MATH 162 – FALL 2006 – FIRST EXAM
SEPTEMBER 18, 2006

SOLUTIONS

1) (4 points) The center and the radius of the sphere given by $x^2 + y^2 + z^2 = 4x + 3y$ are

A) Center $(0, 3/2, 2)$ and radius $3/2$

B) Center $(2, 3/2, 0)$ and radius $3/2$

C) Center $(2, 3/2, 0)$ and radius $5/2$

D) Center $(1, 2, 3)$ and radius $2/3$

E) Center $(2, 2/3, 1)$ and radius $5/2$

Solution: Complete the squares and write $x^2+y^2+z^2 = 4x+3y$ as $(x-2)^2+(y-\frac{3}{2})^2+z^2 = \frac{25}{4}$. So the center of the sphere is $(2, \frac{3}{2}, 0)$ and its radius is $\frac{5}{2}$. Correct answer C.

2) (8 points) The point 1/4 of the way from $(1, -3, 1)$ and $(7, 9, -9)$ is

A) $(4, 3, -4)$

B) $(5/2, 0, -3/2)$

C) $(3/2, 3, -3/2)$

D) $(3/2, 6, -5)$

E) $(11/4, 6, -13/2)$

Solution: The segment of line joining the points $P_1(1, -3, 1)$ and $P_2(7, 9, -9)$ is given by $(1, -3, 1) + t\vec{P_1P_2}$. But $\vec{P_1P_2} =< 6, 12, -10 >= (7, 9, -9) - (1, -3, 1)$. Taking $t = \frac{1}{4}$, we find the point which is 1/4 of the way between the two points. This point is $(\frac{5}{2}, 0, -\frac{3}{2})$. Correct answer B.

3) (8 points) The area of the triangle with vertices $(-1, 1, 1)$, $(2, 0, 2)$ and $(3, 2, 2)$ is

A) $\frac{3\sqrt{6}}{2}$

B) $\frac{5\sqrt{6}}{3}$

C) $2\sqrt{3}$

D) $\sqrt{6}$

E) $\frac{\sqrt{3}}{2}$

Solution: Let $P_1(-1,1,1)$, $P_2(2,0,2)$ and $P_3(3,2,2)$. These three points give two vectors: $\vec{v_1} = \vec{P_1P_2} = \langle 3,-1,1\rangle$ and $\vec{v_2} = \vec{P_1P_3} = \langle 4,1,1\rangle$. The area of the triangle is equal to one half of the area of the parallelogram formed by the vectors. So the area of the triangle is equal to $\frac{1}{2}|\vec{v_1} \times \vec{v_2}|$. We find that $\vec{v_1} \times \vec{v_2} = \langle -2,1,7\rangle$. So $\frac{1}{2}|\vec{v_1} \times \vec{v_2}| = \sqrt{54}2 = \frac{3\sqrt{6}}{2}$. Correct answer A.

4)(8 points) Let $\vec{a} = (-5,4,3)$ and $\vec{b} = (-1,-1,-2)$. Which one of the following is true?

I) $\text{comp}_{\vec{a}}\,\vec{b} = -5/\sqrt{50}$

II) $\text{comp}_{\vec{b}}\,\vec{a} = -5/\sqrt{50}$

III) $\text{comp}_{\vec{b}}\,\vec{a} = -5/\sqrt{6}$

IV) $\text{comp}_{\vec{a}}\,\vec{b} = -5/\sqrt{6}$

A) I is true, II, III and IV are false

B) I and II are true, III and IV are false

C) I and III are true, II and IV are false

D) III is true, I, II and IV are false

E) II and IV are true, I and II are false

Solution: If θ is the angle between $\vec{a}$ and $\vec{b}$, $\text{comp}_{\vec{a}}\,\vec{b} = |\vec{b}|\cos\theta$ and $\text{comp}_{\vec{b}}\,\vec{a} = |\vec{a}|\cos\theta$. Since $\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$, we have $\text{comp}_{\vec{a}}\,\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$ and $\text{comp}_{\vec{b}}\,\vec{a} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$. But $|\vec{a}| = \sqrt{50}$, $|\vec{b} = \sqrt{6}$ and $\vec{a}\cdot\vec{b} = -5$ we find that $\text{comp}_{\vec{a}}\,\vec{b} = \frac{-5}{\sqrt{50}}$ and $\text{comp}_{\vec{b}}\,\vec{a} = \frac{-5}{\sqrt{6}}$. So I and III are correct. Correct answer C.

5)(8 points) The area bounded by the curves $y = 6x^2$, and $y = 6x + 12$ in the interval $[0,3]$ is

A) 3

B) 4

C) 27

D) 31

E) 83

Solution: The curves $y = 6x^2$, and $y = 6x + 12$ intersect at $x = -1$ and at $x = 2$. In the interval $[0, 2]$ the curve $y = 6x + 12$ is above the curve $y = 6x^2$, but in the interval $[2, 3]$ the curve $y = 6x^2$ is above $y = 6x + 1$. So the area of the region is given by

$$A = \int_0^2 (6x + 12 - 6x^2)\, dx + \int_2^3 (6x^2 - 6x - 12)\, dx.$$

Computing these integrals we find that $A = 31$. Correct answer D

6)(8 points) The area bounded by the curves $y = 12 - 6x^2$ and $y = 6|x|$ is

A) 14

B) 7

C) 8

D) 3

E) 5

Solution: The curve $y = 12 - 6x^2$ is a parabola which is concave down and has vertex at the point $(0, 12)$. The curve $y = 6|x|$ looks like a V with vertex at $(0, 0)$. These curves intersect at the points $(-1, 6)$ and $(1, 6)$. So the area of this region is given by

$$A = \int_{-1}^1 (12 - 6x^2 - 6|x|)\, dx = 2\int_0^1 (12 - 6x^2 - 6x)\, dx = 14.$$

Answer A.

7)(8 points) Take the region bounded by the curves $y = x^2$, $y = 2 - x^2$ and $x = 0$, and rotate it about the y-axis. The volume of the solid generated is equal to

A) $\pi/2$

B) $2\pi/3$

C) π

D) $3\pi/2$

E) 2π
Solution: Both curves are parabolas. The one $y = x^2$ is concave up and has vertex at $(0,0)$. The other is concave down and has vertex at $(0,2)$. The curves intersect in the first quadrant when $x = 1$. This is a case where it is better to use method of cylindrical shells. We find that the volume is

$$V = 2\pi \int_0^1 x(2 - 2x^2)\, dx = \pi.$$

Answer C.

8) (8 points) The volume of the solid obtained by rotating the region bounded by the curves $x = -y^2 + 2y$, $x = 1$, $y = 0$ and $y = 2$ about the line $x = 1$ is given by the integral

A) $\pi \int_0^1 (1 - y^2 + 2y)\, dy$
B) $\pi \int_0^2 (1 - y^2 + 2y)\, dy$
C) $\pi \int_0^2 (1 - y^2 + 2y)^2\, dy$
D) $\pi \int_0^1 (1 - y^2 + 2y)^2\, dy$
E) $\pi \int_0^2 (1 + y^2 - 2y)^2\, dy$

Solution: The curve $x = 2y - y^2$ is a parabola which intersects the y-axis, i.e. $\{x = 0\}$ at $(0,0)$ and $(0,2)$. This curve intersects the line $x = 1$ at the point $(1,1)$. This is a case where the method of washers is more suitable. Fixed a point y the radius of the washer, which in this case is a circle is $R(y) = 1 - x = 1 - 2y + y^2$. The area of the washer is $A(y) = \pi R(y)^2 = \pi(1 - 2y + y^2)^2$. So the volume is given by

$$V = \pi \int_0^2 (1 - 2y + y^2)^2\, dy.$$

Correct answer E.

9) (8 points) A conical tank T is h meters high and the radius of its base is R meters long. The base of tank T rests on the ground. If the tank is filled with a liquid of density ρ Kg/m^3, the work necessary to empty it by pumping the liquid through its top is (g is the acceleration of gravity)

A) $\rho\pi g R^2 h$

B) $\rho\pi g R^3 h^2/3$

C) $\rho\pi g R h^2/2$

D) $\rho\pi g R^2 h^2/4$

E) $\rho\pi g R^2 h/4$

Solution: We choose our coordinate axis x with origin at the tip of the cone. Now take a chunk of the cone at height x from the top, and with thickness dx. The weight of this chunk, which is the minimum force required to move it, is equal to its volume times the density of the liquid times g. The volume of this chunk is equal to $V = \pi R(x)^2\ dx$, where $R(x)$ is the radius of the section. To compute $R(x)$ we look at the angle formed by the height of the cone and its side. One one hand, the tangent of this angle is equal to R/h. On the other hand it is equal to $R(x)/x$. So we get that $R(x)/x = R/h$ and so $R(x) = xR/h$. Therefore we conclude that

$$V = \pi \frac{R^2}{h^2} x^2\ dx$$

So the weight of the chunck is

$$dF = \rho g \pi \frac{R^2}{h^2} x^2\ dx,$$

and the work necessary to move this chunk to the top is

$$dW = \rho g \pi \frac{R^2}{h^2} x^3\ dx.$$

The work necessary to empty the tank is:

$$\int_0^h \rho g \pi \frac{R^2}{h^2} x^3\ dx = \pi \rho g \frac{h^2 R^2}{4}.$$

Correct answer: D.

10) (8 points) The integral

$$\int_1^2 x^{-2} \ln x\ dx \quad \text{is equal to}$$

A) $\frac{3}{4} - \frac{\ln 2}{2}$

B) $\frac{(1 - \ln 2)}{2}$

C) $2 - \ln 2$

D) $\ln 2$

E) $\frac{3\ln 2}{4}$

Solution: This is a typical example of integration by parts. Since we want to get rid of the $\ln x$ term we set $u = \ln x$ and $dv = x^{-2}\,dx$. So $du = x^{-1}\,dx$ and $v = -x^{-1}$. So

$$\int x^{-2}\ln x\,dx = -x^{-1}\ln x + \int x^{-2}\,dx = -x^{-1}\ln x - x^{-1}.$$

So

$$\int_1^2 x^{-2}\ln x\,dx = \left(-x^{-1}\ln x - x^{-1}\right)\Big|_1^2 = \frac{1}{2}(1 - \ln 2).$$

Correct answer B.

11) (8 points) The integral

$$\int_0^{\pi/4} x\sin x\,dx \text{ is equal to}$$

A) $\frac{\sqrt{2}}{2}$

B) $\sqrt{2} - \frac{\pi\sqrt{2}}{8}$

C) $\frac{3\sqrt{2}}{2}$

D) $\frac{\sqrt{2}}{4}$

E) $\frac{\sqrt{2}}{2} - \frac{\pi\sqrt{2}}{8}$

Solution: This is another case of integration by parts. Set $u = x$ and $dv = \sin x\,dx$, then $du = dx$ and $v = -\cos x$. So

$$\int x\sin x\,dx = -x\cos x + \int \cos x\,dx = -x\cos x + \sin x.$$

Therefore

$$\int_0^{\pi/4} x\sin x\,dx = (\sin x - x\cos x)]_0^{\frac{\pi}{4}} = \frac{\sqrt{2}}{2} - \frac{\pi\sqrt{2}}{8}.$$

Correct answer: E.

12) (8 points) The integral

$$\int_0^{\frac{\pi}{4}} \tan^3 x \sec^2 x \, dx \quad \text{is equal to}$$

A) 1/3

B) 1

C) 3/4

D) 1/4

E) 2/3

Solution: Here we use that $\tan^x + 1 = \sec^2 x$ and that $\frac{d}{dx} \sec x = \sec x \tan x$ So we write

$$\int \tan^3 x \sec^2 x \, dx = \int \tan^2 x \sec x \tan x \sec x \, dx = \int (\sec^2 x - 1) \sec x \tan x \sec x \, dx$$

Now, makin the substitution $u = \sec x$ in the last integral we obtain

$$\int_0^{\frac{\pi}{4}} \tan^3 x \sec^2 x \, dx = \int_0^{\frac{\pi}{4}} (\sec^2 x - 1) \sec x \tan x \sec x \, dx = \int_1^{\sqrt{2}} u(u^2 - 1) \, du = \frac{u^4}{4} - \frac{u^2}{2}\Big]_1^{\sqrt{2}} = \frac{1}{4}.$$

Correct answer: D.

13) (8 points) If $\int_0^1 x^2 e^x \, du = A$, then $\int_0^1 x^3 \, e^x \, dx$ is equal to

A) $3A$

B) $2A$

C) $e - A$

D) $6 - 2A$

E) $e - 3A$

Solution: This is another question on integration by parts. If we set $u = x^3$ and $dv = e^x \, dx$ we have $du = 3x^2 \, dx$ and $v = e^x$. So

$$\int x^3 e^x \, dx = x^3 e^x - 3 \int x^2 e^x \, dx.$$

Using that $\int_0^1 x^2 e^x \, du = A$, then

$$\int_0^1 x^3 e^x \, dx = x^3 e^x \Big]_0^1 - 3 \int x^2 e^x \, dx = e - 3A.$$

Correct answer: E.