MATH 162 – SPRING 2004 – SECOND EXAM
SOLUTIONS

The following formulas were given on the exam:

Moments and center of mass

$$M_x = \int_a^b \frac{1}{2}\left((f(x))^2 - (g(x))^2\right) dx, \quad M_y = \int_a^b x\,(f(x) - g(x))\,dx$$
$$\overline{x} = \frac{M_y}{M}, \quad \overline{y} = \frac{M_x}{M},$$

Arc length

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\ \ dx$$

Area of a surface of revolution

$$S = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\,dx$$

1) The mass of the region bounded by $f(x) = \frac{1}{2}\sqrt{4 - 2x^2}$, $g(x) = -\frac{1}{2}\sqrt{4 - 2x^2}$ and the y-axis is $\pi\sqrt{2}$. Its center of mass is

A) $(\frac{8}{3\pi\sqrt{2}}, 0)$

B) $(0, \frac{8}{3\pi\sqrt{2}})$

C) $(\frac{1}{2}, \frac{8}{3\pi\sqrt{2}})$

D) $(0, \frac{4}{3\pi\sqrt{2}})$

E) $(\frac{4}{3\pi\sqrt{2}}, 0)$

Solution: According to the formulas above, the center of mass is

$$\overline{x} = \frac{1}{\pi\sqrt{2}}\int_0^{\sqrt{2}} x\sqrt{4 - 2x^2}\,dx,$$
$$\overline{y} = 0 \quad \text{this is because} \quad f(x)^2 = g(x)^2.$$

To compute this integral we just set $u = 4 - 2x^2$. Then $du = -4x\ dx$ and the integral becomes

$$\overline{x} = \frac{1}{\pi\sqrt{2}} \int_0^4 \sqrt{u}\ \frac{du}{4} = \frac{1}{4\pi\sqrt{2}}\ \frac{2}{3}\ u^{\frac{3}{2}}\Big|_0^4 = \frac{1}{4\pi\sqrt{2}}\ \frac{2}{3}\ 8 = \frac{4}{3\pi\sqrt{2}}.$$

The correct answer is E.

2) The improper integral

$$\int_0^1 \ln x\ dx =$$

A) $2\ln 2$

B) $-4\ln 2$

C) $2\ln 2$

D) -1

E) $-\frac{1}{9}$

Solution: By definition of improper integrals

$$\int_0^1 \ln x\ dx = \lim_{a\to 0} \int_a^1 \ln x\ dx.$$

Integration by parts gives

$$\int_a^1 \ln x\ dx = \left.(x\ln x - x)\right|_a^1 = -1 - (a\ln a - a).$$

Now we have to compute

$$\lim_{a\to 0} a\ln a - a = \lim_{a\to 0} a\ln a$$

To do this we use L'Hosptial's rule and write

$$\lim_{a\to 0} a\ln a = \lim_{a\to 0} \frac{\ln a}{\frac{1}{a}} = \lim_{a\to 0} \frac{\frac{1}{a}}{-\frac{1}{a^2}} = \lim_{a\to 0} -a = 0.$$

So

$$\int_0^1 \ln x\ dx = -1$$

The correct answer is D. Unfortunately this question had two identical alternatives, but but both are incorrect.

3) The improper integral

$$\int_0^{\infty} x\ e^{-x^2}\ \ dx \quad \text{is equal to}$$

A) $\frac{1}{3}$

B) $\frac{1}{4}$

C) $\frac{1}{2}$

D) 1

E) 2

Solution: By definition

$$\int_0^{\infty} x\ e^{-x^2}\ \ dx = \lim_{M\to\infty} \int_0^M xe^{-x^2}\ dx.$$

To compute the integral we set $y = x^2$ and so $dy = 2x\ dx$. Therefore

$$\int_0^M xe^{-x^2}\ dx = \frac{1}{2}\int_0^{M^2} e^{-y}\ dy = \frac{1}{2}\left(1 - e^{-M^2}\right).$$

So

$$\int_0^{\infty} x\ e^{-x^2}\ \ dx = \lim_{M\to\infty} \int_0^M xe^{-x^2}\ dx = \lim_{M\to\infty} \frac{1}{2}\left(1 - e^{-M^2}\right) = \frac{1}{2}.$$

The correct answer is C.

4) The length of the curve $y = \frac{x^3}{6} + \frac{1}{2x}$, $1 \le x \le 2$ is

A) $\frac{17}{12}$

B) $\frac{4}{3}$

C) $\frac{3}{2}$

D) 2

E) 1

Solution: The derivative of $f(x) = \frac{x^3}{6} + \frac{1}{2x}$ is $f'(x) = \frac{x^2}{2} - \frac{1}{2x^2}$. So according to the formula given on the first page, the length of the curve is

$$L = \int_1^2 \sqrt{1 + \left(\frac{x^2}{2} - \frac{1}{2x^2}\right)^2}\, dx$$

Notice that

$$1 + \left(\frac{x^2}{2} - \frac{1}{2x^2}\right)^2 = 1 + \frac{x^4}{4} - \frac{1}{2} + \frac{1}{4x^4} = \frac{x^4}{4} + \frac{1}{2} + \frac{1}{4x^4} = \left(\frac{x^2}{2} + \frac{1}{2x^2}\right)^2.$$

Therefore

$$L = \int_1^2 \left(\frac{x^2}{2} + \frac{1}{2x^2}\right) dx = \left(\frac{x^3}{6} - \frac{1}{2x}\right)\Big|_1^2 = \left(\frac{8}{6} - \frac{1}{4}\right) - \left(\frac{1}{6} - \frac{1}{2}\right) = \frac{17}{12}.$$

The correct answer is A.

5) The area of the surface obtained by rotating the curve

$$y = x^3, \quad 0 \le x \le 1$$

about the x-axis is

A) $\pi\sqrt{3}$

B) $\frac{2\pi}{3}$

C) $\frac{\pi}{27}(10\sqrt{10} - 1)$

D) $6\pi(3\sqrt{3} - 1)$

E) $\frac{\pi}{3}(10\sqrt{10} - 1)$

Solution: The derivative of $f(x) = x^3$ is $f'(x) = 3x^2$ so according to the formula on the first page:

$$A = 2\pi \int_0^1 x^3\sqrt{1 + 9x^4}\, dx.$$

Set $u = 1 + 9x^4$. Then $du = 36x^3$ and the integral becomes:

$$A = 2\pi \int_0^1 x^3\sqrt{1 + 9x^4}\, dx = \frac{2\pi}{36}\int_1^{10} \sqrt{u}\, du = \frac{2\pi}{36}\,\frac{2}{3}\,u^{\frac{3}{2}}\Big|_1^{10} = \frac{2\pi}{36}\,\frac{2}{3}(10\sqrt{10} - 1) = \frac{\pi}{27}(10\sqrt{10} - 1).$$

The correct answer is C.

6) Find

$$\lim_{n\to\infty} \frac{\sqrt{2n^4+n^2+1}}{2n^2+1}$$

A) $\frac{1}{2}$

B) 1

C) $\frac{\sqrt{2}}{2}$

D) 0

E) It does not exist.

Solution: We just write

$$\lim_{n\to\infty} \frac{\sqrt{2n^4+n^2+1}}{2n^2+1} = \lim_{n\to\infty} \frac{\sqrt{n^4(2+\frac{1}{n^2}+\frac{1}{n^4})}}{n^2(2+\frac{1}{n^2})} = \lim_{n\to\infty} \frac{\sqrt{(2+\frac{1}{n^2}+\frac{1}{n^4})}}{2+\frac{1}{n^2}} = \frac{\sqrt{2}}{2}.$$

The correct answer is C.

7) The series

$$\sum_{n=1}^{\infty} \frac{\sqrt{2n^4+n^2+1}}{2n^2+1}$$

A) diverges

B) converges conditionally

C) converges by the ratio test

D) converges by the root test

E) converges by the integral test

Solution: We just computed in question 6 that $\lim_{n\to\infty} \frac{\sqrt{2n^4+n^2+1}}{2n^2+1} = \frac{\sqrt{2}}{2}$. Since this limit is not equal to zero, the series diverges. The correct answer is A.

8) Find

$$\lim_{n\to\infty} n \sin\left(\frac{1}{n}\right)$$

A) 0

B) 1

C) 2

D) ∞

E) it does not exist

Solution: We know that

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

We just the write

$$\lim_{n\to\infty} n \sin\left(\frac{1}{n}\right) = \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}}$$

Set $\frac{1}{n} = x$. So when $n \to \infty$, $x \to 0$. Therefore

$$\lim_{n\to\infty} n \sin\left(\frac{1}{n}\right) = \lim_{x\to 0} \frac{\sin x}{x} = 1.$$

The correct answer is B.

9) The series $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$

A) converges by the ratio test

B) diverges by the ratio test

C) converges because $\lim_{n\to\infty} \sin(\frac{1}{n}) = 0$

D) converges by the limit comparison test with $\sum_{n=1}^{\infty} \frac{1}{n}$

E) diverges by the limit comparison test with $\sum_{n=1}^{\infty} \frac{1}{n}$

Solution: The limit comparison theorem states that if $\sum a_n$ and $\sum b_n$ are series of positive terms and and $\lim_{n\to\infty} \frac{a_n}{b_n} = L$, with $L \neq 0$ and $L \neq \infty$, then the series $\sum a_n$ and $\sum b_n$

either converge or diverge simultaneously. That is one cannot converge and the other diverge.

In problem 8 we found that

$$\lim_{n\to\infty} n \sin\frac{1}{n} = \lim_{n\to\infty} \frac{\sin\frac{1}{n}}{\frac{1}{n}} = 1.$$

Since the series $\sum \frac{1}{n}$ diverges, $\sum \sin(\frac{1}{n})$ must diverge as well. The correct answer is E.

We remark that this is problem number 31 of lesson 20, which was assigned in the homework.

10)

$$\sum_{n=0}^{\infty} \frac{2^{n+1} - 3^n}{6^n} =$$

A) 1

B) 5

C) $\frac{7}{3}$

D) $\frac{1}{2}$

E) diverges

Solution: Recall that if $|r| < 1$ then

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

So we write

$$\sum_{n=0}^{\infty} \frac{2^{n+1} - 3^n}{6^n} = \sum_{n=0}^{\infty} 2\frac{2^n}{6^n} - \frac{3^n}{6^n} = 2\sum_{n=0}^{\infty} \frac{1}{3^n} - \sum_{n=0}^{\infty} \frac{1}{2^n} = 2\frac{1}{1-\frac{1}{3}} - \frac{1}{1-\frac{1}{2}} = 3 - 2 = 1.$$

The correct answer is A.

11) What is the smallest number of terms of the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$ that need to be added to compute its sum with error strictly less than 10^{-2}?

A) 3

B) 4

C) 5

D) 6

E) 7

Solution: Recall that for a converging alternating series $\sum_{n=0}^{\infty}(-1)^n b_n$ the difference between the sum of the series and the sum of the first N terms satisfies:

$$\left|\sum_{n=0}^{\infty}(-1)^n b_n - \sum_{n=0}^{N}(-1)^n b_n\right| \leq b_{N+1}.$$

In our case $b_n = \frac{1}{n!}$ and therefore we want N such that $b_{N+1} < 10^{-2}$. That is

$$\frac{1}{(N+1)!} < \frac{1}{100}$$

Hence we want the first N such that

$$(N+1)! > 100$$

That is $N = 4$. So the correct answer is B .

12) Which of the following is a correct statement about the series

$$S_1 = \sum_{n=2}^{\infty} \frac{1}{n \ln n} \quad \text{and} \quad S_2 = \sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}?$$

A) S_1 and S_2 are divergent

B) S_1 converges but S_2 diverges

C) S_1 diverges but S_2 converges conditionally

D) S_1 converges and S_2 converges conditionally

E) S_1 and S_2 converge absolutely

Solution: By the integral test $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ converges if and only if the integral $\int_2^{\infty} \frac{1}{x \ln x}\, dx$ converges. To compute this integral we set $\ln x = u$. Then $du = \frac{1}{x}\, dx$ and hence

$$\int_2^{\infty} \frac{1}{x \ln x}\, dx = \lim_{M \to \infty} \int_{\ln 2}^{\ln M} \frac{1}{u} du = \lim_{M \to \infty} (\ln(\ln M) - \ln(\ln 2)) = \infty.$$

As the integral diverges, so does the series $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$.

On the other hand $b_n = \frac{1}{n \ln n}$ satisfies

i) $b_n \geq 0$

ii) $b_{n+1} \leq b_n$

iii) $\lim_{n\to\infty} b_n = 0$.

Thus by the alternating series test $\sum_{n=2}^{\infty}(-1)^n \frac{1}{n \ln n}$ converges conditionally. The correct answer is C.

13) Find the interval of convergence of the series

$$\sum_{n=1}^{\infty} \frac{x^n}{n\ 3^n}.$$

A) $(-3, 3)$

B) $(-\frac{1}{3}, \frac{1}{3})$

C) $(-3, 3]$

D) $(-\frac{1}{3}, \frac{1}{3}]$

E) $[-3, 3)$

Solution: First we use the ratio test to find the radius of convergence: It says that if

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$$

and $L < 1$, then $\sum |a_n|$ converges. In this case $a_n = \frac{x^n}{n3^n}$. So

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)3^{n+1}} \frac{n3^n}{x^n} \right| = \lim_{n\to\infty} \left| \frac{x}{3} \right| \frac{n}{n+1} = \left| \frac{x}{3} \right|.$$

So the series converges if $\left|\frac{x}{3}\right| < 1$. That is it converges in $(-3, 3)$. Now we need to test the end points of this interval. When $x = 3$ we have

$$\sum_{n=1}^{\infty} \frac{x^n}{n\ 3^n} = \sum_{n=1}^{\infty} \frac{1}{n}$$

and this diverges. When $x = -3$

$$\sum_{n=1}^{\infty} \frac{x^n}{n\ 3^n} = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$$

which converges. So the series converges for x in $[-3, 3)$. The correct answer is E.