,1]$ be the space of all continuous functions over $[-1,1]$ with the inner product

$$< f(t), g(t) >= \int_{-1}^{1} f(t)g(t)dt$$

for any $f(t), g(t) \in C[-1,1]$. Let W be the subspace spanned by the polynomials 1 and t. Find the orthogonal projection of $t^2 + 5t^3$ onto W with respect to the above inner product on $C[-1,1]$.

A. $\frac{1}{3} + t$

B. $\frac{1}{3} - 2t$

C. $\frac{1}{3} + 2t$

D. $\frac{1}{3} - 3t$

E. $\frac{1}{3} + 3t$

25. Performing the Gram-Schmidt process on the **ORDERED** set of vectors $\left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ yields an orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$. What is $\mathbf{u}_3$?

A. $\frac{1}{3}\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

B. $\frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$

C. $\frac{1}{3}\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$

D. $\frac{1}{3}\begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix}$

E. $\frac{1}{3}\begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}$

Scratch paper

Scratch paper