MA26500 FINAL EXAM Spring 2023

GREEN - Test Version 01

NAME ________________ INSTRUCTOR ________________

1. NO CALCULATORS, BOOKS, NOTES, PHONES OR CAMERAS ARE ALLOWED on this exam. Use the back of the test pages for scratch paper. **PLEASE PUT YOUR SCANTRON UNDERNEATH YOUR QUESTION SHEETS WHEN YOU ARE NOT FILLING IN THE SCANTRON SHEET.**
2. Fill in your name and your instructor's name on the question sheets (above).
3. You must use a **#2 pencil** on the mark–sense sheet (answer sheet). Fill in the instructor's name and the course number(**MA265**), fill in the correct TEST/QUIZ NUMBER (**GREEN** is 01), your name, your section number (see below if you are not sure), and your 10-digit PUID(BE SURE TO INCLUDE THE TWO LEADING ZEROS of your PUID.) and shade them in the appropriate spaces. Sign the mark–sense sheet.

172	MWF	9:30AM	Ying Zhang	265	TR	1:30PM	Shiang Tang
173	MWF	10:30AM	Ying Zhang	276	TR	3:00PM	Yiran Wang
185	MWF	10:30AM	Seongjun Choi	277	TR	1:30PM	Yiran Wang
196	MWF	9:30AM	Seongjun Choi	281	MWF	2:30PM	Siamak Yassemi
201	MWF	2:30PM	Jing Wang	282	MWF	1:30PM	Siamak Yassemi
202	TR	4:30PM	Takumi Murayama	283	MWF	11:30AM	Ying Zhang
213	MWF	4:30PM	Eric Samperton	284	MWF	11:30AM	Seongjun Choi
214	MWF	3:30PM	Eric Samperton	285	MWF	7:30AM	Luming Zhao
225	MWF	11:30AM	Farrah Yhee	287	MWF	8:30AM	Luming Zhao
226	MWF	10:30AM	Farrah Yhee	288	MWF	12:30PM	Ping Xu
237	TR	10:30AM	Ying Liang	289	MWF	1:30PM	Ping Xu
238	TR	12:00PM	Ying Liang	290	MWF	11:30AM	Ping Xu
240	MWF	2:30PM	Ayan Maiti	291	MWF	12:30PM	Yevgeniya Tarasova
241	MWF	1:30PM	Ayan Maiti	292	MWF	11:30AM	Yevgeniya Tarasova
252	TR	12:00PM	Vaibhav Pandey	293	MWF	11:30AM	William Heinzer
253	TR	1:30PM	Vaibhav Pandey	294	TR	1:30PM	Guang Lin
264	TR	3:00PM	Shiang Tang	295	MWF	3:30PM	William Heinzer

4. There are 25 questions, each is worth 8 points. Show your work on the question sheets. also **CIRCLE** your answer choice for each problem on the question sheets in case your scantron is lost. Although no partial credit will be given, any disputes about grades or grading will be settled by examining your written work on the question sheets.
5. **Please remain seated during the last 10 minutes of the exam.** When time is called, all students must put down their writing instruments immediately.
6. **Turn in both the mark-sense sheets and the question sheets to your instructor when you are finished**.

1. Consider the linear system

$$\begin{array}{ccccccc} x & + & y & + & 2z & = & 1 \\ & & y & + & az & = & 3 \\ 4x & + & 6y & + & a^2z & = & a+6 \end{array}$$

For which value of a does the system have an infinite number of solutions?

A. $a = -4$

B. $a = 4$

C. $a = -2$

D. $a = 2$

E. $a = 6$

2. Suppose the set $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} a \\ a \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3a \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^3$. Then a satisfies

A. $a \neq 1$ and $a \neq -1$

B. $a \neq 1$ only

C. $a \neq 2$ only

D. $a \neq -1$ only

E. $a \neq 1$ and $a \neq 2$

3. Find all possible c such that the nullity of the following matrix A is 1, where

$$A = \begin{bmatrix} 1 & 2 & 2 \\ c & 4 & 4 \\ 3 & c & 6 \end{bmatrix}$$

A. $c = 1$ only

B. $c = 2$ only

C. $c = 1$ or 6

D. $c = 1$ or 2 or 4

E. $c = 2$ or 6

4. Let A be an 3×5 matrix with $\text{rank}(A) = 3$. Which of the following statements must be **FALSE?**

A. The equation $A\mathbf{x} = \mathbf{b}$ has a solution for any $\mathbf{b}$ in $\mathbb{R}^3$.

B. If B is a 3×5 matrix that is row equivalent to A, then the columns of B^T are linearly dependent.

C. If B is a 3×5 matrix that is row equivalent to A, then $\text{Row}(A) = \text{Row}(B)$.

D. The linear transformation defined by $\mathbf{x} \mapsto A\mathbf{x}$ is onto.

E. The linear transformation defined by $\mathbf{x} \mapsto A^T\mathbf{x}$ is one-to-one.

5. Which of the following statements must be **TRUE**?

(i) The function $T : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $T(x) = 2x + 1$ is a linear transformation.

(ii) The set of points $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ satisfying the equation $x + 3y + 2z = 13$ is a subspace of $\mathbb{R}^3$.

(iii) The set $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^3$.

(iv) The function $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ defined by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} x+y \\ x-y \\ z \end{bmatrix}$ is a linear transformation.

A. (iv) only

B. (i) and (iii)

C. (ii) and (iv)

D. (iii) and (iv)

E. (ii), (iii), and (iv)

6. Let $A = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$ and let its inverse $A^{-1} = [b_{ij}]$. Find b_{12}.

A. -3

B. 3

C. 6

D. 10

E. -10

7. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation whose standard matrix is $\begin{bmatrix} t-2 & -1 \\ 2t+6 & t+3 \end{bmatrix}$ where t is a real number. Find ALL values of t such that L is one-to-one.

A. $t = 1$

B. $t = -3$

C. $t \neq -1$ and $t \neq -3$

D. $t \neq 0$ and $t \neq -3$

E. $t \neq 1$

8. Which of the following sets of vectors in the given vector spaces is linearly dependent?

A. $\{t^2 + 3t - 1, t^3 + 3t^2 - t, 4t + 2\}$ in $\mathbb{P}_3$

B. $\left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 4 \\ 1 \end{bmatrix} \right\}$ in $\mathbb{R}^4$

C. $\left\{ \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \right\}$ in $\mathbb{M}_{2\times 2}$

D. $\left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ -1 \end{bmatrix} \right\}$ in $\mathbb{R}^3$

E. $\{3t^2 - 1, 2t + 10\}$ in $\mathbb{P}_2$

9. Define the linear transformation $T : \mathbb{P}_2 \to \mathbb{P}_3$

$$T(a_0 + a_1 t + a_2 t^2) = a_2 t^3.$$

Then $\mathrm{Ker}(T)$ has a basis

A. $\{t, t^2\}$

B. $\{1, t, t^2, t^3\}$

C. $\{1, t\}$

D. $\{1, t^2\}$

E. $\{t^2, t^3\}$

10. Let

$$A = \begin{bmatrix} 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2 \\ 1 & 2 & 2 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

Denote $a = \det(2A)$ and $b = \det(A^2)$ the determinants, then $a + b$ is equal to

A. 28

B. 30

C. 32

D. 34

E. 36

11. Let $A = \begin{bmatrix} 1 & 0 & -1 & -2 \\ 2 & 1 & 3 & 5 \\ -2 & -1 & 7 & 0 \end{bmatrix}$. Which of the following is always **TRUE?**

A. $Nul(A)$ is a subspace of $\mathbb{R}^3$.

B. The dimension of null space of A is 1.

C. $Col(A)$ is a subspace of $\mathbb{R}^2$.

D. The equation $Ax = 0$ has unique solution.

E. The rank of the matrix A is 4.

12. Which of the following are subspaces of $\mathbb{P}_3$ (the vector space of polynomials of degree at most three)?

A. The set of all polynomials of the form $at + t^2 + (a - b)t^3$ where a and b are real numbers.

B. The set of all polynomials $p(t)$ with $p(1) = 1$.

C. The set of all polynomials $p(t)$ with $p(-1)p(-2) = 0$.

D. The set of all polynomials $p(t)$ with $p(0) = 0$ and $p(2) = 0$.

E. The set of all polynomials $p(t)$ with integer coefficients.

13. Let $A = \begin{bmatrix} -1 & -5 \\ 2 & 1 \end{bmatrix}$. Then $\begin{bmatrix} 3i - 1 \\ 2 \end{bmatrix}$ is

A. an eigenvector of A corresponding to eigenvalue 3.

B. an eigenvector of A corresponding to eigenvalue -3.

C. an eigenvector of A corresponding to eigenvalue $3i$.

D. an eigenvector of A corresponding to eigenvalue $-3i$.

E. an eigenvector of A corresponding to eigenvalue 0.

14. Suppose that A is a 3 by 3 real matrix having *distinct* real eigenvalues $\lambda_1, \lambda_2, \lambda_3$. Which of the following statements are **TRUE?**

(i) A is diagonalizable.

(ii) if $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are eigenvectors for $\lambda_1, \lambda_2, \lambda_3$, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal set.

(iii) if $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are eigenvectors for $\lambda_1, \lambda_2, \lambda_3$, then the matrix $[\mathbf{v}_1 \mathbf{v}_2 \mathbf{v}_3]$ is invertible.

(iv) $\det A = \lambda_1 \lambda_2 \lambda_3$.

A. (i) and (ii) only

B. (i) and (iii) only

C. (ii) and (iv) only

D. (i), (ii) and (iv)

E. (i), (iii) and (iv)

15. Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a linear transformation such that

$$T(\mathbf{e_1}) = \begin{bmatrix} -4 \\ 2 \end{bmatrix} \text{ and } T(\mathbf{e_2}) = \begin{bmatrix} -5 \\ 3 \end{bmatrix}.$$

Which of the following is a basis $\mathcal{B}$ for $\mathbb{R}^2$ such that the matrix $[T]_{\mathcal{B}}$ for T relative to the basis $\mathcal{B}$ is a diagonal matrix?

A. $\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix} \right\}$

B. $\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 5 \\ -2 \end{bmatrix} \right\}$

C. $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ -2 \end{bmatrix} \right\}$

D. $\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \end{bmatrix} \right\}$

E. $\left\{ \begin{bmatrix} -5 \\ 3 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \end{bmatrix} \right\}$

16. Let $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ be the particular solution to the differential equation

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -5 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

satisfying the initial condition

$$\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \end{bmatrix}.$$

Then $x(2) + y(2)$ is

A. $14e^{-6}$

B. $14e^{-6} + 2e^{8}$

C. $14e^{-6} - 2e^{8}$

D. $2e^{8}$

E. $-14e^{-6} + 2e^{8}$

17. Assume that all vectors and subspaces are in $\mathbb{R}^n$. Which of the following statements is FALSE?

A. The best approximation to $\mathbf{y}$ by elements of a subspace W is the vector $\mathbf{y}-$ proj$_W\mathbf{y}$

B. If $\mathbf{z}$ is orthogonal to $\mathbf{u_1}$ and $\mathbf{u_2}$, and if $W = \text{Span}\{\mathbf{u_1}, \mathbf{u_2}\}$, then $\mathbf{z}$ is in $W^\perp$.

C. If $\mathbf{y} = \mathbf{z_1} + \mathbf{z_2}$, where $\mathbf{z_1}$ is in a subspace W and $\mathbf{z_2}$ is in $W^\perp$, then proj$_W\mathbf{y} = \mathbf{z_1}$.

D. If $\mathbf{x}$ is not in a subspace W, then $\mathbf{x}-$proj$_W\mathbf{x}$ is not zero.

E. If the columns of an $m \times n$ matrix A are orthonormal, then the linear mapping $\mathbf{x} \mapsto A\mathbf{x}$ preserves length.

18. A real 2×2 matrix A has an eigenvalue $\lambda = -2 + i$ with corresponding eigenvector $\mathbf{v} = \begin{bmatrix} 3-i \\ 4+2i \end{bmatrix}$. Which of the following is a general **REAL** solution to the system of differential equations $\mathbf{x}'(t) = A\,\mathbf{x}(t)$?

A. $c_1\, e^{-2t} \begin{bmatrix} 3\cos(t) + \sin(t) \\ 4\cos(t) - 2\sin(t) \end{bmatrix} + c_2\, e^{-2t} \begin{bmatrix} -\cos(t) - 3\sin(t) \\ 2\cos(t) - 4\sin(t) \end{bmatrix}$

B. $c_1\, e^{-2t} \begin{bmatrix} 3\cos(t) - \sin(t) \\ 4\cos(t) + 2\sin(t) \end{bmatrix} + c_2\, e^{-2t} \begin{bmatrix} -\cos(t) + 3\sin(t) \\ 2\cos(t) + 4\sin(t) \end{bmatrix}$

C. $c_1\, e^{-2t} \begin{bmatrix} 3\cos(t) - \sin(t) \\ 4\cos(t) + 2\sin(t) \end{bmatrix} + c_2\, e^{-2t} \begin{bmatrix} -\cos(t) - 3\sin(t) \\ 2\cos(t) - 4\sin(t) \end{bmatrix}$

D. $c_1\, e^{-2t} \begin{bmatrix} 3\cos(t) + \sin(t) \\ 4\cos(t) - 2\sin(t) \end{bmatrix} + c_2\, e^{-2t} \begin{bmatrix} -\cos(t) + 3\sin(t) \\ 2\cos(t) + 4\sin(t) \end{bmatrix}$

E. $c_1\, e^{-2t} \begin{bmatrix} 3\cos(t) - \sin(t) \\ 4\cos(t) - 2\sin(t) \end{bmatrix} + c_2\, e^{-2t} \begin{bmatrix} -\cos(t) - 3\sin(t) \\ 2\cos(t) + 4\sin(t) \end{bmatrix}$

19. Find a least-squares solution for $A\mathbf{x} = \mathbf{b}$ where $A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 2 & 2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}$.

A. $\begin{bmatrix} -2 \\ -1 \end{bmatrix}$

B. $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

C. $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

D. $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

E. $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

20. Which of the following matrices is NOT diagonalizable over the real numbers?

A. $\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$

B. $\begin{bmatrix} 1 & 3 & 5 \\ 0 & 4 & 1 \\ 0 & 1 & 4 \end{bmatrix}$

C. $\begin{bmatrix} 1 & -1 & 3 \\ 0 & 2 & -3 \\ 0 & 0 & 10 \end{bmatrix}$

D. $\begin{bmatrix} 5 & 3 & 4 \\ 3 & 2 & 1 \\ 4 & 1 & -2 \end{bmatrix}$

E. $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 8 & 7 \\ 0 & 0 & 8 \end{bmatrix}$

21. Let $\mathbf{v_1} = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}$, $\mathbf{v_2} = \begin{bmatrix} -5 \\ -4 \\ 3 \end{bmatrix}$, and $\mathbf{v_3} = \begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}$ be an orthogonal set that spans $\mathbb{R}^3$. Write the vector $\mathbf{x} = \begin{bmatrix} 10 \\ -10 \\ 20 \end{bmatrix}$ as $c_1\mathbf{v_1} + c_2\mathbf{v_2} + c_3\mathbf{v_3}$. What is c_2?

A. 1

B. 3

C. 5

D. 7

E. 9

22. Suppose that $A = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 4 & 0 \\ 1 & 0 & 3 \end{bmatrix} = PDP^T$ where D is a diagonal matrix and P is an orthogonal matrix. One choice of P and D that satisfies the condition is

A. $D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, $P = \begin{bmatrix} 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

B. $D = \begin{bmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -2 \end{bmatrix}$, $P = \begin{bmatrix} 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

C. $D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, $P = \begin{bmatrix} 0 & -1/\sqrt{10} & 3/\sqrt{10} \\ 1 & 0 & 0 \\ 0 & 3/\sqrt{10} & 1/\sqrt{10} \end{bmatrix}$

D. $D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, $P = \begin{bmatrix} 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

E. $D = \begin{bmatrix} -4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$, $P = \begin{bmatrix} 0 & -1/\sqrt{10} & 3/\sqrt{10} \\ 1 & 0 & 0 \\ 0 & 3/\sqrt{10} & 1/\sqrt{10} \end{bmatrix}$

23. Which of the following statements is always **TRUE?** (All matrices are real matrices.)

(i) If A is a symmetric matrix, then A always has n distinct eigenvalues.

(ii) If A is symmetric, then A^2 is orthogonally diagonalizable.

(iii) If both A and B are orthogonally diagonalizable, then so is AB.

(iv) If A is symmetric, then any two eigenvectors from different eigenspaces are orthogonal.

A. (i) only

B. (ii) only

C. (i) and (iv) only

D. (ii) and (iii) only

E. (ii) and (iv) only

24. For f and g in $C[0,1]$, set

$$\langle f, g \rangle = \int_0^1 f(t)g(t)dt.$$

If one performs Gram-Schmidt Process to the polynomials t^2 and 1, the resulting orthogonal basis of $span\{t^2, 1\}$ is

A. $\{t^2, 1 - \frac{4}{3}t^2\}$

B. $\{t^2, 1 - \frac{3}{5}t^2\}$

C. $\{t^2, 1 - \frac{5}{3}t^2\}$

D. $\{t^2, 1 - \frac{3}{4}t^2\}$

E. $\{t^2, 1 - \frac{5}{4}t^2\}$

25. Let W be the subspace of $\mathbb{R}^3$ spanned by

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

If we apply the Gram-Schmidt procedure to this set to obtain an orthonormal basis for W, we obtain the set

A. $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

B. $\left\{ \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}, \begin{bmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \right\}$

C. $\left\{ \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} -2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \right\}$

D. $\left\{ \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \right\}$

E. $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \right\}$