# MA26600 Final Exam

GREEN VERSION 01

NAME: ______________________________

INSTRUCTOR: ______________________ SECTION/TIME: ____________

- You must use a **#2 pencil** on the mark-sense answer sheet.
- Fill in the **ten digit PUID** (starting with two zeroes) and your **Name** and blacken in the appropriate spaces.
- Fill in the correct **Test/Quiz number** (GREEN is **01**, ORANGE is **02**)
- Fill in the **four digit section number** of your class and blacken the numbers below them. Here they are:

| | | | |
|---|---|---|---|
| 0010 | MWF | 9:30AM | Milana Golich |
| 0022 | MWF | 3:30PM | Iryna Egorova |
| 0034 | MWF | 1:30PM | Ping Xu |
| 0035 | MWF | 10:30AM | Ping Xu |
| 0046 | MWF | 11:30AM | Ping Xu |
| 0061 | MWF | 11:30AM | Yi Wang |
| 0062 | MWF | 12:30PM | Yi Wang |
| 0071 | MWF | 1:30PM | Kaitlyn Hood |
| 0072 | MWF | 10:30AM | Taylor Daniels |
| 0083 | MWF | 10:30AM | Antoine Prouff |
| 0084 | MWF | 8:30AM | Antoine Prouff |
| 0096 | MWF | 1:30PM | Iryna Egorova |
| 0107 | MWF | 2:30PM | Po Chun Kuo |
| 0109 | MWF | 12:30PM | Iryna Egorova |
| 0110 | MWF | 3:30PM | Gert Vercleyen |
| 0111 | MWF | 9:30AM | Taylor Daniels |
| 0112 | MWF | 9:30AM | Po Chun Kuo |
| 0113 | MWF | 4:30PM | Gert Vercleyen |
| 0115 | MWF | 11:30AM | Gayane Poghotanyan |
| 0116 | MWF | 9:30AM | Gayane Poghotanyan |
| 0117 | TR | 1:30PM | Ning Wei |

- Sign the mark-sense sheet.
- Fill in your name and your instructor's name and the time of your class meeting on the exam booklet above.
- There are 20 multiple-choice questions, each worth 10 points. **Blacken in** your choice of the correct answer in the spaces provided for questions 1–20 in the answer sheet. Do all your work on the question sheets, in addition, also **Circle** your answer choice for each problem on the question sheets in case your mark-sense sheet is lost.
- Show your work on the question sheets. Although no partial credit will be given, any disputes about grades or grading will be settled by examining your written work on the question sheets.
- **No calculators, books, electronic devices, or papers are allowed.** Use the back of the test pages for scratch paper.
- Pull off the **table of Laplace transforms** on the last page of the exam for reference. Do not turn it in with your exam booklet at the end.

**1.** A population $P(t)$ is modeled by the logistic equation

$$P' = \frac{1}{12}P(12 - P)$$

and initial value $P(0) = 3$. What will the population $P(t)$ be when $t = \ln(3)$?

A. 3

B. 12

C. 6

D. 36

E. 9

**2.** Consider a first-order equation

$$\frac{dx}{dt} = (x+1)^4 x^3 (x-1)(x-2)^6 (x-3)^5.$$

Which of the following statements about equilibrium points is TRUE?

A. 2 is an unstable point, and 3 is a stable point.

B. $1, 3$ are semistable points.

C. $-1, 2$ are stable points.

D. 0 is an unstable point, and 1 is a stable point.

E. $0, 3$ are stable points.

**3.** Which of the following is an implicit solution to the initial value problem

$$(\pi e^{\pi x} - y + \sin y)dx + (x\cos y - x + e^{-y})dy = 0, \quad y(1) = -\pi.$$

A. $x\sin y - yx + e^{\pi x} + e^{-y} = \pi + 2e^{\pi}$

B. $x\sin y - e^{-y} + e^{\pi x} - yx = \pi$

C. $x\cos y - xy - e^{-y} + \pi e^{\pi x} = (e^{\pi} + 1)(\pi - 1)$

D. $e^{\pi x} - xy + x\sin y - e^{-y} = -\pi$

E. $e^{\pi x} - xy - x\cos y - e^{-y} = \pi + 1$

**4.** A tank of total volume 100 L initially contains 30 L of pure water. A brine solution containing 0.5 kg/L of salt is pumped into the tank at a rate of 10 L/min, and the well-mixed solution in the tank is pumped out at a constant rate. The tank is full after 10 minutes. Let $x(t)$ be the amount of salt (in kg) at time $t$ (in min) before the tank is full. Which differential equation does $x(t)$ satisfy?

[*Hint:* Find the flow out rate first.]

A. $\dfrac{dx}{dt} = 5 - \dfrac{3x}{70 + 3t}$

B. $\dfrac{dx}{dt} = 50 - \dfrac{x}{30 + 7t}$

C. $\dfrac{dx}{dt} = 5 + \dfrac{30x}{100}$

D. $\dfrac{dx}{dt} = 0.5 + \dfrac{3x}{100 - 3t}$

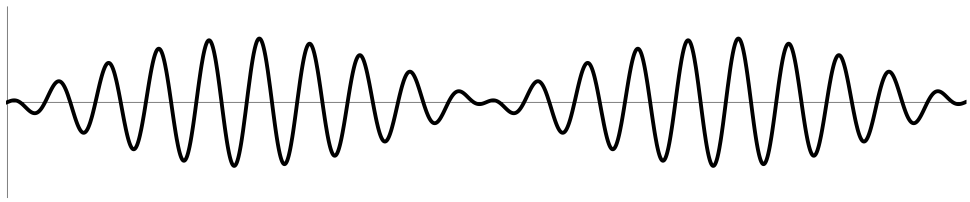
E. $\dfrac{dx}{dt} = 5 - \dfrac{3x}{30 + 7t}$

**5.** Which of the following graphs sketches a solution of the differential equation

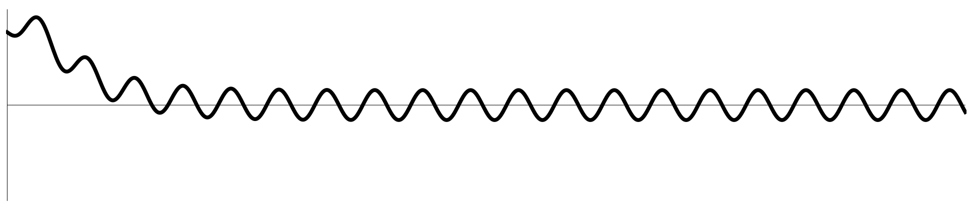
$$y'' + 4y' + 3y = \sin(x)?$$

(In the graphs below, the horizontal axis is the $x$-axis and the vertical axis is the $y$-axis.)

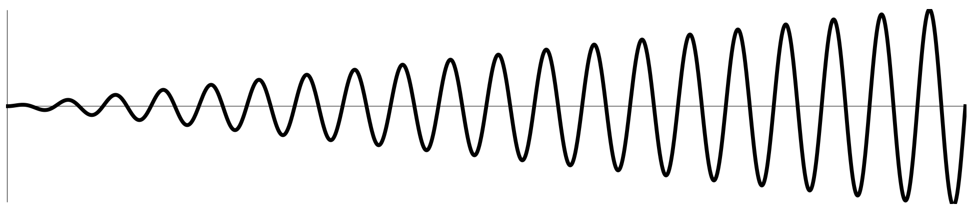
A.



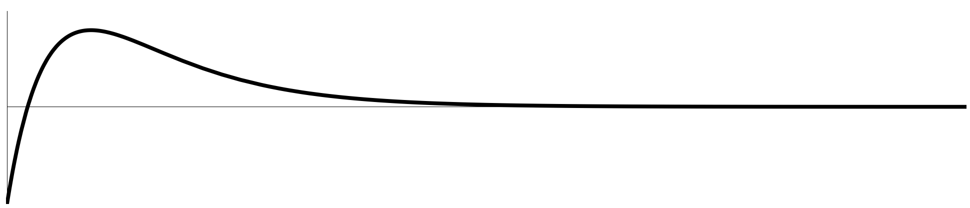
B.



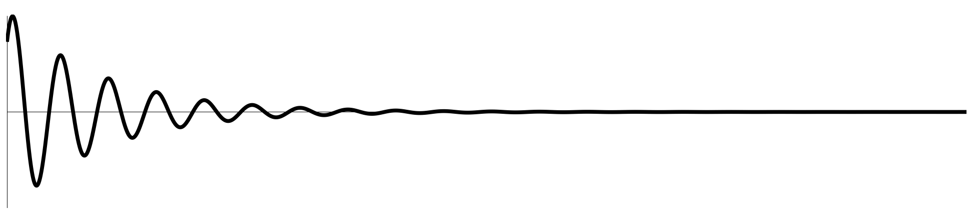
C.



D.



E.



**6.** If $\dfrac{du}{dx} = 3x^2u - u$ and $u(0) = 3$, find $u(2)$.

A. $3e^6$

B. $1 + \dfrac{1}{e^3}$

C. $3 + e^2$

D. $-3 + e^8 - e^2$

E. $\dfrac{1}{e^3}$

**7.** Find the general solution of the equation

$$y'' - 2y' + y = \frac{2e^t}{1+t^2}.$$

A. $y(t) = C_1e^t \ln(t^2+1) + C_2te^t \tan^{-1} t$

B. $y(t) = C_1e^t + C_2te^t - te^t \ln(t^2+1) + 2e^t \tan^{-1} t$

C. $y(t) = C_1e^t + C_2te^t + \dfrac{e^t}{1+t^2}$

D. $y(t) = C_1e^t + C_2te^t - e^t \ln(t^2+1) + 2te^t \tan^{-1} t$

E. $y(t) = C_1e^t(1 - \ln(t^2+1)) + C_2te^t(1 + 2\tan^{-1} t)$

**8.** A body with a mass of 250 g is attached to the end of a spring that is stretched 50 cm by a force of 2 N. There is no damping. The body is set in motion by pulling it from the equilibrium position and letting it go. Find the period of motion $T$ (in seconds) of the body.

(Recall that a force of 1 N gives a mass of 1 kg an acceleration of 1 m/s$^2$; 1 kg = 1000 g; 1 m = 100 cm.)

A. $T = \pi$
B. $T = \dfrac{\pi}{2}$
C. $T = 2\pi$
D. $T = \dfrac{\pi}{4}$
E. $T = 4\pi$

**9.** Determine the appropriate form for a particular solution $y_p$ of the following fifth order nonhomogeneous equation, using the method of undetermined coefficients:

$$y^{(5)} + 6y^{(4)} + 13y''' + 14y'' + 12y' + 8y = x(e^{-2x} + \cos x).$$

Use that $r^5 + 6r^4 + 13r^3 + 14r^2 + 12r + 8 = (r+2)^3(r^2+1)$.

A. $y_p = x^3\big[(Ax + B)e^{-2x} + (Cx + D)\cos x + (Ex + F)\sin x\big]$
B. $y_p = (Ax + B)e^{-2x} + (Cx + D)\cos x + (Ex + F)\sin x$
C. $y_p = (Ax^3 + Bx^2)e^{-2x} + (Cx^2 + Dx)\cos x + (Ex^2 + Fx)\sin x$
D. $y_p = (Ax^4 + Bx^3)e^{-2x} + (Cx^2 + Dx)\cos x$
E. $y_p = (Ax^4 + Bx^3)e^{-2x} + (Cx^2 + Dx)\cos x + (Ex^2 + Fx)\sin x$

**10.** Let $y$ be the solution to the initial value problem

$$y''' + 4y' = 0; \qquad y(0) = 1,\ y'(0) = 2,\ y''(0) = 3.$$

What is the value of $y(\pi/2)$?

A. $5/2$

B. $3/2$

C. $1/2$

D. $7/2$

E. $-1/2$

**11.** Consider

$$\mathbf{x}(t) = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t + C_2 \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) e^t.$$

Then $\mathbf{x}(t)$ is a general solution of the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ with

A. $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

B. $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

C. $\mathbf{A} = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$

D. $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

E. $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$

**12.** Find the appropriate form of a particular solution $\mathbf{x}_p(t)$ of the nonhomogeneous linear system

$$\mathbf{x}' = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} te^t \\ e^{2t} \end{bmatrix}$$

by using the method of undetermined coefficients.

A. $\mathbf{x}_p(t) = (\mathbf{a}t + \mathbf{b})e^t + \mathbf{c}e^{2t}$

B. $\mathbf{x}_p(t) = (\mathbf{a}t + \mathbf{b})e^t + (\mathbf{c}t + \mathbf{d})e^{2t}$

C. $\mathbf{x}_p(t) = (\mathbf{a}t + \mathbf{b})e^t + \mathbf{c}te^{2t}$

D. $\mathbf{x}_p(t) = (\mathbf{a}t^2 + \mathbf{b}t + \mathbf{c})e^t + \mathbf{d}e^{2t}$

E. The system has no solution, so no form is appropriate.

**13.** What value(s) (or range of values) for the parameter $b$ cause the system

$$\mathbf{x}' = \begin{bmatrix} 1 & 2b \\ -1 & 1 \end{bmatrix} \mathbf{x}$$

to have a phase-plane portrait that is a saddle-point?

A. $b = -\frac{1}{2}$ and $b = \frac{1}{2}$

B. $-\frac{1}{2} < b < 0$

C. $b < -\frac{1}{2}$

D. $b = 0$

E. $0 < b < \frac{1}{2}$

**14.** Consider a $2 \times 2$ matrix $\mathbf{A} = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$. Then, a general solution of the linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is

A. $\mathbf{x} = c_1 \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} e^{2t}$.

B. $\mathbf{x} = c_1 \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} + c_2 \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$.

C. $\mathbf{x} = c_1 \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} e^{-2t}$.

D. $\mathbf{x} = c_1 \begin{bmatrix} \sin t \\ \cos t \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} e^{2t}$.

E. $\mathbf{x} = c_1 \begin{bmatrix} \sin t \\ \cos t \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} e^{-2t}$.

**15.** Determine the trajectory and the direction of motion for the following system in $xy$-plane.

$$\begin{cases} x' = -2y, \\ y' = \dfrac{1}{2}x. \end{cases}$$

A. Circle, clockwise

B. Ellipse, clockwise

C. Ellipse, counterclockwise

D. Parabola, counterclockwise

E. None of the above

**16.** Find the Laplace transform of the function $f(t) = \begin{cases} 0, & t < 2\pi, \\ e^t \sin t, & t \geq 2\pi. \end{cases}$

A. $\dfrac{-e^{-\pi s}}{s^2 + \pi^2}$

B. $\dfrac{-e^{-2\pi s + 2\pi}}{(s+1)^2 + 1}$

C. $\dfrac{e^{-2\pi s + \pi}(s-1)}{(s-1)^2 + 1}$

D. $\dfrac{e^{-2\pi s + 2\pi}}{(s-1)^2 + 1}$

E. $\dfrac{e^{-2\pi s + \pi}}{(s-1)^2 + 1}$

**17.** After applying the Laplace Transform to the differential equation:

$$x'' + 4x' + 3x = 2e^{-3t}; \qquad x(0) = a,\ x'(0) = b,$$

one obtains that $X(s) = \mathcal{L}\{x(t)\}$ is given by the algebraic formula:

$$X(s) = \frac{2}{(s+1)(s+3)^2} + \frac{3s+8}{(s+1)(s+3)}.$$

Find the values of $a$ and $b$.

A. $a = 0,\ b = 3$

B. $a = 1,\ b = -1$

C. $a = 3,\ b = -8$

D. $a = 1,\ b = -9$

E. $a = 3,\ b = -4$

**18.** Which of the following functions solves the initial value problem

$$y''' = f(t); \quad y(0) = y'(0) = y''(0) = 0.$$

[*Hint:* Use the Laplace Transform.]

A. $y(t) = \dfrac{1}{2}\displaystyle\int_0^t \tau^2 f(t-\tau)d\tau$

B. $y(t) = \dfrac{1}{6}\displaystyle\int_0^t \tau^3 f(t-\tau)d\tau$

C. $y(t) = \dfrac{1}{6}\displaystyle\int_0^t \tau^2 f(t-\tau)d\tau$

D. $y(t) = 2\displaystyle\int_0^t \tau^2 f(t-\tau)d\tau$

E. $y(t) = 6\displaystyle\int_0^t \tau^3 f(t-\tau)d\tau$

**19.** Consider an initial value problem

$$x'' + x = \delta\,(t-\pi) - 5\delta\,(t-2\pi)\,; \quad x(0)=0,\ x'(0)=0.$$

Find the value of $x\left(\frac{3\pi}{2}\right)$.

A. 6

B. 5

C. $-4$

D. 1

E. $-5$

**20.** Find the Inverse Laplace Transform of $F(s) = \dfrac{3s+1}{(s-3)^2+16}$.

A. $e^{3t}\left[3\cos(4t) + \sin(4t)\right]$

B. $e^{3t}\left[3\cos(4t) + \dfrac{5}{2}\sin(4t)\right]$

C. $e^{4t}\left[\dfrac{5}{2}\cos(3t) + 3\sin(3t)\right]$

D. $e^{-3t}\left[\dfrac{5}{2}\cosh(4t) - 3\sinh(4t)\right]$

E. $e^{3t}\left[3\cos(4t) + 10\sin(4t)\right]$

# Table of Laplace Transforms

| | $f(t) = \mathcal{L}^{-1}\{F(s)\}$ | $F(s) = \mathcal{L}\{f(t)\}$ |
|---|---|---|
| 1. | $1$ | $\dfrac{1}{s}, \quad s > 0$ |
| 2. | $e^{at}$ | $\dfrac{1}{s-a}, \quad s > a$ |
| 3. | $t^n, \quad n = \text{positive integer}$ | $\dfrac{n!}{s^{n+1}}, \quad s > 0$ |
| 4. | $t^p, \quad p > -1$ | $\dfrac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$ |
| 5. | $\sin at$ | $\dfrac{a}{s^2+a^2}, \quad s > 0$ |
| 6. | $\cos at$ | $\dfrac{s}{s^2+a^2}, \quad s > 0$ |
| 7. | $\sinh at$ | $\dfrac{a}{s^2-a^2}, \quad s > \|a\|$ |
| 8. | $\cosh at$ | $\dfrac{s}{s^2-a^2}, \quad s > \|a\|$ |
| 9. | $e^{at} \sin bt$ | $\dfrac{b}{(s-a)^2+b^2}, \quad s > a$ |
| 10. | $e^{at} \cos bt$ | $\dfrac{s-a}{(s-a)^2+b^2}, \quad s > a$ |
| 11. | $t^n e^{at}, \quad n = \text{positive integer}$ | $\dfrac{n!}{(s-a)^{n+1}}, \quad s > a$ |
| 12. | $u(t-c)$ | $\dfrac{e^{-cs}}{s}, \quad s > 0$ |
| 13. | $u(t-c)f(t-c)$ | $e^{-cs}F(s)$ |
| 14. | $e^{ct}f(t)$ | $F(s-c)$ |
| 15. | $f(ct)$ | $\dfrac{1}{c} F\left(\dfrac{s}{c}\right), \quad c > 0$ |
| 16. | $\displaystyle\int_0^t f(t-\tau)\, g(\tau)\, d\tau$ | $F(s)\, G(s)$ |
| 17. | $\delta(t-c)$ | $e^{-cs}$ |
| 18. | $f^{(n)}(t)$ | $s^n F(s) - s^{n-1} f(0) - \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0)$ |
| 19. | $t^n f(t)$ | $(-1)^n F^{(n)}(s)$ |