MA 16200 Exam 2 Spring 2009

1. Evaluate the integral:

$$\int_0^2 x^2\sqrt{4-x^2}\,dx$$

A. $\pi/2$

B. π

C. 2π

D. 4π

E. 8π

$4-x^2 \rightarrow$ Let $x = 2\sin\theta$. Then $dx = 2\cos\theta\, d\theta$
and $\theta = \sin^{-1}\left(\frac{x}{2}\right)$, so $\theta(0)=0$ and $\theta(2)=\frac{\pi}{2}$

$$\int_0^2 x^2\sqrt{4-x^2}\,dx = \int_0^{\pi/2}(4\sin^2\theta)(2\cos\theta)(2\cos\theta\, d\theta)$$
$$= 16\int_0^{\pi/2}\sin^2\theta\cos^2\theta\, d\theta = 16\int_0^{\pi/2}\left(\frac{1-\cos 2\theta}{2}\right)\left(\frac{1+\cos 2\theta}{2}\right)d\theta$$
$$= 4\int_0^{\pi/2}(1-\cos^2 2\theta)\,d\theta = 4\int_0^{\pi/2}\left(1-\left(\frac{1+\cos 4\theta}{2}\right)\right)d\theta$$
$$= 4\int_0^{\pi/2}\left(\frac{1}{2}-\frac{1}{2}\cos 4\theta\right)d\theta = 4\left(\frac{1}{2}\theta - \frac{1}{8}\sin 4\theta\right)\Big|_0^{\pi/2}$$
$$= 4\left[\left(\frac{\pi}{4}-0\right)-(0-0)\right] = \pi$$

2. $\int \frac{dx}{4-x^2}$

A. $\frac{1}{2}\frac{\ln|2+x|}{\ln|2-x|}+C$

B. $\frac{1}{2}\ln|2-x| - \frac{1}{2}\ln|2+x| + C$

C. $\frac{1}{2}\ln|2+x| - \frac{1}{2}\ln|2-x| + C$

D. $\frac{1}{4}\ln|2+x| - \frac{1}{4}\ln|2-x| + C$

E. $\frac{1}{4}\ln|2-x| - \frac{1}{4}\ln|2+x| + C$

$$\frac{1}{4-x^2} = \frac{1}{(2-x)(2+x)} = \frac{A}{2-x} + \frac{B}{2+x}$$
$$\rightarrow 1 = A(2+x) + B(2-x)$$

$x=2 \rightarrow 1 = 4A + 0\cdot B \rightarrow A = \frac{1}{4}$

$x=-2 \rightarrow 1 = 0\cdot A + 4B \rightarrow B = \frac{1}{4}$

$$\int \frac{1}{4-x^2}\,dx = \int\left(\frac{\frac{1}{4}}{2-x} + \frac{\frac{1}{4}}{2+x}\right)dx = \frac{1}{4}(-\ln|2-x|) + \frac{1}{4}\ln|2+x| + C$$
$$= \frac{1}{4}\ln|2+x| - \frac{1}{4}\ln|2-x| + C$$

7. Find the length of the curve, $y = \ln(\cos x)$, $0 \le x \le \frac{\pi}{4}$.

A. $\ln\sqrt{3}$

B. $\ln(\sqrt{3}+1)$

C. $\ln(\sqrt{3}+2)$

D. $\ln\sqrt{2}$

E. $\ln(\sqrt{2}+1)$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\left(\frac{-\sin x}{\cos x}\right)^2} = \sqrt{1+\tan^2 x}$$

$$= \sqrt{\sec^2 x} = \sec x$$ since $\sec x > 0$ for $0 \le x \le \pi/4$.

$$\text{arc length} = \int_0^{\pi/4} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^{\pi/4} \sec x\,dx$$

$$= \ln|\sec x + \tan x| \Big|_0^{\pi/4} = \ln|\sqrt{2}+1| - \ln|1+0|$$

$$= \ln(\sqrt{2}+1) - 0$$

8. Which integral represents the area of the surface obtained by revolving the curve, $y = e^{2x}$, $0 \le x \le 1$, about the y-axis?

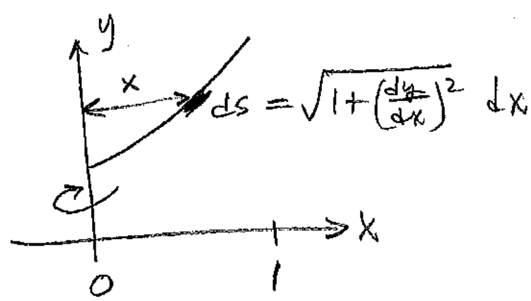
A. $\int_0^1 2\pi x e^{2x}\,dx$

B. $\int_0^1 2\pi x\sqrt{1+e^{4x}}\,dx$

C. $\int_0^1 2\pi x\sqrt{1+4e^{4x}}\,dx$

D. $\int_0^1 2\pi e^{2x}\sqrt{1+e^{4x}}\,dx$

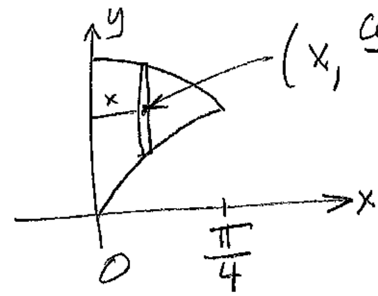
E. $\int_0^1 2\pi e^{2x}\sqrt{1+4e^{4x}}\,dx$



$$\text{Surface Area} = \int_0^1 2\pi x \sqrt{1+\left(2e^{2x}\right)^2}\,dx$$

$$= \int_0^1 2\pi x \sqrt{1+4e^{4x}}\,dx$$

9. Which of the following represents the y–coordinate of the centroid of the bounded region bounded by $y = \sin x$, $y = \cos x$, $x = 0$, and $x = \frac{\pi}{4}$, where A is the area of the region?



$\left(x, \frac{\cos x + \sin x}{2}\right)$

A. $\frac{1}{A}\int_0^{\frac{\pi}{4}} \frac{1}{2}(\cos^2 x - \sin^2 x)\, dx$

B. $\frac{1}{A}\int_0^{\frac{\pi}{4}} \frac{1}{2}(\sin^2 x - \cos^2 x)\, dx$

C. $\frac{1}{A}\int_0^{\frac{\pi}{4}} x(\cos x - \sin x)\, dx$

D. $\frac{1}{A}\int_0^{\frac{\pi}{4}} x(\sin x - \cos x)\, dx$

E. $\frac{1}{A}\int_0^{\frac{\pi}{4}} \frac{1}{2}x(\cos x - \sin x)^2 dx$

$$\bar{y} = \cancel{\frac{M_y}{A}} = \cancel{\frac{M_{x=0}}{A}} = \frac{M_x}{A}$$

$$= \frac{\int_0^{\pi/4} \cancel{x}(\cos x - \sin x)\, dx}{A}$$

$\frac{\cos x + \sin x}{2}$

$$= \frac{1}{A}\int_0^{\pi/4} \frac{1}{2}\cancel{x}(\cos^2 x - \sin^2 x)\, dx$$

10. $\sum_{n=1}^{\infty} \frac{2^n + 3^n}{4^n} =$

A. 1

B. 2

C. 3

D. 4

E. The series diverges.

$$\sum_{n=1}^{\infty} \frac{2^n}{4^n} = \sum_{n=1}^{\infty} \frac{2}{4}\left(\frac{2}{4}\right)^{n-1} = \sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{1}{2}\right)^{n-1}$$

$$= \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1.$$

and $\sum_{n=1}^{\infty} \frac{3^n}{4^n} = \sum_{n=1}^{\infty} \frac{3}{4}\left(\frac{3}{4}\right)^{n-1} = \frac{\frac{3}{4}}{1-\frac{3}{4}} = 3.$

$$\therefore \sum_{n=1}^{\infty} \frac{2^n+3^n}{4^n} = \sum_{n=1}^{\infty} \frac{2^n}{4^n} + \sum_{n=1}^{\infty} \frac{3^n}{4^n} = 1+3 = 4.$$

11. Which of the following series converge?

a. $\sum_{n=1}^{\infty} \frac{3^n}{1+3^n}$

b. $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}$

c. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

A. Only a.
B. Only b.
C. Only c.
D. None of them.
E. All of them.

a. diverges since $\lim_{n\to\infty} \frac{3^n}{1+3^n} = 1 \neq 0$.

b. diverges since $\sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$ is a p-series with $p=\frac{1}{3}$ and $\frac{1}{3}<1$.

c. $\int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx = \int_2^{\infty} (\ln x)^{-2}\frac{1}{x}\,dx$

$= \lim_{t\to\infty} \int_2^t (\ln x)^{-2}\frac{1}{x}\,dx = \lim_{t\to\infty}\left(-(\ln x)^{-1}\Big|_2^t\right)$

$= \lim_{t\to\infty}\left(-\frac{1}{\ln t}+\frac{1}{\ln 2}\right) = \frac{1}{\ln 2} \Rightarrow$ series converges

12. Which of the following statements are true?

I. If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$ TRUE

II. If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$ TRUE

III. If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges. FALSE

A. I only
B. I and II only
C. I and III only
D. II and III only
E. All of them.

I: $-|a_n| \le a_n \le |a_n|$

$\lim_{n\to\infty} -|a_n| = \lim_{n\to\infty} |a_n| = 0$

$\Rightarrow \lim_{n\to\infty} a_n = 0$ by squeeze theorem.

III: $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges and $\lim_{n\to\infty} \frac{1}{n} = 0$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges and $\lim_{n\to\infty} \frac{1}{n^2} = 0$