MA161 - Fall 2009 - FINAL EXAM

Solutions

(1) The domain of the function $f(x) = \sqrt{\ln(x+5)}$ is:

(a) $[0, \infty)$

(b) $[-5, \infty)$

(c) $[5, \infty)$

(d) $[-4, \infty)$

(e) $[4, \infty)$

$\ln(x+5) \geq 0$

$\rightarrow e^{\ln(x+5)} \geq e^{0}$

$\rightarrow x+5 \geq 1$

$\rightarrow x \geq -4$

$\rightarrow [-4, \infty)$

(2) Let $f(x) = \sqrt[3]{2-x}$. Which of the following is $f^{-1}(x)$?

(a) $(2-x)^3$

(b) $(x-2)^3$

(c) $2+x^3$

(d) x^3-2

(e) $2-x^3$

Interchange x and y $(y = f(x))$

$\rightarrow x = \sqrt[3]{2-y}$

Solve for y:

$x^3 = 2-y$

$\rightarrow y = 2-x^3$

(3) If $f(x) = \begin{cases} x^2+9 & \text{for } x \le 1 \\ 12x - ax^2 & \text{for } x > 1 \end{cases}$ determine all values of a so that $f(x)$ is continuous at all values of x.

(a) $a = 0$

(b) $a = 1$

(c) $a = 2$

(d) $a = 3$

(e) There are no such values of a

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x^2 + 9 = 10$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 12x - ax^2 = 12 - a$$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) \Rightarrow 10 = 12 - a$$

$$\rightarrow a = 2$$

(4) If $f(x) = x^2 \tan x$, the slope of the tangent line at $(\frac{\pi}{3}, f(\frac{\pi}{3}))$ is:

(a) $\dfrac{4\pi^2 + 6\sqrt{3}\pi}{27}$

(b) $\dfrac{4\pi^2 + 6\sqrt{3}\pi}{9}$

(c) $\dfrac{2\sqrt{3}\pi^2 + 2\sqrt{3}\pi}{9}$

(d) $\dfrac{8\pi}{3}$

(e) $\dfrac{4\pi\sqrt{3}}{3}$

$$f'(x) = 2x \tan x + x^2 \sec^2 x$$

$$f'\left(\frac{\pi}{3}\right) = \frac{2\pi}{3} \tan\frac{\pi}{3} + \left(\frac{\pi}{3}\right)^2 \left(\sec\frac{\pi}{3}\right)^2$$

$$= \frac{2\pi}{3}(\sqrt{3}) + \frac{\pi^2}{9}(2)^2$$

$$= \frac{2\sqrt{3}\pi}{3} + \frac{4\pi^2}{9}$$

$$= \frac{6\sqrt{3}\pi + 4\pi^2}{9}$$

(5) If $f(x) = \dfrac{x^3 - 2x}{x^2 + 1}$, then $f'(2) =$

(a) 34/25

(b) 66/25

(c) 5/2

(d) 1/5

(e) −2/25

$$f'(x) = \frac{(3x^2-2)(x^2+1) - (x^3-2x)(2x)}{(x^2+1)^2}$$

$$f'(2) = \frac{(10)(5) - (4)(4)}{25}$$

$$= \frac{34}{25}$$

(6) If $f(x) = e^{4x}$, evaluate $\lim\limits_{h \to 0} \dfrac{f(3+h) - f(3)}{h}$.

(a) e^{12}

(b) e^7

(c) $4e^{12}$

(d) $4e^7$

(e) ∞

$$= \lim_{h \to 0} \frac{e^{12+4h} - e^{12}}{h}$$

$$= \lim_{h \to 0} \frac{e^{12}(e^{4h} - 1)}{h}$$

$$\overset{\frac{0}{0}}{=} \lim_{h \to 0} e^{12} \cdot \frac{4e^{4h}}{1}$$

$$= e^{12} \cdot \frac{4 \cdot 1}{1}$$

$$= 4e^{12}$$

(7) The function $f(x) = \dfrac{x^2+1}{x^3+8}$ has:

(a) no vertical or horizontal asymptotes

(b) 1 vertical and 1 horizontal asymptote

(c) 2 vertical and 1 horizontal asymptote

(d) 1 vertical and 2 horizontal asymptotes

(e) 1 vertical and no horizontal asymptotes

vertical asymptotes:

$x^3+8=0 \to x=-2$

$\to$ one vertical asymptote, $x=-2$

horizontal asymptote:

$\lim_{x\to\pm\infty} \dfrac{x^2+1}{x^3+8} = 0$

$\to$ $y=0$ is the only horizontal asymptote

(8) A particle moves on a line with velocity $v(t) = t - \ln(t^2+1)$. What is its maximum velocity on the interval $0 \le t \le 2$?

(a) $1 - \ln 2$

(b) 0

(c) $2 - \ln 5$

(d) $\ln 2 - 1$

(e) $\ln 5 - 2$

$v'(t) = 1 - \dfrac{2t}{t^2+1} = \dfrac{t^2+1-2t}{t^2+1} = \dfrac{(t-1)^2}{t^2+1}$

$v'(t) = 0 \to t = 1$

v continuous on a closed interval.

$v(0) = 0 - \ln(1) = 0$

$\left.\begin{array}{l} v(1) = 1 - \ln 2 > 0 \\ v(2) = 2 - \ln 5 > 0 \end{array}\right\}$ which is larger?

Use $v'(t)$ to determine behavior of v.

$v'(t) = \dfrac{(t-1)^2}{t^2+1} > 0$ for all $t \neq 1$.

Therefore v is increasing on $[0,2]$ and $v(2)$ is max value.

(9) Assume that f and g are differentiable functions defined on $(-\infty, \infty)$, $f(0) = 6, f'(0) = 10, f(2) = 5, f'(2) = 4, g(0) = 2$, and $g'(0) = 3$. Let $h(x) = f(g(x))$. What is $h'(0)$?

(a) 4

(b) 8

(c) 10

(d) 12

(e) 30

$$h'(x) = f'(g(x)) \cdot g'(x)$$
$$h'(0) = f'(g(0)) \cdot g'(0)$$
$$= f'(2) \cdot 3$$
$$= 4 \cdot 3$$
$$= 12$$

(10) Assume that y is defined implicitly as a differentiable function of x by the equation $2x^3 + x^2y - xy^3 = 2$. Find $\dfrac{dy}{dx}$ at $(1, 1)$.

(a) $\dfrac{-3}{2}$

(b) $\dfrac{7}{2}$

(c) 0

(d) -3

(e) -4

$$\frac{d}{dx} \rightarrow 6x^2 + (2x)(y) + (x^2)\left(\frac{dy}{dx}\right) - \left[(1)(y^3) + (x)\left(3y^2\frac{dy}{dx}\right)\right] = 0$$
$$(1,1) \rightarrow 6 + 2 + \frac{dy}{dx} - \left[1 + 3\frac{dy}{dx}\right] = 0$$
$$\rightarrow -2\frac{dy}{dx} = -7$$
$$\rightarrow \frac{dy}{dx} = \frac{7}{2}$$

(11) Evaluate $\lim_{x\to 0} \dfrac{\cos(2x)-1}{x^2}$. $= \dfrac{1-1}{0} = \dfrac{0}{0}$

(a) -2

(b) -1

(c) 0

(d) 1

(e) 2

$\overset{0/0}{=} \lim_{x\to 0} \dfrac{-2\sin(2x)}{2x}$

$\overset{0/0}{=} \lim_{x\to 0} \dfrac{-4\cos(2x)}{2}$

$= \dfrac{-4}{2}$

$= -2$

(12) Water is withdrawn at the constant rate of 2 ft^3 / min from a cone-shaped reservoir which has its vertex down. The diameter of the top of the tank measures 4 feet and the height of the tank is 8 feet . How fast is the water level falling when the depth of the water in the reservoir is 2 feet? (Recall that the volume of a cone of height h and radius r is $V = \frac{\pi}{3}r^2h$).

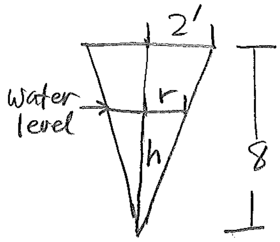
(a) $\dfrac{2}{\pi}$ ft/min

(b) $\dfrac{6}{\pi}$ ft/min

(c) $\dfrac{4}{\pi}$ ft/min

(d) $\dfrac{8}{\pi}$ ft/min

(e) $\dfrac{16}{\pi}$ ft/min



tank cross-section

know: $\dfrac{dV}{dt} = -2 \ \dfrac{ft^3}{min}$

(V = volume of water in tank)

want: $\left|\dfrac{dh}{dt}\right|$ when $h = 2'$

note $\dfrac{2}{8} = \dfrac{r}{h}$

$V = \frac{\pi}{3}r^2h$ and $r = \frac{1}{4}h$

$V(h) = \frac{\pi}{3}\left(\frac{h}{4}\right)^2 h = \frac{\pi}{48}h^3 \longrightarrow \frac{dV}{dt} = \frac{\pi}{16}h^2\frac{dh}{dt}$

$-2 = \frac{\pi}{16}2^2\frac{dh}{dt} \longrightarrow \frac{dh}{dt} = (-2)\left(\frac{16}{4\pi}\right) = -\frac{8}{\pi}$

$\left|\dfrac{dh}{dt}\right| = \dfrac{8}{\pi}\ \dfrac{ft}{min}$

(13) At the beginning of an experiment a colony has N bacteria. Two hours later it has 4N bacteria. How many hours, measured from the beginning, does it take for the colony to have 10N bacteria?

(a) $\frac{\ln 5N}{\ln 2}$

(b) $\frac{N \ln 5}{2 \ln 2}$

(c) $\frac{\ln 5}{\ln 2}$

(d) $4\frac{\ln N}{\ln 2}$

(e) $\frac{\ln 10}{\ln 2}$

$P(t) = P(0)e^{kt}$, $P(0) = N$

$P(2) = Ne^{k2} = 4N \to e^{k2} = 4$

$\to 2k = \ln 4 \to k = \frac{\ln 4}{2}$

Therefore $P(t) = Ne^{\left(\frac{\ln 4}{2}\right)t}$

Solve $10N = Ne^{\left(\frac{\ln 4}{2}\right)t}$ for t.

$\to 10 = e^{\left(\frac{\ln 4}{2}\right)t} \to \ln 10 = \frac{\ln 4}{2} \cdot t$

$\to t = 2\frac{\ln 10}{\ln 4} = \frac{1}{\frac{1}{2}}\left(\frac{\ln 10}{\ln 4}\right)$

$= \frac{\ln 10}{\frac{1}{2}\ln 4} = \frac{\ln 10}{\ln 4^{1/2}} = \frac{\ln 10}{\ln 2}$

(14) The approximate value of $(16.32)^{\frac{1}{4}}$ given by linear approximation is equal to

(a) 2.01

(b) 2.10

(c) 2.02

(d) 2.20

(e) 2.06

Let $f(x) = x^{\frac{1}{4}}$ and $a = 16$

$L(x) = f(16) + f'(16)(x-16)$

$f'(x) = \frac{1}{4}x^{-3/4}$. $f'(16) = \frac{1}{4}(16)^{-3/4} = \frac{1}{32}$

$L(x) = 2 + \frac{1}{32}(x-16)$

$L(16.32) = 2 + \frac{1}{32}(16.32-16)$

$= 2 + \frac{1}{32}(0.32)$

$= 2 + 0.01$

$= 2.01$

(15) Find the critical numbers of $f(x) = e^x \sin x$ for $0 \leq x \leq 2\pi$.

(a) $\pi/4$ and $5\pi/4$

(b) $3\pi/4$ and $7\pi/4$

(c) $\pi/4$ and $3\pi/4$

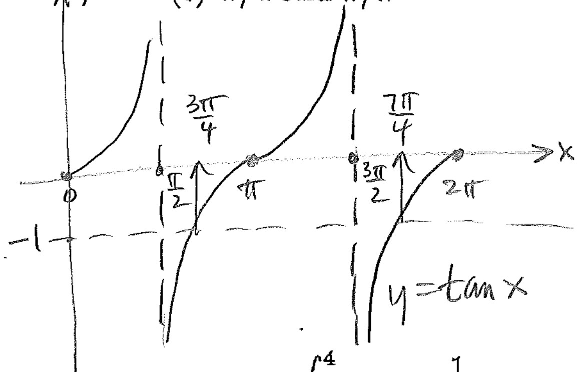
(d) $\pi/4$ and $7\pi/4$

(e) $\pi/4$ and $\pi/2$

$$f'(x) = e^x \sin x + e^x \cos x$$
$$= e^x(\sin x + \cos x)$$

$f'(x)$ exists for all numbers x.

$$f'(x) = 0 \longrightarrow \sin x + \cos x = 0$$
$$\longrightarrow \sin x = -\cos x$$
$$\longrightarrow \frac{\sin x}{\cos x} = -1$$
$$\longrightarrow \tan x = -1$$
$$\longrightarrow x = \frac{3\pi}{4} \text{ and } \frac{7\pi}{4}$$



(16) Compute $\int_1^4 (\sqrt{x} - \frac{1}{\sqrt{x}})\, dx$

(a) $2\sqrt{2} - 10/3$

(b) $\sqrt{2} - 1/3$

(c) $\sqrt{2} + 4/3$

(d) $2\sqrt{2} + 14/3$

(e) $8/3$

$$= \int_1^4 \left(x^{1/2} - x^{-1/2}\right) dx$$
$$= \left(\frac{2}{3} \cdot x^{3/2} - 2x^{1/2}\right)\Big|_1^4$$
$$= \left(\frac{2}{3} \cdot 4^{3/2} - 2 \cdot 4^{1/2}\right) - \left(\frac{2}{3} \cdot 1^{3/2} - 2 \cdot 1^{1/2}\right)$$
$$= \frac{2}{3} \cdot 8 - 2 \cdot 2 - \left(\frac{2}{3} - 2\right)$$
$$= \frac{16}{3} - 4 - \frac{2}{3} + 2$$
$$= \frac{14}{3} - 2$$
$$= \frac{14}{3} - \frac{6}{3}$$
$$= \frac{8}{3}$$

(17) Evaluate $\frac{d}{dx}\left(\int_0^{2x} \arctan t \, dt\right)$ at $x = \frac{1}{2}$.

(a) $\pi/3$

(b) 1

(c) $\pi/4$

(d) $\pi/2$

(e) 2

$$\frac{d}{dx}\left(\int_0^{2x} \arctan t \, dt\right) = (\arctan 2x)(2)$$

$$\text{at } x = \tfrac{1}{2} \longrightarrow (\arctan 1)(2) = \left(\frac{\pi}{4}\right)(2) = \frac{\pi}{2}$$

(18) A certain function $f(x)$ satisfies $f''(x) = 2 - 3x$. We also know that $f'(0) = -1$ and $f(0) = 1$. Compute $f(2)$.

(a) -1

(b) -3

(c) 3

(d) 1

(e) -2

$$f''(x) = 2 - 3x$$

$$\longrightarrow f'(x) = 2x - \tfrac{3}{2}x^2 + C$$

$$f'(0) = -1 = 0 - 0 + C \rightarrow C = -1$$

$$\longrightarrow f'(x) = 2x - \tfrac{3}{2}x^2 - 1$$

$$\longrightarrow f(x) = x^2 - \tfrac{3}{2}\cdot\tfrac{1}{3}x^3 - x + C_1$$

$$f(0) = 1 = 0 - 0 - 0 + C_1 \rightarrow C_1 = 1$$

$$f(x) = x^2 - \tfrac{1}{2}x^3 - x + 1$$

$$f(2) = 4 - 4 - 2 + 1 = -1$$

(19) Compute $\lim_{x\to 0}(1-x)^{\frac{5}{x}}$. $= 1^{\infty}$

(a) 1

(b) e^3

(c) e^{-5}

(d) e^{-3}

(e) e^5

$$\lim_{x\to 0}(1-x)^{5/x} = \lim_{x\to 0} e^{\ln(1-x)^{5/x}}$$

$$= e^{\boxed{\lim_{x\to 0}\ln(1-x)^{5/x}}} \longleftarrow \text{what is this limit ?}$$

$$\lim_{x\to 0}\ln(1-x)^{5/x} = \lim_{x\to 0}\frac{5}{x}\ln(1-x)$$

$$= \lim_{x\to 0}\frac{5\ln(1-x)}{x} \overset{\frac{0}{0}}{=} \lim_{x\to 0}\frac{\frac{-5}{1-x}}{1} = \frac{-5}{1} = -5$$

Therefore $\lim_{x\to 0}(1-x)^{5/x} = e^{-5}$

(20) The derivative of a function f is given by $f'(x) = (x-1)^2(x-2)^3(x-3)$. Which of the following are correct?

T I) $f(2)$ is a local maximum and $f(3)$ is a local minimum of $f(x)$.

F II) $f(x)$ is increasing on the interval $(1, 3)$.

F III) $f(x)$ is decreasing on $(-\infty, 1)$ and increasing on $(1, \infty)$.

(a) only I is correct

(b) only I and III are correct

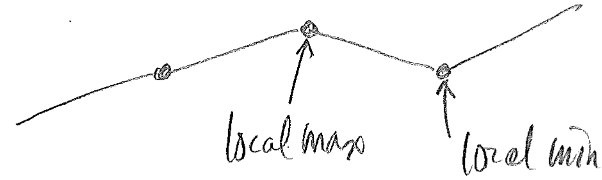
(c) only II is correct

(d) only II and III are correct

(e) only III is correct

	$-\infty$ to 1	1 to 2	2 to 3	3 to ∞
$(x-1)^2$	+	+	+	+
$(x-2)^3$	−	−	+	+
$x-3$	−	−	−	+
$f'(x)$	+	+	−	+
	Inc	Inc	dec	Inc

f



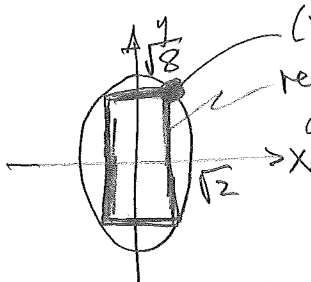
$$\frac{x^2}{2}+\frac{y^2}{8}=1$$

and $y^2=8-4x^2$

(21) A rectangle is centered at the origin, its sides are parallel to the axes and all of its vertices lie on the curve $4x^2+y^2=8$. What is the maximum area of such rectangle?

$(x,y)=(x,\sqrt{8-4x^2})$

rectangle center at origin with all vertices on $4x^2+y^2=8$



(a) 4

(b) 8

(c) $4\sqrt{2}$

(d) $2\sqrt{2}$

(e) 2

Area of rectangle $= A = (2x)(2\sqrt{8-4x^2})$, $0\le x\le\sqrt{2}$

$$A(x) = 4x\sqrt{8-4x^2}$$

$$\frac{dA}{dx} = 4\sqrt{8-4x^2} + 4x\left(\frac{-8x}{2\sqrt{8-4x^2}}\right)$$

$$= \frac{8(8-4x^2)-32x^2}{2\sqrt{8-4x^2}} = \frac{64-64x^2}{2\sqrt{8-4x^2}} = 0 \rightarrow x=1$$

$$A(1) = 4\sqrt{8-4} = 4\sqrt{4} = 8$$

(22) Compute $\int_0^1 \frac{3x^2}{\sqrt{x^3+1}}\,dx$

$$\int_0^1 (x^3+1)^{-1/2}(3x^2)\,dx = *$$

(a) $3\sqrt{2}-3$

(b) $2(\sqrt{3}-1)$

let $u=x^3+1$. Then $du=3x^2\,dx$

(c) 2

and $u(0)=1$ and $u(1)=2$

(d) $2(\sqrt{2}-1)$

(e) $6\sqrt{3}-4$

$$* = \int_1^2 u^{-1/2}\,du = 2u^{1/2}\Big|_1^2$$

$$= 2\sqrt{2}-2\sqrt{1}$$

$$= 2\sqrt{2}-2$$

$$= 2(\sqrt{2}-1)$$

(23) On what intervals is the graph of $f(x) = x^4 + 4x^3 - 18x^2 - 6x$ concave downward?

(a) on $(-3, 1)$ and $(2, 3)$

(b) on $(-\infty, -3)$ and $(1, \infty)$

(c) only on $(-\infty, 11)$

(d) only on $(3, \infty)$

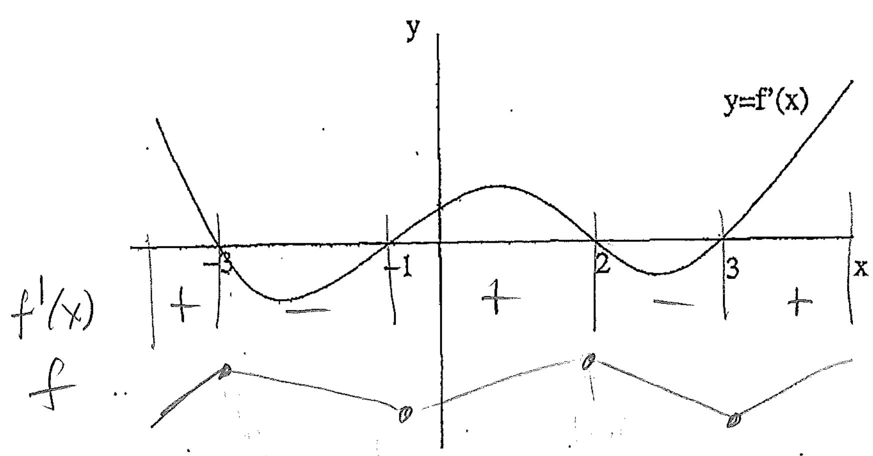
(e) on $(-3, 1)$

$$f'(x) = 4x^3 + 12x^2 - 36x - 6$$
$$f''(x) = 12x^2 + 24x - 36$$
$$= 12(x^2 + 2x - 3)$$
$$= 12(x+3)(x-1)$$
$$= 0 \rightarrow x = -3, 1$$

	$-\infty$ to -3	-3 to 1	1 to ∞
12	+	+	+
$x+3$	−	0 +	+
$x-1$	−	−	0 +
$f''(x)$	+	−	+
f		concave down	

(24) The figure below illustrates the graph of the derivative of a differentiable function f which is defined in $(-4, 4)$. We can conclude that $f(x)$ achieves local maxima and minima at the following points:

(a) local maxima at -3 and 2 and local minima at -1 and 3
(b) local maxima at -1 and 3 and local minima at -3 and 2
(c) local maxima at -1 and 3 and local minimum at 2
(d) local maxima at -3 and 2 and local minimum at -1
(e) local maximum at a point between -3 and -1 and a local minimum at 0.

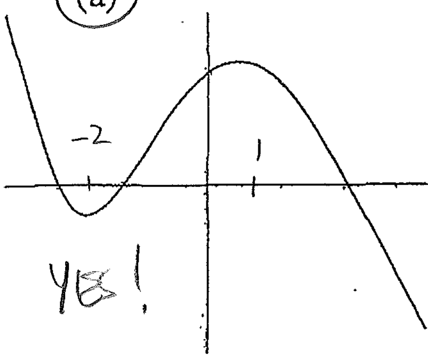


local max at
$x = -3, 2$

local min at
$x = -1, 3$

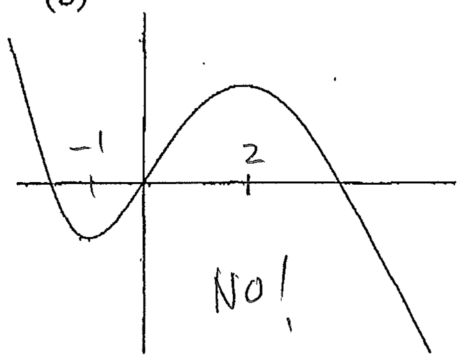
(25) The graph of the function $f(x) = -\frac{1}{3}x^3 - \frac{1}{2}x^2 + 2x + 2$ looks mostly like

(a)



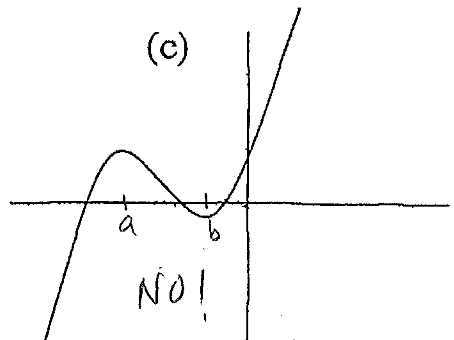
YES!

(b)



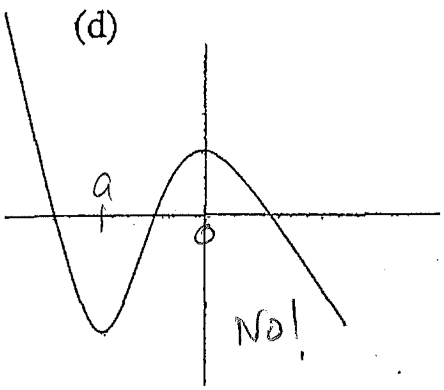
No!

(c)



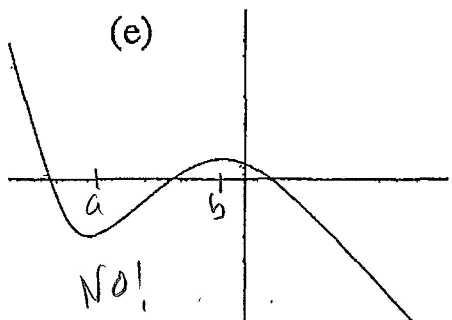
No!
$a<0$ and $b<0$

(d)



No!

(e)



No!
$a<0$ and $b<0$

$f'(x) = -x^2 - x + 2 = (-x+1)(x+2)$

$f'(x) = 0 \rightarrow x = -2, 1,$

	$-\infty$ to -2	-2 to 1	1 to ∞ (x)
$1-x$	+	+	0 −
$x+2$	− 0	+	+
$f'(x)$	−	+	−
	dec	inc	dec

f

$\rightarrow$ only graph (a) has min and max points located at right x-values.