1. Let $m \geq 1$ and $n \geq 1$ be two natural numbers such that $m > n$. Which of the following is/are true?

   (i) A linear system of m equations in n variables is always consistent.

   (ii) A linear system of n equations in m variables is always consistent.

   (iii) A homogeneous linear system of m equations in n variables always has infinitely many solutions.

   (iv) A homogeneous linear system of n equations in m variables always has infinitely many solutions.

   A. (ii) only

   B. (iv) only

   C. (ii) and (iv) only

   D. (ii), (iii) and (iv) only

   E. (i), (ii), (iii) and (iv)

2. Consider the linear system of equations

$$\begin{cases} 2x + y - z & = a \\ x - y + 2z & = 1 \\ 4x - y + 3z & = a^2. \end{cases}$$

   Under which condition will the system be consistent?

   A. $a = 0$

   B. $a = -2$

   C. $a = 0$ or 1

   D. $a = -1$ or 2

   E. $a = 1$

3. Let A, B be two 4×4 matrices. Which of the following statements is ALWAYS true?

A. If $A + B$ is singular, then either A or B is singular.

B. If AB is symmetric, then BA is also symmetric.

C. If A and B are similar matrices, then $\det(A) = \det(B)$.

D. If $AB = AC$, then $B = C$.

E. If $A\mathbf{x} = \mathbf{b}$ is consistent for some $\mathbf{b} \neq \mathbf{0}$, then A is non-singular.

4. Assume that a 4×4 matrix A is row equivalent to

$$B = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

which of the following statements is true?

A. B is not the reduced row echelon form of A.

B. $\det(A) \neq 0$.

C. The null space of A has dimension 3.

D. The column space of A has dimension 2.

E. If $A\mathbf{x} = \mathbf{b}$ is consistent for some $\mathbf{b} \neq \mathbf{0}$, then it has infinitely many solutions.

5. Consider the linear system

$$\begin{array}{rrrr} x+ & y+ & 3z = & 3 \\ & y+ & az = & -1 \\ 2x+ & 3y+ & a^2z = & a+2 \end{array}$$

For which value of a does the system have an infinite number of solutions?

A. $a = 0$

B. $a = 2$

C. $a = -2$

D. $a = 3$

E. There is no a with such a property

6. If the determinant of

$$\begin{bmatrix} 2c_1 & a_1 & 3b_1 \\ 2c_2 & a_2 & 3b_2 \\ \frac{1}{3}c_3 & \frac{1}{6}a_3 & \frac{1}{2}b_3 \end{bmatrix}$$

is 6, find the determinant of

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

A. 6

B. -1

C. 1

D. -36

E. 36

7. Compute the determinant of the following matrix:

$$\begin{bmatrix} 0 & 3 & 2 & 0 & 0 \\ 1 & 2 & 0 & 2 & 3 \\ 0 & 0 & 1 & 0 & 0 \\ 3 & 2 & 0 & 2 & 4 \\ 0 & 3 & 2 & 0 & 5 \end{bmatrix}$$

A. 60

B. 20

C. 0

D. -20

E. -60

8. If $A = \begin{bmatrix} 0 & 0 & 4 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ then the $(2, 1)$-entry of A^{-1} is:

A. 1

B. 1/4

C. 0

D. $-1/4$

E. -1

9. Consider the vector space P_2 of all polynomials of degree at most 2. Find all real numbers a such that $t^2 - at + 1 - a^2$ is in the span of $t^2 - 2t + 1$, $t - 1$ and $3t^2 - 2t - 1$.

A. a can be any real number.

B. $a = \pm 2$.

C. $a = -1$ or $a = 2$.

D. $a = -2$ or $a = 1$.

E. $a = 0$.

10. Consider the linear system

$$\begin{bmatrix} 1 & 1 & 4 \\ 1 & 2 & 3 \\ 7 & 0 & 1 \\ 2 & 0 & a \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -b \\ 0 \\ 0 \\ b \end{bmatrix}.$$

Suppose that the solution set of this system is a **linear subspace** of $\mathbb{R}^3$. Which of the following conditions must hold:

A. $a = 0$

B. $a = b = 0$

C. $a \neq 0$

D. $a = b$

E. $b = 0$

11. Determine all the values for c such that the following vectors form a basis of $\mathbb{R}^3$:

$$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ c \end{bmatrix}.$$

(a) $c = 2$.

(b) $c = 4$.

(c) $c \neq 1$.

(d) $c \neq 3$.

(e) $c \neq 2$.

12. Determine the rank of the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 3 & 2 \\ 4 & 7 & 2 & 1 \\ -1 & -10 & 7 & 5 \end{bmatrix}$$

A. 0

B. 1

C. 2

D. 3

E. 4

13. Suppose that W consists of all vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ such that $x + 3y - 2z = 0$.

Which of the following is a basis for $W^{\perp}$?

A. $\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

B. $\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$

C. $\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

D. $\frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

E. $\begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$

14. Suppose that W consists of all vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ such that $x + 3y - 2z = 0$.

What is the distance from $v = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ to W?

A. $\dfrac{\sqrt{14}}{2}$

B. $\sqrt{5}$

C. 4

D. $\dfrac{\sqrt{5}}{2}$

E. 5

15. Consider the following inconsistent linear system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & -2 \\ 1 & 0 \\ 1 & 2 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

The *least squares solution* of the linear system is

A. $\hat{x} = \begin{bmatrix} 7/8 \\ 2 \end{bmatrix}$

B. $\hat{x} = \begin{bmatrix} -1/3 \\ 0 \end{bmatrix}$

C. $\hat{x} = \begin{bmatrix} 1 \\ -2/3 \end{bmatrix}$

D. $\hat{x} = \begin{bmatrix} 7/3 \\ 3/4 \end{bmatrix}$

E. None of the above.

16. Let $V \subset \mathbb{R}^2$ be the set of vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ with $x \cdot y \geq 0$. Which of the following statements is true.

A. V is a subspace.

B. V is not a subspace since it does not contain the zero vector.

C. V is not a subspace since it is not closed under vector addition.

D. V is not a subspace since it is not closed under scalar multiplication.

E. V is not a subspace since it is not closed under the dot product.

17. Let A be an $m \times n$ matrix. If W is the column space of A^T, then $W^\perp$ (the orthogonal complement of W) is

(a) the null space of A

(b) the column space of A

(c) the null space of A^T

(d) the column space of A^T

(e) none of the above

18. Let

$$A = \begin{bmatrix} 1 & 0 & 3 & 0 \\ 2 & 1 & -4 & -1 \\ 3 & 2 & 4 & 0 \\ 0 & 3 & -1 & 0 \end{bmatrix}.$$

Then cofactor A_{32} of A is

A. -2

B. -1

C. 0

D. 1

E. 2

19. Let P_4 be the space of all polynomials of degree at most 4. The subspace V of P_4 consists of all polynomials $p(t)$ such that $p(0) = 2p(1)$. Dimension of V is

A. 2

B. 3

C. 4

D. 5

E. 1

20. Let

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & c & 2 \\ 1 & -3 & c \end{bmatrix}.$$

Find all values of c for which the system $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution.

A. $c = -1, -2$

B. $c = 1, -2$

C. $c = 1, 2$

D. $c = -1, 1$

E. $c = -1, 2$

21. Find a nonsingular matrix P such that

$$P^{-1}\begin{bmatrix} 2 & -2 & 1 \\ -1 & 3 & -1 \\ 2 & -4 & 3 \end{bmatrix} P \;=\; \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 6 \end{bmatrix}.$$

A. $P = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$

B. $P = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

C. $P = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}$

D. $P = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}$

E. $P = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & -2 \end{bmatrix}$

22. Let $A = \begin{bmatrix} 1 & -2 & -6 \\ 0 & 2 & -5 \\ 0 & 1 & 8 \end{bmatrix}$. Find the eigenvalues λ of A.

A. $\lambda = 1, 2, 8$

B. $\lambda = 1, 3, 7$

C. $\lambda = 1, 2, 7$

D. $\lambda = 2, 3, 7$

E. $\lambda = -1, 3, 7$

23. Which one of the following matrices is **NOT** diagonalizable?

A. $\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 4 \\ 0 & 0 & 3 \end{bmatrix}$

B. $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 5 & 0 \\ 0 & 6 & 9 \end{bmatrix}$

C. $\begin{bmatrix} 4 & -2 & 0 \\ -2 & 5 & 6 \\ 0 & 6 & 1 \end{bmatrix}$

D. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

E. $\begin{bmatrix} 5 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 7 \end{bmatrix}$

24. Which statement(s) about symmetric matrices with real entries is(are) TRUE?

(I)	Every symmetric matrix is nonsingular.
(II)	All the eigenvalues of each symmetric matrix are real numbers.
(III)	There exists a symmetric matrix which is not diagonalizable.
(IV)	Symmetric matrices are diagonalizable.

A. (II) only

B. (IV) only

C. (I) and (II)

D. (II) and (III)

E. (II) and (IV)

25. Consider the following linear system of differential equations:

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}.$$

The general solution of the linear system of differential equations is

A. $\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = b_1 \begin{bmatrix} e^{3t} \\ e^{3t} \end{bmatrix} + b_2 \begin{bmatrix} e^{2t} \\ e^{2t} \end{bmatrix}$ for arbitrary constants b_1, b_2.

B. $\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = b_1 \begin{bmatrix} 3e^{3t} \\ 2e^{3t} \end{bmatrix} + b_2 \begin{bmatrix} 2e^{2t} \\ 3e^{2t} \end{bmatrix}$ for arbitrary constants b_1, b_2.

C. $\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = b_1 \begin{bmatrix} e^{t} \\ -e^{t} \end{bmatrix} + b_2 \begin{bmatrix} e^{5t} \\ e^{5t} \end{bmatrix}$ for arbitrary constants b_1, b_2.

D. $\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = b_1 \begin{bmatrix} e^{-t} \\ e^{-t} \end{bmatrix} + b_2 \begin{bmatrix} e^{-5t} \\ -e^{-5t} \end{bmatrix}$ for arbitrary constants b_1, b_2.

E. $\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = b_1 \begin{bmatrix} e^{6t} \\ e^{6t} \end{bmatrix} + b_2 \begin{bmatrix} e^{5t} \\ e^{5t} \end{bmatrix}$ for arbitrary constants b_1, b_2.