MA 161M EXAM 3 Nov. 21, 2002

Name *Key*

(10 pts) 1) Find the absolute maximum and minimum of $f(x) = x/(x+1)$ on the interval $[0, 2]$.

$f'(x) = \frac{(1+x) - x}{(1+x)^2} = \frac{1}{1+x^2} > 0$ on $[0,2]$

$\therefore$ f is increasing on $[0,2]$. $\therefore$ Min occurs at $x=0$ Max occurs at $x=2$

$f(0) = 0 \quad f(2) = 2/3$

MIN 0 at $x=0$ MAX $2/3$ at $x=2$

(10 pts) 2) Show that $3x^5 + 30x + 5 = 0$ has a root in the interval $(-1, 0)$ and that this is the only real root.

Let $f(x) = 3x^5 + 30x + 5$. Then $f(0) = 5 \quad f(-1) = -3 - 30 + 5 = -28$

$\therefore$ By intermediate value theorem there is a root in $(-1,0)$

If there were a root at another point c, then $f'(c) = 0$

But $f'(x) = 15x^4 + 30 = 15(x^4+2)$ has NO root

$\therefore$ there is only one root

3) Find

(5 pts) (a) $\lim_{x\to 0} \frac{\sin x}{e^x}$.

$\lim_{x\to 0} \sin x = 0$

$\lim_{x\to 0} e^x = 1$

$\therefore \lim_{x\to 0} \frac{\sin x}{e^x} = \frac{0}{1} = 0$

Ans. 0

(10 pts) (b) $\lim_{x\to 1} x^{\frac{1}{1-x}}$.

$\lim_{x\to 1} (x) = 1 \quad \lim_{x\to 1} \frac{1}{1-x} = \pm\infty$

$\therefore$ Apply L'Hospital.

Let $y = x^{\frac{1}{1-x}}$ Then $\ln y = \frac{1}{1-x} \ln x$

and $\lim_{x\to 1} (\ln y) = \lim_{x\to 1} \frac{\ln x}{1-x} = \lim_{x\to 1} \frac{\frac{1}{x}}{-1} = -1$

Ans. e^{-1}

$\therefore \lim_{x\to 1} y = \lim_{x\to 1} e^{\ln y} = e^{-1}$

4) Let $f(x) = xe^x$.

(5 pts) (a) Find asymptotes, if any of f. As $x \to +\infty$, $e^x \to +\infty$ so $xe^x \to +\infty$

As $x \to -\infty$, $e^{+x} \to 0$ so write $xe^x = \frac{x}{e^{-x}}$ so as $x \to -\infty$ $x \to -\infty$ $e^{-x} \to +\infty$

$\therefore$ Apply L'Hopital's rule + get

$\lim_{x \to -\infty} xe^x = \lim_{x \to -\infty} \frac{x}{e^{-x}} = \lim_{x \to -\infty} \frac{1}{-e^{-x}} = 0$; Moreover $xe^x \to 0$ through negative values

Ans. $y = 0$ (x-axis)

(5 pts) (b) Find intervals of increase and decrease of f and find local maxima and minima, if any.

$f'(x) = xe^x + e^x = e^x(x+1)$ $\therefore f'(x) < 0$ for $x < -1$ (interval of decrease)

$f'(x) > 0$ for $x > -1$ (" " increase)

$f'(-1) = 0 \Rightarrow$ ~~Rel.~~ (local min)

$f(-1) = -e^{-1}$

No local max

INCREASE $x > -1$ DECREASE $x < -1$ MAX NONE MIN $(-1, -e^{-1})$

(5 pts) (c) Find intervals of concavity and inflection points, if any.

$f''(x) = e^x(x+1) + e^x = e^x(x+2)$

$f''(x) < 0$ if $x < -2$ (concave down)

$f''(x) > 0$ if $x > -2$ (" up)

$f''(-2) = 0$ $f(-2) = -2e^{-2}$

INFLECTION PT. $(-2, -2e^{-2})$

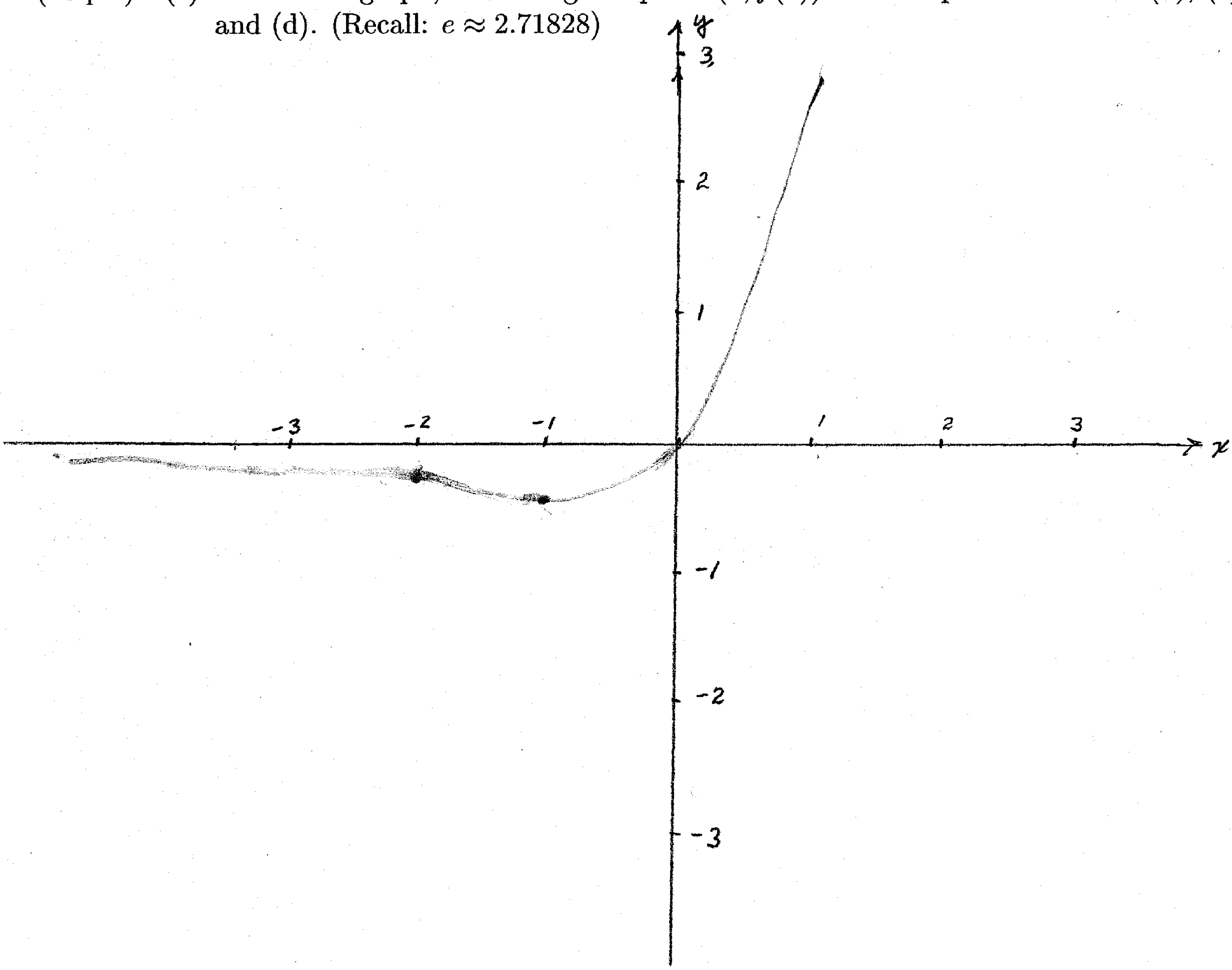
CONCAVE UP $x > -2$ CONCAVE DOWN $x < -2$ INFLECTION PTS $(-2, -2e^{-2})$

5 pts) (d) Find all intercepts.

y - intercept. $x = 0 \Rightarrow y = 0$
x intercept $y = 0 \Rightarrow x = 0$
only intercept is origin

Ans. $(0,0)$

(10 pts) (e) Sketch the graph, indicating the point $(1, f(1))$ and the points found in (b), (c) and (d). (Recall: $e \approx 2.71828$)



(15 pts) 5) Find the two points on the parabola $2y = x^2 - 8$ that are closest to the point $(0, 4)$.

$\rho = d^2 = x^2 + (y-4)^2$

Pt. on Parabola $\Rightarrow x^2 = 2y + 8$

$\therefore \rho = (2y+8) + (y-4)^2$

$\frac{d\rho}{dy} = 2 + 2(y-4)$ $\quad \frac{d^2\rho}{dy^2} = 2 > 0 \Rightarrow \frac{d\rho}{dy} = 0$ will give min

$\frac{d\rho}{dy} = 0 \Leftrightarrow 2 + 2(y-4) = 0 \Leftrightarrow (y-4) = -1$ or $y = 3$

Subs. into eqn for parabola gives

$2(3) = x^2 - 8$

or $x^2 = 14$ $\quad x = \pm\sqrt{14}$

Ans. $(\pm\sqrt{14}, 3)$

(10 pts) 6) If $g(x) = \int_{x^2}^{x^3} \sin t \, dt$, find $g'(x)$.

$g(x) = \int_{x^2}^{0} \sin t \, dt + \int_{0}^{x^3} \sin t \, dt = -\int_{0}^{x^2} \sin t \, dt + \int_{0}^{x^3} \sin t \, dt$

$g'(x) = (-2x)(\sin x^2) + 3x^2(\sin x^3)$

Ans. $3x^2 \sin x^3 - 2x \sin x^2$

(10 pts) 7) Find the value of a such that the area under the curve $y = \sin x$, $0 \le x \le \pi$ equals the area under the curve $y = e^x$, $0 \le x \le a$.

A_1 (under sine curve) $\quad A_1 = \int_0^{\pi} \sin t \, dt = -\cos t \Big|_0^{\pi} = 2$

A_2 (area under e^x) $\quad A_2 = \int_0^a e^t \, dt = e^t \Big|_0^a = e^a - 1$

For $A_1 = A_2$ require

$e^a - 1 = 2$

$e^a = 3$

$a = \ln 3$

Ans. $a = \ln 3$