NAME GRADING KEY

STUDENT ID ______________________

RECITATION INSTRUCTOR ______________________

RECITATION TIME ______________________

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TOTAL	/100

DIRECTIONS

1. Write your name, student ID number, recitation instructor's name and recitation time in the space provided above. Also write your name at the top of pages 2, 3 and 4.
2. The test has four (4) pages, including this one.
3. Write your answers in the boxes provided.
4. You must show sufficient work to justify all answers unless otherwise stated in the problem. Correct answers with inconsistent work may not be given credit.
5. Credit for each problem is given in parentheses in the left hand margin.
6. No books, notes or calculators may be used on this exam.

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(8) 1. Find the absolute maximum and absolute minimum values of the function $f(x) = \frac{x}{x+1}$ on the interval $[1, 2]$.

$$f'(x) = \frac{(x+1)1 - x\cdot 1}{(x+1)^2} = \frac{1}{(x+1)^2} \quad ①$$

$f'(x) = 0$: $\frac{1}{(x+1)^2} = 0$ no solution

$\therefore$ no critical numbers ①

$f(1) = \frac{1}{2}$ $\quad f(2) = \frac{2}{3}$

abs. max. $\boxed{f(\overset{②}{2}) = \overset{①}{\frac{2}{3}}}$

abs. min. $\boxed{f(\underset{②}{1}) = \underset{①}{\frac{1}{2}}}$

$\boxed{8}$

(4) 2. Explain why the function $f(x) = x^{2/3}$ does not satisfy the hypotheses of the Mean Value Theorem on the interval $[-2, 3]$.

$f'(x) = \frac{2}{3}x^{-\frac{1}{3}}$ for $x \neq 0$; $f'(0)$ DNE

f is not differentiable for all $x \in (-2, 3)$

$\boxed{4}$

(8) 3. Use calculus to find a positive number such that the sum of the number and its reciprocal is as small as possible.

Let $S = \overset{③}{x} + \frac{1}{x}$, $x \overset{①}{>} 0$. Find abs. min. of S on $(0, \infty)$

$$\frac{dS}{dx} = 1 - \frac{1}{x^2} = \frac{x^2-1}{x^2} \quad ②$$

$\frac{dS}{dx} = 0$: $\frac{x^2-1}{x^2} = 0 \rightarrow x = 1$ ②

$\frac{dS}{dx}$: - - - 0 + + + + (0, 1, x)

$\therefore$ S has abs. min. when $x = 1$

$\boxed{1}$

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0 credit if answer is correct but there is no work or work is wrong

(20) 4. Find each of the following as a real number, +∞, −∞ or write DNE (does not exist).

(a) $\lim_{x\to 0}\dfrac{e^x+e^{-x}-2}{x^2} \overset{L'H}{=} \lim_{x\to 0}\dfrac{e^x-e^{-x}}{2x} \overset{L'H}{=} \lim_{x\to 0}\dfrac{e^x+e^{-x}}{2} = \dfrac{2}{2} = 1$

$\frac{0}{0}$ $\frac{0}{0}$

$\boxed{1}$ $\boxed{5}$

(b) $\lim_{x\to 0}\dfrac{\sin^{-1}x}{x} \overset{L'H}{=} \lim_{x\to 0}\dfrac{\frac{1}{\sqrt{1-x^2}}}{1} = 1$

$\frac{0}{0}$

$\boxed{1}$ $\boxed{5}$

(c) $\lim_{x\to\infty}(x-\sqrt{x^2-1}) = \lim_{x\to\infty}\left(x-\sqrt{x^2-1}\right)\dfrac{(x+\sqrt{x^2-1})}{(x+\sqrt{x^2-1})}$

$\infty-\infty$

$= \lim_{x\to\infty}\dfrac{x^2-(x^2-1)}{x+\sqrt{x^2-1}} = \lim_{x\to\infty}\dfrac{1}{x+\sqrt{x^2-1}} = 0$

$\boxed{0}$ $\boxed{5}$

(d) $\lim_{x\to 0^+}(1+x)^{1/x} = \lim_{x\to 0^+} e^{\ln(1+x)^{1/x}} = \lim_{x\to 0^+} e^{\frac{1}{x}\ln(1+x)} = e^{\lim_{x\to 0^+}\frac{\ln(1+x)}{x}}$

$\lim_{x\to 0^+}\dfrac{\ln(1+x)}{x} \overset{L'H}{=} \lim_{x\to 0^+}\dfrac{\frac{1}{1+x}}{1} = 1$

$\frac{0}{0}$

$\therefore \lim_{x\to 0^+}(1+x)^{\frac{1}{x}} = e^1 = e$

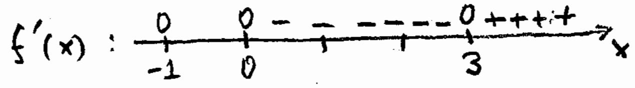
$\boxed{e}$ $\boxed{5}$

(14) 5. The numbers 3 and −1 are critical numbers of the function $f(x) = 2x^5 - 5x^4 - 10x^3$. Showing all necessary work, decide whether f has a local maximum or a local minimum

(a) at 3 using the first derivative test,

$f'(x) = 10x^4 - 20x^3 - 30x^2 = 10x^2(x^2-2x-3) =$ ②

$= 10x^2(x+1)(x-3)$ ①



f' changes from negative to positive at 3 ②

∴ f has a local minimum at 3 ②

loc. min. ⑦

(b) at −1 using the second derivative test.

$f''(x) = 40x^3 - 60x^2 - 60x$ ②

$f''(-1) = 40(-1)^3 - 60(-1)^2 - 60(-1)$ ② ①

$= -40 - 60 + 60 = -40 < 0$

∴ f has a local maximum at -1 ②

loc. max $\boxed{7}$

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(16) 6. Let $f(x) = \frac{1}{1+e^{-x}}$. Give all the requested information and sketch the graph of the function on the axes below. Give both coordinates of the intercepts, local extrema and points of inflection, and give an equation for each asymptote. Write NONE where appropriate.

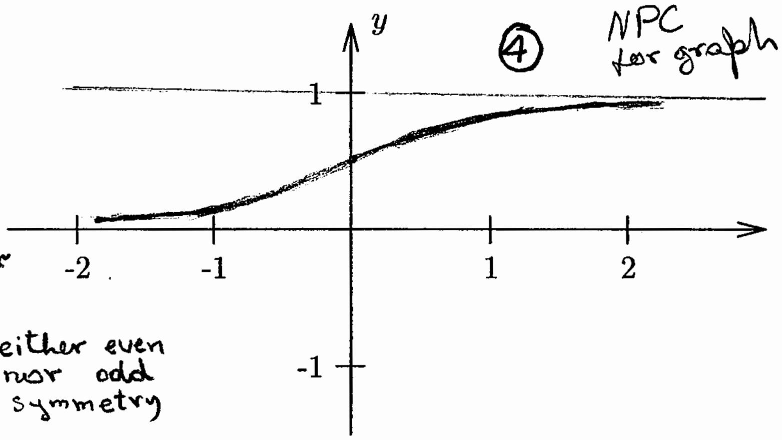
Domain: all x

$y = \frac{1}{1+e^{-x}}$

$x=0 \rightarrow y = \frac{1}{2}$ $(0, \frac{1}{2})$

$y=0 \rightarrow \frac{1}{1+e^{-x}} = 0$ never

$f(-x) = \frac{1}{1+e^{x}}$ ∴ f is neither even nor odd ∴ no symmetry



④ NPC for graph

f is finite for every x ∴ no V.A.

$\lim_{x\to\infty} \frac{1}{1+e^{-x}} = 1$ ∴ y=1 is H.A.

$\lim_{x\to-\infty} \frac{1}{1+e^{-x}} = 0$ ∴ y=0 is H.A.

$$f'(x) = \frac{(1+e^{-x})\cdot 0 - 1(-e^{-x})}{(1+e^{-x})^2} = \frac{e^{-x}}{(1+e^{-x})^2}$$

$f'(x) > 0$ for all x ∴ f is incr. on $(-\infty, \infty)$

no crit. numbers ∴ no loc max or loc. min.

$$f''(x) = \frac{(1+e^{-x})^2(-e^{-x}) - e^{-x}2(1+e^{-x})(-e^{-x})}{(1+e^{-x})^4}$$

$$= \frac{-(1+e^{-x})e^{-x} + 2e^{-x}e^{-x}}{(1+e^{-x})^3}$$

$$= \frac{-e^{-x} - e^{-2x} + 2e^{-2x}}{(1+e^{-x})^3}$$

$$= \frac{e^{-x}(e^{-x}-1)}{(1+e^{-x})^3}$$

$f''(x) = 0 \rightarrow e^{-x} = 1 \rightarrow x = 0$

$f''(x)$ ++ + ++0 − − − − − (x-axis, 0)

domain	$(-\infty, \infty)$	①
intercepts	$(0, \frac{1}{2})$	①
symmetry	NONE	①
horizontal asymptotes	$y=1$, $y=0$	①
vertical asymptotes	NONE	①
intervals of increase	$(-\infty, \infty)$	①
intervals of decrease	NONE	①
local maxima	NONE	①
local minima	NONE	①
intervals of concave down	$(0, \infty)$	①
intervals of concave up	$(-\infty, 0)$	①
points of inflection	$(0, \frac{1}{2})$	①

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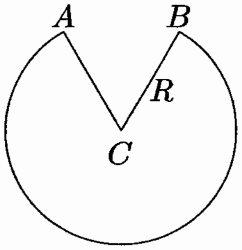
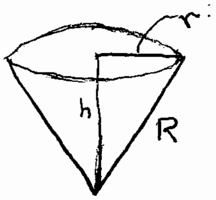
(14) 7. A conical drinking cup is made from a circular piece of paper of radius R by cutting out a sector and joining the edges CA and CB. Find the maximum capacity of such a cup.





$V = \frac{1}{3}\pi r^2 h,$ ②

$r^2 + h^2 = R^2$ ②

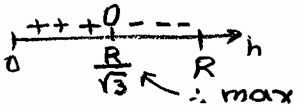
$\therefore V = \frac{1}{3}\pi(R^2 - h^2)h$

$V = \frac{1}{3}\pi R^2 h - \frac{1}{3}\pi h^3,$ ④ $0 < h < R$

$\frac{dV}{dh} = \frac{1}{3}\pi R^2 - \pi h^2 = \frac{\pi}{3}(R^2 - 3h^2)$ ②

$\frac{dV}{dx} = 0 \; : \; \frac{\pi}{3}(R^2 - 3h^2) = 0 \longrightarrow h = \frac{R}{\sqrt{3}}$ ②

$\frac{dV}{dh}$:



$\max V = \frac{1}{3}\pi \frac{R^3}{\sqrt{3}} - \frac{1}{3}\pi \frac{R^3}{3\sqrt{3}}$

$= \frac{\pi R^3}{3\sqrt{3}}\left(1 - \frac{1}{3}\right)$

②

$$\boxed{\frac{2\pi R^3}{9\sqrt{3}}}$$

14

(5) 8. Find the most general antiderivative of $f(x) = 5e^x - \frac{1}{1+x^2}$.

② ② ①

$$\boxed{5e^x - \tan^{-1}x + C}$$

5

(6) 9. Find $f(x)$ if $f'(x) = 2\sin x + \sec^2 x$ and $f(0) = 3$.

$f(x) = -2\cos x + \tan x + C$

$x=0 : \; f(0) = -2\cos 0 + \tan 0 + C$

$3 = -2 + 0 + C \longrightarrow C = 5$

② ② ②

$$\boxed{f(x) = -2\cos x + \tan x + 5}$$

6

(5) 10. If $\int_2^8 f(x)dx = 1.7$ and $\int_5^8 f(x)dx = 2.5$, find $\int_2^5 f(x)dx$.

$\int_2^5 f(x)dx = \int_2^8 f(x)dx - \int_5^8 f(x)dx$ ③

$= 1.7 - 2.5 = -0.8$

②

$$\boxed{-0.8}$$

5