MA 161/MA 161E FINAL EXAM Fall 2007

Name SOLUTIONS

ten–digit Student ID number ____________________

RECITATION Division and Section Numbers ____________________

Recitation Instructor ____________________

Instructions:

1. Fill in all the information requested above and on the scantron sheet.
2. This booklet contains 25 problems, each worth points.

   The maximum score is 200 points.
3. For each problem mark your answer on the scantron sheet and also circle it in this booklet.
4. Work only on the pages of this booklet.
5. Books, notes, calculators are not to be used on this test.
6. At the end turn in your exam and scantron sheet to your recitation instructor.

1. The graph of $y = f(x)$ is first translated 2 units to the left, then stretch vertically by a factor of 3 and then finally shrunk horizontally by a factory of 4. The resulting graph has equation

$y = f(x+2) = g(x)$

$y = 3g(x) = 3f(x+2) = h(x)$

$y = h(4x) = 3f(4x+2)$

A. $y = 3f\left(\frac{x}{4} + 2\right)$

B. $y = 3f(4x+2)$

C. $y = \frac{1}{3}f(4x+2)$

D. $y = \frac{1}{3}f\left(\frac{x}{4} + 2\right)$

E. $y = 3f(4(x+2))$

2. $\lim_{x \to 4} \frac{x^3 - 4x^2}{x^2 - x - 12} = \frac{64-64}{16-4-12} = \frac{0}{0}$

$= \lim_{x \to 4} \frac{x^2\cancel{(x-4)}}{\cancel{(x-4)}(x+3)} = \frac{4^2}{4+3} = \frac{16}{7}$

A. 0

B. 16

C. $\frac{16}{7}$

D. $\frac{2}{7}$

E. ∞

3. What is the domain of the function $f(x) = \ln(2 - \ln x)$?

$\ln(2-\ln x) \rightarrow 2-\ln x > 0$

$\rightarrow \ln x < 2$

$\rightarrow x < e^2$

Also $\ln x \rightarrow x > 0$

Therefore domain is $0 < x < e^2$

A. $0 < x < e^2$

B. $x > e$

C. $x < e^2$

D. $e < x < e^2$

E. $x > 0$

4. If $f(x) = x^2 + 3x + 1$ and $g(x) = \sqrt{x^2+5}$,then $(f \circ g)(-2) =$

$(f \circ g)(-2) = f(g(-2))$

$= f(3)$

$= 19$

A. 5

B. 3

C. $\sqrt{6}$

D. 19

E. None of these

5. Evaluate

$$\lim_{x\to\infty} \frac{1+x-3x^3}{2x^2-x^3}. \quad = \frac{-3}{-1} = 3$$

A. 3

B. $\frac{1}{2}$

C. 0

D. $-\infty$

E. ∞

OR $\lim_{x\to\infty} \frac{\cancel{x^3}\left(\frac{1}{x^3}+\frac{1}{x^2}-3\right)}{\cancel{x^3}\left(\frac{2}{x}-1\right)}$

$= \frac{0+0-3}{0-1}$

$= 3$

6. If $x = \tan y$, then $\frac{dy}{dx} =$

A. $\sec^2 y$

B. $1-\sec^2 y$

C. $\cos^2 y$

D. $\sin^2 y$

E. $\sec^2 y - 1$

$\frac{d}{dx} \rightarrow \quad 1 = (\sec^2 y)\left(\frac{dy}{dx}\right)$

$\rightarrow \quad \frac{dy}{dx} = \frac{1}{\sec^2 y} = \cos^2 y$

7.

A function f has derivative $f'(x) = \frac{1}{\sqrt{x^2+5}}$. If $h(x) = f(2x-1)$, find $h'(2)$.

$$h'(x) = (f'(2x-1))(2)$$

$$h'(2) = (f'(3))(2)$$

$$= \frac{1}{\sqrt{14}} \cdot 2$$

$$= \frac{2}{\sqrt{14}}$$

A. $\frac{1}{3}$

B. $\frac{2}{3}$

C. $\frac{1}{\sqrt{14}}$

D. $\frac{2}{\sqrt{14}}$

E. $\frac{\sqrt{14}}{3}$

8. If $f(x) = \frac{x+1}{x-1}$, evaluate $\lim_{x\to 2} \frac{f(x)-f(2)}{x-2}$.

Note: $\lim_{x\to 2} \frac{f(x)-f(2)}{x-2} = f'(2)$.

$$f'(x) = \frac{(1)(x-1)-(x+1)(1)}{(x-1)^2} =$$

$$f'(2) = \frac{(1)(1)-(3)(1)}{(1)^2} = \frac{-2}{1} = -2$$

A. ∞

B. -2

C. $\frac{2}{(x-1)^2}$

D. 2

E. 0

alternate solution: $\lim_{x\to 2} \frac{f(x)-f(2)}{x-2} = \lim_{x\to 2} \frac{\frac{x+1}{x-1} - 3}{x-2} = \lim_{x\to 2} \frac{\frac{x+1-3x+3}{x-1}}{x-2}$

$$= \lim_{x\to 2} \frac{\frac{-2x+4}{x-1}}{\frac{x-2}{1}} = \lim_{x\to 2} \frac{-2\cancel{(x-2)}}{x-1} \cdot \frac{1}{\cancel{x-2}} = -\frac{2}{1} = -2$$

9. Find $\dfrac{dy}{dx}$ if $y = 3x^2 \cos x$.

$$\frac{dy}{dx} = 6x \cos x + 3x^2(-\sin x)$$

A. $3x^2 \sin x + 6x \cos x$

B. $-3x^2 \sin x$

C. $-3x^2 \sin x + 6x \cos x$

D. $-6x \cos x$

E. $6x \cos x$

10. Find $f''(x)$ if $f(x) = \dfrac{1-x}{1+x}$.

$$f'(x) = \frac{(-1)(1+x) - (1-x)(1)}{(1+x)^2}$$

$$= \frac{-1-x-1+x}{(1+x)^2} = \frac{-2}{(1+x)^2}$$

$$= -2(1+x)^{-2}$$

$$f''(x) = 4(1+x)^{-3} = \frac{4}{(1+x)^3}$$

A. $\dfrac{4}{(1+x)^3}$

B. $-\dfrac{4}{(1+x)^3}$

C. $-\dfrac{4x}{(1+x)^3} + \dfrac{2}{(1+x)^2}$

D. $\dfrac{2(1+x)^2 - 2x(1+x)}{(1+x)^4}$

E. -1

11. If $y = \sin\sqrt{x^4+3}$, then $\dfrac{dy}{dx} =$

$$\frac{dy}{dx} = \left(\cos\sqrt{x^4+3}\right)\left(\frac{1}{2\sqrt{x^4+3}}\right)\left(4x^3\right)$$

$$= \left(\cos\sqrt{x^4+3}\right)\left(\frac{\cancel{4}^{2}x^3}{\cancel{2}\sqrt{x^4+3}}\right)$$

A. $-(\cos\sqrt{x^4+3})\dfrac{2x^3}{\sqrt{x^4+3}}$

B. $4x^3\cos\sqrt{x^4+3}$

C. $\dfrac{\cos\sqrt{x^4+3}}{2\sqrt{x^4+3}}$

D. $(\cos\sqrt{x^4+3})\dfrac{2x^3}{\sqrt{x^4+3}}$

E. $\cos\left(\dfrac{4x^3}{\sqrt{x^4+3}}\right)$

12. The half–life of a certain radioactive substance is 1340 years. A sample of the substance has a mass of 75 mg. When will the mass be reduced to 40 mg.?

$$A(t) = A(0)e^{kt}$$

$$A(1340) = \tfrac{1}{2}\cancel{A(0)} = \cancel{A(0)}e^{k(1340)}$$

$$\rightarrow \tfrac{1}{2} = e^{k(1340)} \rightarrow \ln\tfrac{1}{2} = k(1340)$$

$$\rightarrow k = \frac{1}{1340}\ln\frac{1}{2}$$

$$A(t) = 75e^{\left(\frac{1}{1340}\ln\frac{1}{2}\right)t}$$

$$A(t) = 40 = 75e^{\left(\frac{1}{1340}\ln\frac{1}{2}\right)t}$$

$$\rightarrow \ln\frac{40}{75} = \left(\frac{1}{1340}\ln\frac{1}{2}\right)t$$

$$\rightarrow t = 1340\left(\frac{\ln\frac{40}{75}}{\ln\frac{1}{2}}\right)$$

A. $1340\dfrac{ln(1/2)}{ln(40/75)}$ years

B. $1340\dfrac{ln(1/2)}{ln(75/40)}$ years

C. $1340\dfrac{ln(40/75)}{ln(1/2)}$ years

D. $1340\dfrac{ln(75/40)}{ln(1/2)}$ years

E. $\dfrac{1}{1340}\dfrac{ln(40/75)}{ln(1/2)}$ years

13. A linear approximation of $\sqrt{4.12}$ is

let $f(x) = \sqrt{x}$ and $a = 4$

$f'(x) = \frac{1}{2\sqrt{x}}$

$f'(4) = \frac{1}{4}$

$L(x) = f(4) + f'(4)(x-4)$

$= 2 + \frac{1}{4}(x-4)$

$L(4.12) = 2 + \frac{1}{4}(4.12-4)$

$= 2 + \frac{1}{4}(0.12)$

$= 2 + 0.03$

$= 2.03$

A. 2.03
B. 2.029778
C. 2.029
D. 2.031
E. 2.12

14. Given $x^2 - 3y^3 + 2x = 5$ and $\frac{dy}{dt} = -2$, then for $x = 2$, $y = 1$,

we have $\frac{dx}{dt} =$

$\frac{d}{dt} \rightarrow 2x\frac{dx}{dt} - 9y^2\frac{dy}{dt} + 2\frac{dx}{dt} = 0$

$\left.\begin{matrix} x=2 \\ y=1 \\ \frac{dy}{dt}=-2 \end{matrix}\right\} \rightarrow 4\frac{dx}{dt} + 18 + 2\frac{dx}{dt} = 0$

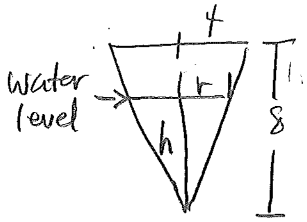
$\rightarrow 6\frac{dx}{dt} = -18$

$\rightarrow \frac{dx}{dt} = -3$

A. 3
B. 2
C. 1
D. −2
E. −3

15. Water is poured into a conical paper cup at the rate of 2 cubic centimeters per second. If the cup is 8 cm tall and the top has a radius of 4 cm, how fast is the water level rising (in cm/sec) when the water is 6 cm deep? (volume of cone: $V = \frac{1}{3}\pi r^2 h$)

Know: $\frac{dV}{dt} = 2$. Want: $\frac{dh}{dt}$ when $h = 6$



$\frac{r}{h} = \frac{4}{8} \rightarrow r = \frac{h}{2}$

$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{\pi}{12} h^3$

$\rightarrow \frac{dV}{dt} = \frac{\pi}{4} h^2 \frac{dh}{dt}$

$h = 6 \rightarrow 2 = \frac{\pi}{4} \cdot 6^2 \cdot \frac{dh}{dt} \rightarrow \frac{dh}{dt} = 2 \cdot \frac{4}{\pi} \cdot \frac{1}{36} = \frac{2}{9\pi}$

A. $\frac{\pi}{9}$

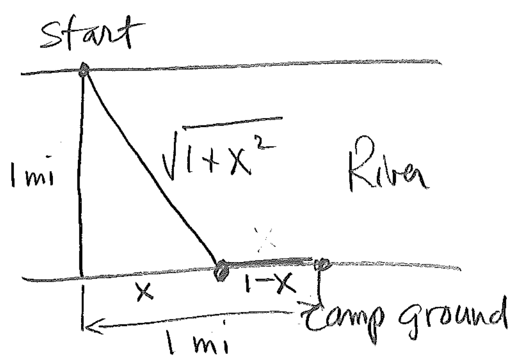
B. $\frac{2\pi}{9}$

C. $\frac{1}{9\pi}$

D. $\frac{2}{9\pi}$

E. $\frac{18}{\pi}$

16. You are standing at the edge of a slow-moving river which is one mile wide and wish to return to your campground on the opposite side of the river. You can swim at 2 mph and walk at 3 mph. You must first swim across the river to any point on the opposite bank. From there walk to the campground, which is one mile from the point directly across the river from where you start your swim. How far (in miles) from the campground is the point across the river to which you should swim in order to take the least amount of time to get back to the campground?



minimize time T of trip.

time = $\frac{\text{distance}}{\text{rate}}$

A. $\frac{2}{\sqrt{5}}$

B. $\frac{\sqrt{3}}{2}$

C. $\frac{2}{\sqrt{3}}$

D. 1

E. $1 - \frac{2}{\sqrt{5}}$

$T = \frac{\sqrt{1+x^2}}{2} + \frac{1-x}{3}, \quad 0 \le x \le 1.$

$\frac{dT}{dx} = \frac{1}{\cancel{2}} \cdot \frac{1}{2\sqrt{1+x^2}} \cdot \cancel{2}x - \frac{1}{3} = 0 \rightarrow \frac{x}{2\sqrt{1+x^2}} = \frac{1}{3} \rightarrow 3x = 2\sqrt{1+x^2}$

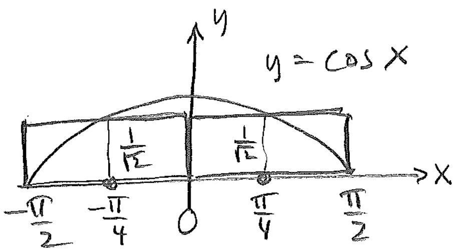
$\rightarrow 9x^2 = 4(1+x^2) \rightarrow 9x^2 = 4 + 4x^2 \rightarrow 5x^2 = 4 \rightarrow x^2 = \frac{4}{5} \rightarrow x = \frac{2}{\sqrt{5}}$

Distance from campground is $1 - \frac{2}{\sqrt{5}}$

17. The midpoint Riemann sum to estimate the area of the region under the graph of $f(x) = \cos x$ and

above the x-axis,

from $x = -\frac{\pi}{2}$ to $x = \frac{\pi}{2}$, using two approximating rectangles is



A. 2

B. $\frac{3}{\pi}$

C. $\frac{\pi\sqrt{2}}{2}$

D. $\frac{2\pi}{3}$

E. $\frac{\pi(1+\sqrt{2})}{4}$

$$\left(\cos\left(-\frac{\pi}{4}\right)\right)\left(\frac{\pi}{2}\right) + \left(\cos\left(\frac{\pi}{4}\right)\right)\left(\frac{\pi}{2}\right)$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{\pi}{2} + \frac{1}{\sqrt{2}} \cdot \frac{\pi}{2} = 2\left(\frac{\pi}{\sqrt{2} \cdot 2}\right) = \frac{\pi}{\sqrt{2}} = \frac{\sqrt{2}\,\pi}{2}$$

18. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{n}{n^2 + i^2}$ is a limit of Riemann sums on the interval $[0, 1]$. Its value is

A. 1

B. $\frac{\pi}{4}$

C. $\frac{\pi}{3}$

D. $\frac{1+\sqrt{2}}{2}$

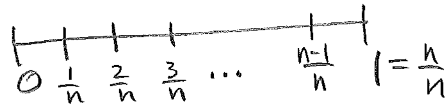
E. None of the above

This Riemann Sum uses right-hand endpoints and $\Delta x = \frac{1-0}{n} = \frac{1}{n}$

Note: $\frac{n}{n^2+i^2} = \frac{n^2}{n^2+i^2} \cdot \frac{1}{n}$

$$= \frac{1}{\frac{n^2+i^2}{n^2}} \cdot \frac{1}{n} = \frac{1}{1+\left(\frac{i}{n}\right)^2} \cdot \frac{1}{n}$$

$$= \left(f\left(\frac{i}{n}\right)\right) \cdot \frac{1}{n}$$



Thus $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{1+\left(\frac{i}{n}\right)^2} \cdot \frac{1}{n} = \int_0^1 \frac{1}{1+x^2}\,dx = \arctan x \Big|_0^1$

$$= \frac{\pi}{4} - 0$$

$$= \pi/4$$

19. Given $g(x) = \int_0^{x^2} \frac{u}{u^2+2}\, du$ the value of $g'(2)$ is

$$g'(x) = \frac{x^2}{x^4+2} \cdot 2x$$

$$g'(2) = \frac{4}{16+2} \cdot 4 = \frac{16}{18} = \frac{8}{9}$$

A. $\frac{8}{9}$

B. $\frac{4}{9}$

C. $\frac{3}{2}$

D. $\frac{3}{4}$

E. 3

20. Given $g(x) = \int_0^{x} \frac{u}{u^2+2}\, du$ the value of $g(1)$ is

$$g'(x) = \frac{x}{x^2+2}$$

$$g'(1) = \frac{1}{1^2+2} = \frac{1}{3}$$

A. $\frac{1}{2}$

B. 1

C. $\ln\frac{3}{2}$

D. $\frac{1}{2}\ln\frac{3}{2}$

E. None of the above

21. Given a differentiable function g such that $g(-1) = 2$ and $g(2) = 3$, the mean value theorem guarantees that

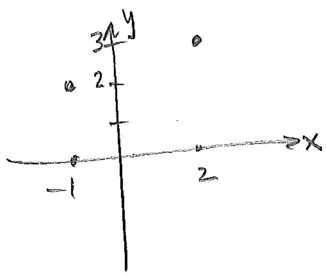
A. for every $d \in (2, 3)$ there is $c \in (-1, 2)$ such that $g(c) = d$

B. there is $c \in (-1, 2)$ such that $g(c) = \frac{1}{3}$

C. there is $c \in (2, 3)$ such that $g'(c) = \frac{1}{3}$

D. there is $c \in (-1, 2)$ such that $g'(c) = \frac{1}{3}$

E. None of the above



Mean Value Theorem $\rightarrow \dfrac{g(2)-g(-1)}{2-(-1)} = g'(c)$, $-1 < c < 2$

$$\frac{g(2)-g(-1)}{2-(-1)} = \frac{3-2}{2+1} = \frac{1}{3}$$

$\Rightarrow$ there is a "c" in $(-1, 2)$ such that $g'(c) = \frac{1}{3}$

22. Assume f is differentiable on $(\frac{1}{2}, 3)$. Given $f'(x) = 7x - \frac{2}{x}$ and $f(1) = 7$, it follows that $f(2) =$

A. $\frac{7}{2}$

B. $\frac{35}{2} - \ln 4$

C. $\frac{7}{2} - \ln 4$

D. $\frac{35}{2} - \ln 2$

E. cannot be determined from the information given

$f'(x) = 7x - \frac{2}{x}$

$\rightarrow f(x) = \frac{7}{2}x^2 - 2\ln x + C$

$f(1) = 7 = \frac{7}{2} - 2\ln 1 + C$

$\rightarrow 7 = \frac{7}{2} + C \rightarrow C = \frac{7}{2}$

$\rightarrow f(x) = \frac{7}{2}x^2 - 2\ln x + \frac{7}{2}$

So $f(2) = \frac{7}{2} \cdot 4 - 2\ln 2 + \frac{7}{2}$

$= 14 - \ln 4 + \frac{7}{2}$

$= \frac{28}{2} - \ln 4 + \frac{7}{2} = \frac{35}{2} - \ln 4$

23. At $t = -\frac{1}{3}$ the function $R(t) = t^3 + t^2 - 2t$ has a

$R(-\frac{1}{3}) = -\frac{1}{27} + \frac{1}{9} + \frac{1}{6} \neq 0$

A. local maximum

B. local minimum

C. discontinuity

D. point of inflection

E. zero

$R'(t) = 3t^2 + 2t - 2$

$R'(t) = 0 \rightarrow t = \frac{-2 \pm \sqrt{2^2 - 4(3)(-2)}}{6} = -\frac{1}{3} \pm \frac{\sqrt{28}}{6}$

$R'(t) = 6t + 2 = 0 \rightarrow t = \frac{1}{3}$

	$-\infty$	$\frac{1}{3}$	∞
$6t+2$	$-$	0	$+$
R	concave down		concave up

R has a point of inflection at $t = \frac{1}{3}$

24. It takes 40 minutes to fill a gas storage tank at a gas station at a rate of $100+t$ gallons per minute, $0 \le t \le 40$. Find the capacity of the tank in gallons.

A. 5600

B. 140

C. 2440

D. 4800

E. None of the above

$$\int_0^{40} (100 + t)\, dt = \left(100t + \frac{1}{2}t^2\right)\Big|_0^{40}$$

$$= (4000 + 800) - (0 + 0)$$

$= 4800$ gallons

25. The function $f(x) = |x|(x-1)$ has

A. a local maximum at $x = 1$

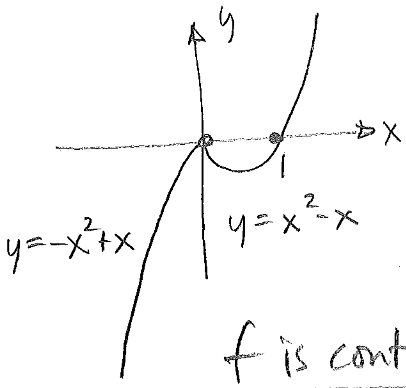
B. a local minimum at $x = 0$

C. a local minimum at $x = 1$

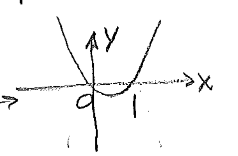
D. a local maximum at $x = 0$

E. a discontinuity at $x = 0$

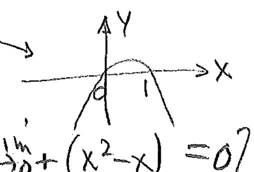
$$f(x) = |x|(x-1) = \begin{cases} x(x-1) = x^2 - x & \text{for } x \geq 0 \\ -x(x-1) = -x^2 + x & \text{for } x < 0 \end{cases}$$



$y = x^2 - x = x(x-1) \rightarrow$



$y = -x^2 + x = x(-x+1) \rightarrow$



f is continuous at $x=0$.

$$\left.\begin{array}{l} \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (x^2 - x) = 0 \\ \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (-x^2+x) = 0 \end{array}\right\} \lim_{x\to 0} f(x) = 0 = f(0)$$

$$f'(x) = \begin{cases} 2x-1 & \text{for } x \geq 0 \\ -2x+1 & \text{for } x < 0 \end{cases}$$

$2x - 1 = 0 \rightarrow x = \frac{1}{2}$ $(x > 0)$

$-2x+1 = 0 \rightarrow x = \frac{1}{2}$ not in interval $(-\infty, 0)$

	$-\infty$ to 0	0 to $\frac{1}{2}$	$\frac{1}{2}$ to ∞
$x \geq 0$, $2x-1$		$-$	$+$
$x < 0$, $-2x+1$	$+$		
$f'(x)$	$+$	$-$	$+$
	inc.	dec.	inc.

f

f has local max at $x = 0$

f has local min at $x = \frac{1}{2}$