MA 266 Final Exam

Fall 2008, version 1

Print YOUR LAST NAME: ____________________ FIRST NAME: ____________________

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Instructions:

1. This exam contains 21 pages, including the cover page and a table of Laplace transforms. The last two pages are left intentially blank, which you may use as scrap paper.
2. This exam consists of two parts: (a) 17 Multiple Choice Questions and (b) 7 Written Answer Questions:
   (a) Each of Problems # 1–17 contains a multiple choice question. Circle the answers to the Multiple Choice Questions and fill in the corresponding answers on your scantron. Each question worths 10 points.
   (b) Each of Problems # 11–17 contains an additional written answer question, labelled **Written Answer Question (a)–Written Answer Question (g)**. Show *detailed work* for these questions below the questions and circle your final answers. No credit will be given to no work. Your instructor will hand grade these questions and their weights were previously announced in class.

   Turn in *both* the scantron and the exam to your instructor in the end of the exam.
3. Put away books, notes, calculators, cell phones, i-pods and other electronic devices. No discussion during the exam.

1. Find the Laplace transform of

$$f(t) = \begin{cases} 0 & \text{when } t < \pi, \\ t - \pi & \text{when } \pi \le t < 2\pi, \\ 0 & \text{when } t \ge 2\pi. \end{cases}$$

A. $e^{-\pi s}\dfrac{1}{s^2} - e^{-2\pi s}\dfrac{1}{s^2} - \pi e^{-2\pi s}\dfrac{1}{s}$

B. $e^{-\pi s}\dfrac{1}{s^2} - e^{-2\pi s}\dfrac{1}{s^2}$

C. $e^{\pi s}\dfrac{1}{s^2} - e^{2\pi s}\dfrac{1}{s^2} - \pi e^{2\pi s}\dfrac{1}{s^2}$

D. $\dfrac{1}{s}\left(e^{-\pi s} - e^{-2\pi s}\right)$

E. None of the above.

2. Which of the following is an implicit solution to the initial value problem

$$\big(x^2 - 3y^2 - \sin(x+y)\big)y' + 2xy - \sin(x+y) = 0, \quad y(-1) = 1?$$

A. $x^3/3 - 3xy^2 + \cos(x+y) = 14/3$,

B. $x^2y + \cos(x+y) - y^3 = 1$,

C. $xy^2 + \cos(x+y) = 0$,

D. $x^2y - \cos(x+y) + 2 = 0$,

E. None of the above.

3. Find the Laplace transform of the function

$$f(t) = \int_0^t \cos(t-\tau)e^{\tau} \sin(\tau)\, d\tau.$$

A. $\dfrac{s}{(s^2+1)((s-1)^2+1)}$,

B. $\dfrac{s-1}{(s^2+1)((s-1)^2+1)}$,

C. $\dfrac{s}{((s-1)^2+1)^2}$,

D. $\dfrac{s-1}{((s-1)^2+1)^2}$,

E. $\dfrac{s}{(s^2+1)^2}$.

4. How many of the following differential equations are linear?

(1) $ty' + y - t^2 = 0$.

(2) $ty' + t - y^2 = \sin t$.

(3) $ty' + y = \sin y$.

(4) $ty'' + (\sin t)y = e^t$.

(5) $yy'' = \cos t$.

(6) $y' = y^{-1}$.

Answer:

A. None,

B. 1,

C. 2,

D. 3,

E. more than 3.

5. How many asymptotically unstable equilibrium solution(s) does the following differential equation have?

$$y' = (y^2 + 1)(y^2 - 1)(y + 2).$$

A. 0,

B. 1,

C. 2,

D. 3,

E. None of the above.

6. The largest interval in which the solution of the initial value problem

$$(\cos t)y'' + t^2 y' - \frac{5}{t}y = \frac{e^t}{t-3}, \quad y(1) = 2, \quad y'(1) = 0,$$

is guaranteed to exist by the Existence and Uniqueness Theorem is:

A. $(0, \infty)$

B. $\left(\frac{\pi}{2}, 3\right)$

C. $\left(0, \frac{\pi}{2}\right)$

D. $(0, \pi)$

E. $(0, 3)$

7. Find the general solution of the system

$$\mathbf{x}' = \begin{pmatrix} -3 & -4 \\ 1 & 1 \end{pmatrix} \mathbf{x}.$$

A. $c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{t} + c_2 \left[\begin{pmatrix} 2 \\ 1 \end{pmatrix} te^{t} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{t} \right],$

B. $c_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-t} + c_2 \left[\begin{pmatrix} -2 \\ 1 \end{pmatrix} te^{-t} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} e^{-t} \right],$

C. $c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-t} + c_2 \left[\begin{pmatrix} 2 \\ 1 \end{pmatrix} te^{-t} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-t} \right],$

D. $c_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-t} + c_2 \left[\begin{pmatrix} -2 \\ 1 \end{pmatrix} te^{-t} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-t} \right],$

E. $c_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{t} + c_2 \left[\begin{pmatrix} -2 \\ 1 \end{pmatrix} te^{t} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{t} \right].$

8. Which of the following is the general solution to $y'' + 4y = e^{2t} + 12\sin(2t)$?

A. $y(t) = c_1 \cos(2t) + c_2 \sin(2t) + \frac{1}{8}e^{2t} - 3t\cos(2t)$

B. $y(t) = c_1 e^{2t} + c_2 e^{-2t} + \frac{1}{4}t\,e^{2t} - 3t\cos(2t)$

C. $y(t) = c_1 + c_2 e^{-4t} + \frac{1}{12}\,te^{2t} - 3t\cos(2t)$

D. $y(t) = c_1 \cos(2t) + c_2 \sin(2t) + \frac{1}{8}e^{2t} + 3\sin(2t)$

E. None of the above.

9. According to the method of undetermined coefficients, what is the proper form of a particular solution Y to the following differential equation?

$$y^{(4)} - 4y'' = 24t^2 - 4 - 3te^t.$$

A. $Y(t) = At^2 + Bte^t$.

B. $Y(t) = At^2 + Bt + C + Dte^t + Ee^t$.

C. $Y(t) = At^3 + Bt^2 + Ct + D + Ete^t + Fe^t$.

D. $Y(t) = At^4 + Bt^3 + Ct^2 + Dte^t + Ee^t$.

E. None of the above.

10. A mass weighing 4 lb, when hung from the ceiling, stretches a spring by 6 inches (in) at equilibrium. The mass is attached to a viscous damper with a damping constant of 5 lb·sec/ ft, and is set in motion from its equilibrium position with a upward velocity of 2 in/sec. Suppose that t seconds later, the mass is displaced from its equilibrium position by $u(t)$ ft, taking the downward direction as the positive direction. Then $u(t)$ satisfies the initial value problem:

A. $\frac{1}{8}u'' + 5u' + 8u = 0, \quad u(0) = 0, \quad u'(0) = -\frac{1}{6}$

B. $\frac{1}{8}u'' + 5u' + 8u = 0, \quad u(0) = 0, \quad u'(0) = \frac{1}{6}$

C. $4u'' + 5u' + \frac{20}{3}u = 0, \quad u(0) = 0, \quad u'(0) = -2$

D. $4u'' + 5u' + 8u = 0, \quad u(0) = 0, \quad u'(0) = -2$

E. $u'' + 40u' + 64u = 0, \quad u(0) = 0, \quad u'(0) = -\frac{1}{6}$

Hint: The gravitional constant $g = 32$ ft/sec^2. One feet is 12 inches.

11. Which of the following functions below is a particular solution to the differential equation

$$y'' + y = \frac{1}{\cos t}?$$

A. $y(t) = \frac{1}{2}e^t \int \frac{e^{-t}}{\cos t} dt - \frac{1}{2}e^{-t} \int \frac{e^t}{\cos t} dt$

B. $y(t) = \cos^2 t + t \sin t$

C. $y(t) = \ln|\cos t| + t \sin t$

D. $y(t) = \cos t \, \ln|\cos t| + t \sin t$

E. None of the above.

Written Answer Question (a): Find the *general* solution to the above differential equation using the method of variation of parameters.

Hint: $\displaystyle\int (\cos t)^n \sin t \, dt = \begin{cases} -\dfrac{(\cos t)^{n+1}}{n+1} + C & \text{when } n \neq -1 \\ -\ln|\cos t| + C & \text{when } n = -1 \end{cases}.$

12. Consider the initial value problem for $y = y(t)$, $t > 0$:

$$ty' + y - t^2 = 0, \quad y(1) = 1.$$

Then $y(2) =?$

A. $\frac{5}{4}$,

B. $-\frac{5}{3}$,

C. $\frac{5}{3}$,

D. $-22 + 15\, e^{-1}$.

E. None of the above.

Written Answer Question (b): Write down your solution for $y(t)$.

13. Find the solution of the initial value problem

$$y'' - 3y' + 2y = \delta(t-2); \quad y(0) = 0, \quad y'(0) = 1.$$

A. $-e^{2t} + e^t + e^t\, u_2(t)\,(e^{2t-4} + e^{t-2})$,

B. $e^{2t} + e^t + u_2(t)(e^{2t-4} - e^{t-2})$,

C. $e^{2t} - e^t + u_2(t)(e^{2t-4} - e^{t-2})$,

D. $e^{2t} - e^t + u_2(t)(e^{2t-4} + e^{t-2})$,

E. $-e^{2t} + e^t + u_2(t)\, e^{t-2}$.

Written Answer Question (c): What is $\mathcal{L}\{y\}$?

14. A ball of mass $m = 0.1$ kg is dropped from a tower of 100 m high. (The intial velocity is 0). Suppose the magnitude of the air resistance force is $mg(2|v|/10 - v^2/100)$ kg$\cdot$ m/sec^2. Which of the following function describes the velocity of the ball? (The gravitational constant is g =9.8 m/sec^2).

A. $v(t) = \dfrac{10gt}{10+gt}$ m/sec.

B. $v(t) = gt$ m/sec.

C. $v(t) = 5(1 - e^{-gt/5})$ m/sec.

D. $v(t) = 5(1 + e^{gt/5})$ m/sec.

E. None of the above.

Written Answer Question (d): Write down the initial value problem describing $v(t)$.

15. Which of the following is a particular solution to the system $\mathbf{x}' = \mathbf{A}\mathbf{x} + e^{2t}\mathbf{g}$, where

$$\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}, \qquad \mathbf{g} = \begin{pmatrix} -1 \\ -3 \end{pmatrix}.$$

A. $(e^t - 2e^{2t}/3)\begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{2t}\begin{pmatrix} 1/9 \\ -1/3 \end{pmatrix}$,

B. $(e^t - e^{2t})\begin{pmatrix} 1 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}$,

C. $\begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t} - \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{2t}/2$,

D. $\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t} - \begin{pmatrix} 1/3 \\ 1 \end{pmatrix} e^{2t}$,

E. None of the above.

Written Answer Question (e): Write down the *general* solution to the above system.

Hint: $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ are eigenvectors of $\mathbf{A}$ with eigenvalues 1 and -1 respectively.

16. The function $y_1 = e^x$ is a solution to

$$(x-1)y'' - 2xy' + (x+1)y = 0,\ x > 1.$$

Use reduction of order to find a function y_2 so that the pair $y_1,\ y_2$ form a fundamental set of solutions to the differential equation.

A. $y_2 = (x-1)^3$,

B. $y_2 = (x-1)^3 e^x$,

C. $y_2 = (x-1)^2$,

D. $y_2 = (x-1)^2 e^x$

E. None of the above.

Written Answer Question (f): Show detailed solution for the above problem using the reduction of order method.

17. In the phase portrait of the system

$$\mathbf{x}' = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix} \mathbf{x},$$

the origin is a:

A. saddle point.

B. asymptotically stable node.

C. asymptotically unstable node.

D. asymptotically stable spiral point.

E. asymptotically unstable spiral point.

Written Answer Question (g): Find the general solution to the above system.

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