MA 16200 FINAL EXAM 01 Spring 2009

Name SOLUTIONS

ten–digit Student ID number

Lecture Time

Recitation Instructor

Section Number

Instructions:

1. Fill in all the information requested above. On the scantron sheet fill in your name, student ID number, and the section number of your recitation with an extra 0 at the left. See list below. Blacken the correct circles.
2. On the bottom under Test/Quiz Number, write 01 and fill in the little circles.
3. This booklet contains 25 problems, each worth 8 points. The maximum score is 200 points.
3. For each problem mark your answer on the scantron sheet and also circle it in this booklet.
4. Work only on the pages of this booklet.
5. Books, notes, calculators are not to be used on this test.
6. At the end turn in your exam and scantron sheet to your recitation instructor.

TA	Lecture time	Rec. time	Sect. #	TA	Lecture time	Rec. time	Sect. #
Hyojung Lee	11:30	7:30	0022	Ritesh Nagpal	2:30	8:30	0010
		8:30	0001			11:30	0013
Kwangho Choiy	11:30	9:30	0002				
		10:30	0003	Matthew Barrett	2:30	9:30	0011
Sungmun Cho	11:30	11:30	0019			10:30	0012
		12:30	0004	Jishnu Jaganathan		12:30	0023
Hyungyu Choo	11:30	1:30	0006			1:30	0015
		2:30	0007	Botong Wang	2:30	2:30	0016
Yean Su Kim	11:30	3:30	0008				
		4:30	0009	Young Su Kim	2:30	3:30	0017
						4:30	0018

Some Useful Formulas

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n$$

1. The equation $x^2 + 4y^2 - 2x - 4y = 7$ in the plane describes
   A. a circle with radius 3 and a center $(1, 1)$
   B. a circle with radius 3 and center $(1, \frac{1}{2})$
   C. a circle with radius 9 and center $(1, 1)$
   D. a circle with radius 9 and center $(1, \frac{1}{2})$
   E. not a circle

Complete the squares

$$(x^2 - 2x + \underline{1}) + 4(y^2 - y + \underline{\tfrac{1}{4}}) = 7 + \underline{1} + \underline{1}$$

$$\to (x-1)^2 + 4(y - \tfrac{1}{2})^2 = 9$$

$$\to \frac{(x-1)^2}{9} + \frac{(y-\frac{1}{2})^2}{\frac{9}{4}} = 1$$

This is an ellipse.

2. Determine whether the given pairs of vectors are orthogonal, parallel or neither

$$\vec{a}_1 = \langle 1, -1, 1 \rangle \qquad \vec{b}_1 = \langle 1, 1, 1 \rangle$$
$$\vec{a}_2 = \langle 4, 6 \rangle \qquad \vec{b}_2 = \langle -6, -9 \rangle$$
$$\vec{a}_3 = -\vec{i} + 2\vec{j} + 5\vec{k} \qquad \vec{b}_3 = 3\vec{i} + 4\vec{j} - \vec{k}$$

   A. $\vec{a}_1, \vec{b}_1$ are neither, $\vec{a}_2, \vec{b}_2$ are orthogonal, $\vec{a}_3, \vec{b}_3$, are parallel.
   B. $\vec{a}_1, \vec{b}_1$ are orthogonal, $\vec{a}_2, \vec{b}_2$ are parallel, $\vec{a}_3, \vec{b}_3$ are orthogonal.
   C. $\vec{a}_1, \vec{b}_1$ are neither, $\vec{a}_2, \vec{b}_2$ are parallel, $\vec{a}_3, \vec{b}_3$ are orthogonal.
   D. $\vec{a}_1, \vec{b}_1$ are neither, $\vec{a}_2, \vec{b}_2$ are parallel, and $\vec{a}_3, \vec{b}_3$ are parallel.
   E. $\vec{a}_1, \vec{b}_1$ are orthogonal, $\vec{a}_2, \vec{b}_2$ are orthogonal, and $\vec{a}_3, \vec{b}_3$ are parallel.

$\vec{a}_1 \cdot \vec{b}_1 = 1 - 1 + 1 = 1 \neq 0 \to$ not orthogonal

$\vec{a}_1 \neq k\vec{b}_1 \Rightarrow$ not parallel

$-\frac{3}{2} a_2 = -\frac{3}{2}\langle 4, 6 \rangle = \langle -6, -9 \rangle = \vec{b} \Rightarrow$ parallel

$\vec{a}_3 \cdot \vec{b}_3 = -3 + 8 - 5 = 0 \Rightarrow$ orthogonal

3. Let $\vec{a}, \vec{b}, \vec{c}$ be three vectors in $\mathbb{R}^3$. Then

$$((\vec{a}+\vec{b}) \times (2\vec{a}-\vec{b})) \cdot (-5\vec{a}+7\vec{b}+\vec{c})$$

equals

A. 0

B. $(\vec{a} \times \vec{b}) \times \vec{c}$

C. $(\vec{a} \times \vec{b}) \cdot \vec{c}$

D. $7(\vec{a} \times \vec{b}) \cdot \vec{c}$

E. $-3(\vec{a} \times \vec{b}) \cdot \vec{c}$

First, $(\vec{a}+\vec{b}) \times (2\vec{a}-\vec{b})$

$$= (\vec{a} \times 2\vec{a}) - (\vec{a} \times \vec{b}) + (\vec{b} \times 2\vec{a}) - (\vec{b} \times \vec{b})$$
$$= \vec{0} - (\vec{a} \times \vec{b}) - 2(\vec{a} \times \vec{b}) - \vec{0}$$
$$= -3(\vec{a} \times \vec{b})$$

Second, $-3(\vec{a} \times \vec{b}) \cdot (-5\vec{a}+7\vec{b}+\vec{c})$

$$= -3\left[(\vec{a} \times \vec{b}) \cdot (-5\vec{a}) + (\vec{a} \times \vec{b}) \cdot (7\vec{b}) + (\vec{a} \times \vec{b}) \cdot (\vec{c})\right]$$
$$= -3\left[0 + 0 + (\vec{a} \times \vec{b}) \cdot \vec{c}\right]$$
$$= -3(\vec{a} \times \vec{b}) \cdot \vec{c}$$

4. The area between the curves $x = 1 - y^2$ and $x = y^2 - 1$ is

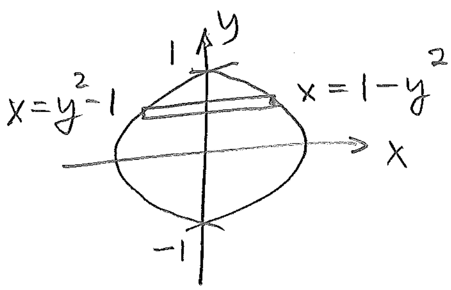
A. $\frac{2}{3}$

B. $\frac{4}{3}$

C. $\frac{6}{3}$

D. $\frac{8}{3}$

E. $\frac{10}{3}$



$$\text{area} = \int_{-1}^{1} \left[(1-y^2) - (y^2-1)\right] dy$$
$$= \int_{-1}^{1} \left[2 - 2y^2\right] dy$$
$$= \left(2y - \tfrac{2}{3}y^3\right)\Big|_{-1}^{1}$$
$$= \left(2 - \tfrac{2}{3}\right) - \left(-2 + \tfrac{2}{3}\right)$$
$$= 4 - \tfrac{4}{3}$$
$$= \frac{8}{3}$$

5. A spring has a natural length of 2m. If a force of 25 N is needed to keep it stretched to a length of 5m, how much work is required to stretch it from 2m to 4m?

   A. 25J

   B. 50J

   C. $\frac{25}{2}$ J

   D. $\frac{25}{3}$ J

   E. $\frac{50}{3}$ J

$F(x) = kx$. $25 = k3 \rightarrow k = \frac{25}{3}$

$$\text{work} = \int_0^2 \frac{25}{3} x \, dx$$

$$= \frac{25}{6} x^2 \Big|_0^2$$

$$= \frac{25}{6}(4-0)$$

$$= \frac{100}{6} = \frac{50}{3} \text{ Newton-meters}$$

6. If the region bounded by $y = 3 + 2x - x^2$ and $x + y = 3$ is rotated about the y–axis, then the resulting solid will have volume

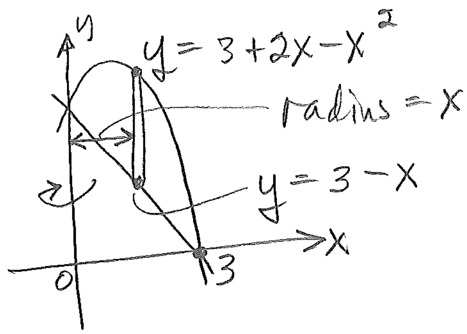
   A. $\frac{16}{3}\pi$

   B. $\frac{9}{2}\pi$

   C. $\frac{27}{2}\pi$

   D. 8π

   E. 9π



(Shells) $$\text{Volume} = \int_0^3 2\pi x \left((3+2x-x^2) - (3-x)\right) dx$$

$$= \int_0^3 2\pi x (3x - x^2) \, dx$$

$$= \int_0^3 2\pi (3x^2 - x^3) \, dx = 2\pi \left(x^3 - \frac{1}{4}x^4 \right)\Big|_0^3$$

$$= 2\pi\left(27 - \frac{81}{4}\right) = 2\pi\left(\frac{108-81}{4}\right)$$

$$= \frac{27\pi}{2}$$

7. Evaluate the integral

$$\int_0^{\pi} t \sin 5t dt$$

$\int u \cdot dv = uv - \int v\, du$

Let u be first in L.I.A.T.E.

A. $-\frac{1}{25}$

B. $\frac{\pi}{5}$

C. $\frac{1}{25}$

D. $\frac{1}{25} - \frac{\pi}{5}$

E. $-\frac{\pi}{5}$

let $u = t$ and $dv = \sin 5t\,dt$

then $du = dt$ and $v = -\frac{1}{5}\cos 5t$

$$\int_0^{\pi} t \sin 5t\,dt = (t)\left(-\tfrac{1}{5}\cos 5t\right)\Big|_0^{\pi} - \int_0^{\pi} -\tfrac{1}{5}\cos 5t\,dt$$

$$= \left(-\tfrac{1}{5}t\cos 5t + \tfrac{1}{25}\sin 5t\right)\Big|_0^{\pi}$$

$$= \left(\tfrac{\pi}{5} + 0\right) - (0 + 0)$$

$$= \frac{\pi}{5}$$

8. Evaluate the integral

$$\int_0^{\pi/4} \tan^2 x dx$$

A. $1 + \frac{\pi}{4}$

B. $-\frac{\pi}{4}$

C. $\frac{\sqrt{2}}{2} - \frac{\pi}{4}$

D. $1 - \pi/4$

E. $\frac{\sqrt{2}}{2} + \frac{\pi}{4}$

$$= \int_0^{\pi/4} (\sec^2 x - 1)\,dx$$

$$= (\tan x - x)\Big|_0^{\pi/4}$$

$$= \left(1 - \tfrac{\pi}{4}\right) - (0 - 0)$$

$$= 1 - \frac{\pi}{4}$$

9. After the trigonometric substitution $x = 4\sin\theta$, the integral

$$\int_0^{2\sqrt{3}} \frac{x^3}{\sqrt{16-x^2}}\,dx$$

$x = 4\sin\theta$

$\rightarrow dx = 4\cos\theta\, d\theta$

and $\sqrt{16-x^2} = 4\cos\theta$

is transformed into the following integral:

A. $\displaystyle\int_0^{\pi/3} 4^3 \sin^3\theta d\theta$

B. $\displaystyle\int_0^{\frac{\pi}{3}} \frac{4^2\sin^3\theta}{\cos\theta}\,d\theta$

C. $\displaystyle\int_0^{\pi/6} 4^3 \sin^3\theta d\theta$

D. $\displaystyle\int_0^{\pi/6} \frac{4^2\sin^3\theta}{\cos\theta}\,d\theta$

E. $\displaystyle\int_0^{\pi/3} 4^2 \sin^3\theta d\theta$

$\theta = \sin^{-1}\left(\frac{x}{4}\right)$

$\theta(0) = \sin^{-1}(0) = 0$

$\theta(2\sqrt{3}) = \sin^{-1}\left(\frac{2\sqrt{3}}{4}\right) = \frac{\pi}{3}$

$$\int_0^{2\sqrt{3}} \frac{x^3}{\sqrt{16-x^2}}\,dx = \int_0^{\pi/3} \frac{64\sin^3\theta}{4\cos\theta}\, 4\cos\theta\, d\theta$$

$$= \int_0^{\pi/3} 64\sin^3\theta\, d\theta$$

10. Evaluate

$$\int \frac{x^2+2x+5}{x^2+1}\,dx = *$$

$$\begin{array}{r|l} & 1 \\ \hline x^2+1 & x^2+2x+5 \\ & -(x^2 \quad\quad +1) \\ \hline & 2x+4 \end{array}$$

A. $x + (2x+4)\tan^{-1}x + C$

B. $x + \ln(x^2+1) + 4\tan^{-1}x + C$

C. $(x^2+2x+5)\tan^{-1}x + C$

D. $x + 2x\ln(x^2+1) + 4\tan^{-1}x + C$

E. $x + 2\ln(x^2+1) + 4\tan^{-1}x + C$

$* = \int \left(1 + \frac{2x+4}{x^2+1}\right)dx$

$= \int \left(1 + \frac{2x}{x^2+1} + \frac{4}{x^2+1}\right)dx$

$= x + \ln(x^2+1) + 4\tan^{-1}x + C$

11. Which of the following integrals converge?

diverges (I) $\displaystyle\int_{-\infty}^{0} \frac{1}{2x-5}\,dx = \lim_{t\to-\infty}\int_t^0 \frac{1}{2x-5}\,dx = \lim_{t\to-\infty}\left(\frac{1}{2}\ln|2x-5|\Big|_t^0\right)$

conv. (II) $\displaystyle\int_{2}^{3} \frac{1}{\sqrt{3-x}}\,dx$ $\displaystyle = \lim_{t\to-\infty}\left(\frac{1}{2}\ln 5 - \frac{1}{2}\ln|2t-5|\right) = \frac{1}{2}\ln 5 - \infty$

conv. (III) $\displaystyle\int_{0}^{\infty} \frac{x}{x^3+1}\,dx$

(II) $\displaystyle = \lim_{t\to 3^-}\int_2^t \frac{1}{\sqrt{3-x}}\,dx = \lim_{t\to 3^-}\left(-2(3-x)^{1/2}\Big|_2^t\right)$

$\displaystyle = \lim_{t\to 3^-}\left(-2(3-t)^{1/2} + 2\right) = 0+2$

A. All of them

B. (I) and (II) only

C. (II) and (III) only

D. (I) and (III) only

E. none

(III) note: $0 \le \frac{x}{x^3+1} < 1$ for $0 \le x \le 1$

Therefore $\displaystyle\int_0^1 \frac{x}{x^3+1}\,dx < \infty$

also note: $0 < \frac{x}{x^3+1} < \frac{x}{x^3} = \frac{1}{x^2}$ for $1 < x < \infty$

and $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^2}\,dx = \lim_{t\to\infty}\left(\frac{-1}{x}\Big|_1^t\right) = \lim_{t\to\infty}\left(-\frac{1}{t}+1\right) = 1$

Therefore $\displaystyle\int_0^\infty \frac{x}{x^3+1}\,dx$ converges.

12. Let $(\overline{x}, \overline{y})$ be the centroid of the region bounded by the curves $y = 1/x$, $y = 0$, $x = 1$, $x = 2$. Then the value of $\overline{x}$ is given by

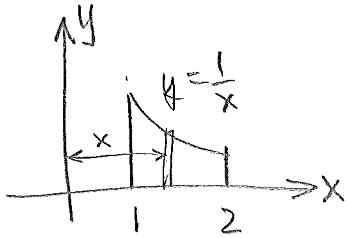
A. 1

B. 3/2

C. $\ln 2$

D. $\dfrac{1}{4\ln 2}$

E. $\dfrac{1}{\ln 2}$



$$\overline{x} = \frac{M_y}{m} = \frac{\int_1^2 x\left(\frac{1}{x}-0\right)dx}{\int_1^2 \left(\frac{1}{x}-0\right)dx} = \frac{\int_1^2 1\,dx}{\ln x\Big|_1^2} = \frac{x\Big|_1^2}{\ln 2} = \frac{2-1}{\ln 2} = \frac{1}{\ln 2}$$

13. If $a = \lim_{n\to\infty} \cos(\frac{n}{2})$ and $b = \lim_{n\to\infty} \cos(\frac{2}{n})$, then

A. $a = 0$ and $b = 1$

B. $a = 1$ and $b = 0$

C. $a = 1$ and b does not exist

D. a does not exist and $b = 1$

E. Neither a nor b exists.

$a = \lim_{n\to\infty} \cos\left(\frac{n}{2}\right)$ does not exist.

(cosine $\left(\frac{n}{2}\right)$ oscillates between -1 and 1 as its argument $\frac{n}{2} \to \infty$)

$b = \lim_{n\to\infty} \cos\left(\frac{2}{n}\right) = \cos(0) = 1$

14. Find the sum of the series $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{3^n}$ and find the set of values for which your answer is valid.

A. $f(x) = \dfrac{x}{3-x}$ for $-3 < x < 3$

B. $f(x) = \dfrac{x}{3-x}$ for $-3 \le x < 3$

C. $f(x) = \dfrac{1}{3-x}$ for $-3 < x < 3$

D. $f(x) = \dfrac{1}{3-x}$ for $-3 \le x < 3$

E. $f(x) = \dfrac{1}{3-x}$ for $x \ne 3$

$$f(x) = \sum_{n=1}^{\infty} \left(\frac{x}{3}\right)^n = \sum_{n=1}^{\infty} \left(\frac{x}{3}\right)\left(\frac{x}{3}\right)^{n-1}$$

$$= \frac{\frac{x}{3}}{1-\frac{x}{3}} = \frac{\frac{x}{3}}{\frac{3-x}{3}} = \frac{x}{3-x}$$

for $\left|\frac{x}{3}\right| < 1 \to -3 < x < 3$.

Note: $\sum_{n=1}^{\infty} \frac{x^n}{3^n}$ is a geometric series.

15. $\sum_{n=1}^{\infty} \frac{n^2}{n^3+1}$ is

A. Convergent by the integral test

B. Convergent by the ratio test

C. Divergent by the ratio test

D. Divergent by the limit comparison test

E. Divergent by the root test

Note: the root and ratio tests have limit 1 and hence are inconclusive.

$$\lim_{n\to\infty} \frac{\frac{n^2}{n^3+1}}{\frac{1}{n}} = 1 > 0$$

and $\sum^{\infty} \frac{1}{n}$ diverges.

16. If we know that $\ln 2 = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$, what is the least number of terms of the series to use to be sure that we have approximated $\ln 2$ to within 10^{-2}?

A. 9

B. 99

C. 999

D. 9,999

E. 999,999

$10^{-2} = \frac{1}{100}$

n	$\left\|\frac{(-1)^{n-1}}{n}\right\| = \frac{1}{n}$
99	$\frac{1}{99} > \frac{1}{100}$
100	$\frac{1}{100} \leq \frac{1}{100}$

Therefore $\left| \sum_{n=1}^{99} \frac{(-1)^{n-1}}{n} - \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \right| \leq \frac{1}{100}$

17. Find the interval of convergence of $\sum_{n=1}^{\infty} \frac{10^n x^n}{n^3}$.

A. $(-\infty, \infty)$

B. $(-10, 10)$

C. $\left(-\frac{1}{10}, \frac{1}{10}\right)$

D. $\left[-\frac{1}{10}, \frac{1}{10}\right)$

E. $\left[-\frac{1}{10}, \frac{1}{10}\right]$

Root Test: $\lim_{n\to\infty} \left| \frac{10^n x^n}{n^3} \right|^{1/n}$

$= \lim_{n\to\infty} \frac{10|x|}{(n^{1/n})^3} = 10|x|.$

$10|x| < 1 \longrightarrow -\frac{1}{10} < x < \frac{1}{10}$

Endpts: $x = -\frac{1}{10} \rightarrow \sum_{n=1}^{\infty} \frac{10^n\left(-\frac{1}{10}\right)^n}{n^3} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$ converges.

$x = \frac{1}{10} \rightarrow \sum_{n=1}^{\infty} \frac{10^n\left(\frac{1}{10}\right)^n}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n^3}$ converges

18. Find a power series representation for $f(x) = \frac{x}{2x^2+1}$ and find its radius of convergence R.

A. $f(x) = \sum_{n=0}^{\infty} (-1)^n 2^n x^{2n+1}$, $R = \frac{1}{2}$

B. $f(x) = \sum_{n=0}^{\infty} (-1)^n 2^n x^{2n+1}$, $R = \frac{1}{\sqrt{2}}$

C. $f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{2^n}$, $R = 2$

D. $f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2^n}$, $R = 2$

E. $f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2^n}$, $R = \sqrt{2}$

$f(x) = \frac{x}{1-(-2x^2)} = x \sum_{n=0}^{\infty} (-2x^2)^n$

$= \sum_{n=0}^{\infty} (-1)^n 2^n x^{2n+1}$, $|-2x^2| < 1$

$\longrightarrow x^2 < \frac{1}{2} \rightarrow -\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$

$\longrightarrow R = \frac{1}{\sqrt{2}}$

19. The first three terms of the McLaurin series of $f(x) = x(1-x^2)^{-\frac{1}{2}}$ are

A. $1 + \frac{1}{2}\,x^2 + \frac{3}{8}\,x^4$

B. $1 - \frac{1}{2}\,x^2 + \frac{3}{8}\,x^4$

C. $x + \frac{1}{2}\,x^3 + \frac{3}{8}\,x^5$

D. $x - \frac{1}{2}\,x^3 + \frac{3}{8}\,x^5$

E. $x + \frac{1}{2}\,x^3 - \frac{1}{8}\,x^5$

$$f(x) = x \cdot \left(1 + (-x^2)\right)^{-1/2}$$

$$= x\left(1 + \left(-\tfrac{1}{2}\right)(-x^2) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}(-x^2)^2 + \cdots\right)$$

$$= x + \tfrac{1}{2}x^3 + \tfrac{3}{8}x^5 + \cdots$$

<u>Binomial Series</u>

20. The Taylor series of $f(x) = \cos x$ at $a = \frac{\pi}{2}$ is

A. $\sum_{n=0}^{\infty} (-1)^{n-1} \frac{(x-\frac{\pi}{2})^{2n+1}}{(2n+1)!}$

B. $\sum_{n=0}^{\infty} (-1)^{n} \frac{(x-\frac{\pi}{2})^{2n}}{(2n)!}$

C. $\sum_{n=0}^{\infty} (-1)^{n} \frac{(x-\frac{\pi}{2})^{2n+1}}{(2n+1)!}$

D. $\sum_{n=0}^{\infty} (-1)^{n} \frac{(x+\frac{\pi}{2})^{2n}}{(2n)!}$

E. $\sum_{n=0}^{\infty} (-1)^{n-1} \frac{(x+\frac{\pi}{2})^{2n+1}}{(2n+1)!}$

$f(x) = \cos x$ $\quad f(\frac{\pi}{2}) = 0$

$f'(x) = -\sin x$ $\quad f'(\frac{\pi}{2}) = -1$

$f''(x) = -\cos x$ $\quad f''(\frac{\pi}{2}) = 0$

$f'''(x) = \sin x$ $\quad f'''(\frac{\pi}{2}) = 1$

$f^{(4)}(x) = \cos x$ $\quad f^{(4)}(\frac{\pi}{2}) = 0$

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!}\left(x-\tfrac{\pi}{2}\right)^{2n+1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{(2n+1)!}\left(x-\tfrac{\pi}{2}\right)^{2n+1}$$

represent cos x by its Maclaurin Series.

21. $\lim_{x\to 0} \dfrac{\cos x - 1 + \frac{x^2}{2}}{x^4} =$

A. 0

B. 1/4

C. 1/12

D. 1/24

E. None of the above

$$\lim_{x\to 0} \frac{\left(\cancel{1} - \cancel{\frac{x^2}{2!}} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) - \cancel{1} + \cancel{\frac{x^2}{2}}}{x^4}$$

$$= \lim_{x\to 0} \frac{x^4\left(\frac{1}{4!} - \frac{x^2}{6!} + \cdots\right)}{x^4\left(\quad 1 \quad\right)}$$

$$= \frac{1}{4!} = \frac{1}{24}.$$

(or use l'Hôpital's Rule)

22. Find the points on the curve

$$x = 2t^3 + 3t^2 - 12t,\ y = 2t^3 + 3t^2 + 1$$

where the tangent is horizontal.

A. $(20, -3)$ and $(-7, 6)$

B. $(-2, 0)$ and $(1, 0)$

C. $(0, 1)$ and $(13, 2)$

D. $(0, 0)$

E. $(0, -2)$ and $(0, 1)$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{6t^2+6t}{6t^2+6t-12} = 0$$

$$\rightarrow 6t(t+1) = 0 \rightarrow t = 0 \text{ or } -1$$

$$t=0 \rightarrow x=0,\ y=1 \rightarrow (x,y) = (0,1)$$

$$t=-1 \rightarrow x=13,\ y=2 \rightarrow (x,y) = (13,2)$$

23. Identify the curve. Hint: Find a Cartesian equation for it.

$$r = 3\sin\theta$$

A. a circle of radius $\sqrt{3}$ centered at $(0,0)$

B. a parabola with vertex $(0,0)$

C. a half–line through $(0,0)$

D. a cycloid

E. a circle of radius $3/2$ centered at $(0, \frac{3}{2})$

$r = 3\sin\theta$

$\rightarrow r^2 = 3r\sin\theta$

$\rightarrow x^2 + y^2 = 3y$

$\rightarrow x^2 + y^2 - 3y = 0$

$\rightarrow x^2 + \left(y^2 - 3y + \frac{9}{4}\right) = \frac{9}{4}$

$\rightarrow x^2 + \left(y - \frac{3}{2}\right)^2 = \left(\frac{3}{2}\right)^2$

24. For which values of t is the curve

$$x = t^3 - 12t, \quad y = t^2 - 1$$

concave upward?

A. $t < -2$

B. $t < -2$ or $t > 2$

C. $t > 2$

D. $t > 4$

E. $-2 < t < 2$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{3t^2-12}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{(2)(3t^2-12)-(2t)(6t)}{(3t^2-12)^2}}{3t^2-12}$$

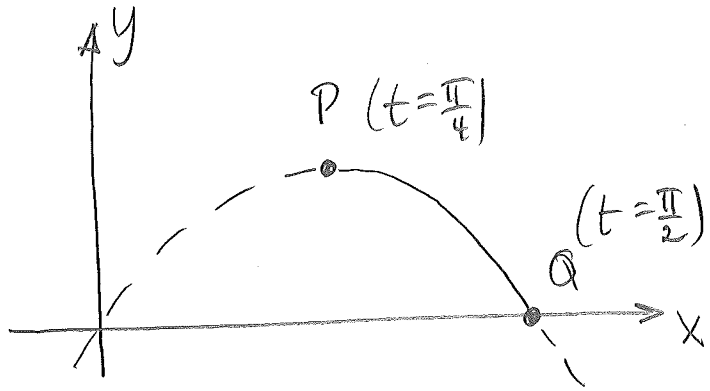
$$= \frac{6t^2 - 24 - 12t^2}{3t^2-12} = \frac{-6t^2-24}{3t^2-12}$$

$$= \frac{-6(t^2+4)}{3(t^2-4)} = \frac{-2(t^2+4)}{t^2-4}$$

t: $-\infty$ to ∞	$(-\infty, -2)$	$(-2, 2)$	$(2, \infty)$
-2	$-$	$-$	$-$
t^2+4	$+$	$+$	$+$
t^2-4	$+$	$-$	$+$
d^2y/dx^2	$-$	$+$	$-$

concave upward

25. A part of the curve $x = 3t,\ y = \sin 2t$ is sketched below, where P is the highest point on the arc shown. Then the length of the arc of the curve from P to Q is given by



at Q, $\sin 2t = 0$
$\rightarrow 2t = \pi$
$\rightarrow t = \frac{\pi}{2}$
Therefore $P = \frac{1}{2}\left(\frac{\pi}{2}\right)$
$= \frac{\pi}{4}$
at highest point on curve where
$\sin(2t) = 1$
$\rightarrow 2t = \frac{\pi}{2}$
$\rightarrow t = \frac{\pi}{4}$

A. $\displaystyle\int_{\pi/2}^{\pi} \sqrt{1 + 4\cos^2 2t}\ dt$

B. $\displaystyle\int_{\pi/4}^{\pi/2} \sqrt{9 + 4\cos^2 2t}\ dt$

C. $\displaystyle\int_{\frac{\pi}{2}}^{\pi} \sqrt{9 + \cos^2 2t}\ dt$

D. $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sqrt{1 + 4\cos^2 2t}\ dt$

E. $\displaystyle\int_{0}^{\pi} \sqrt{9 + \cos^2 2t}\ dt$

arc length $\displaystyle = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$

$\displaystyle = \int_{\pi/4}^{\pi/2} \sqrt{(3)^2 + (2\cos 2t)^2}\ dt$