NAME Grading Key

10–digit PUID

RECITATION INSTRUCTOR

RECITATION TIME

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Page 4	/24
TOTAL	/100

DIRECTIONS

1. Write your name, 10–digit PUID, recitation instructor's name and recitation time in the space provided above. Also write your name at the top of pages 2, 3 and 4.
2. The test has four (4) pages, including this one.
3. Write your answers in the boxes provided.
4. You must show sufficient work to justify all answers unless otherwise stated in the problem. Correct answers with inconsistent work may not be given credit. Please write neatly. Remember, if we cannot read your work and follow it logically, you may receive no credit.
5. Credit for each problem is given in parentheses in the left hand margin.
6. No books, notes, calculators or any electronic device may be used on this exam.

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(10) 1. Find the absolute maximum and absolute minimum values of the function $f(x) = x^3 - 12x + 1$ on the interval $[0, 3]$.

Critical number :

$0 = f'(x) = 3x^2 - 12 = 3(x+2)(x-2)$ (2)

$\Rightarrow x = \cancel{-2}, 2.$ (2) (If -2 is not excluded then -1 pt.)

$$\begin{cases} f(0) = 1 \\ f(2) = -15 \\ f(3) = -8 \end{cases}$$

abs. max. $\boxed{f(0) = 1}$ (3)

abs. min. $\boxed{f(2) = -15}$ (3)

(10) 2. Suppose f is continuous on the closed interval $[2, 5]$ and differentiable on the open interval $(2, 5)$. NPC.

(i) Complete the following statement, and give the name of the theorem used:

Statement: There exists c in the interval $(2, 5)$ such that $f'(c) = \boxed{\dfrac{f(5) - f(2)}{5 - 2}}$ (3)

Name of theorem used:	Mean Value Theorem

(3)

(ii) Suppose $f'(x) \geq 2$ for all x on the open interval $(2, 5)$. Find the smallest possible value for $f(5)$ when $f(2) = 4$.

$f'(c) = \dfrac{f(5) - f(2)}{5 - 2} \geqq 2$ $\therefore f(5) \geqq f(2) + 6 = 10$ $\boxed{10}$ (4)

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(10) 3. The only critical numbers for a continuous function $y = f(x)$ defind on the interval $(-3, 3)$ are -2 and 0. We know that

$$f'(x)\begin{cases} > 0 & \text{if } -3 < x < -2 \\ < 0 & \text{if } -2 < x < 0 \\ < 0 & \text{if } 0 < x < 3. \end{cases}$$

Determine whether $f(-2)$ and $f(0)$ are local maximum, local minimum or neither.

$f'(x)$: $+ + + +$ on $(-3,-2)$; $- - - - - -$ on $(-2, 0)$; $- - - - - - -$ on $(0, 3)$

$-3 \quad -2 \quad 0 \quad 3$

NPC

$f(-2)$ is local maximum 5

$f(0)$ is neither 5

For #4, 0 credit if answer is correct but there is no work

(20) 4. Find each of the following as a real number, $+\infty$, $-\infty$ or write DNE (does not exist).

(a) $\lim\limits_{x \to (\pi/2)^+} \left(\dfrac{\cos x}{1 - \sin x} \right) = \lim\limits_{x \to (\frac{\pi}{2})^+} \dfrac{-\sin x}{-\cos x} = -\infty$ 2

$\boxed{-\infty}$ 3

(b) $\lim\limits_{x \to 0} \left(\dfrac{e^x - 1 - x}{x^2} \right) = \lim\limits_{x \to 0} \dfrac{e^x - 1}{2x} = \lim\limits_{x \to 0} \dfrac{e^x}{2} = \dfrac{1}{2}$ 1 1

$\boxed{\frac{1}{2}}$ 3

(c) $\lim\limits_{x \to 0^+} x(\ln x) = \lim\limits_{x \to 0^+} \dfrac{\ln x}{\frac{1}{x}} = \lim\limits_{x \to 0^+} \dfrac{\frac{1}{x}}{-\frac{1}{x^2}} = 0$ 1 1

$\boxed{0}$ 3

(d) $\lim\limits_{x \to \infty} \left(\sqrt{x^2 + x} - x \right) = \lim\limits_{x \to \infty} \dfrac{(\sqrt{x^2+x} - x)(\sqrt{x^2+x} + x)}{\sqrt{x^2+x} + x}$ 2

$= \lim\limits_{x \to \infty} \dfrac{x}{\sqrt{x^2+x} + x} = \lim\limits_{x \to \infty} \dfrac{1}{\sqrt{1 + \frac{1}{x}} + 1} = \dfrac{1}{2}$

$\boxed{\frac{1}{2}}$ 3

(8) 5. Compute $\lim\limits_{x \to \infty} x^{\frac{1}{x}}$

Let $y = x^{1/x}$. $\ln y = \frac{1}{x} \ln x$ 2

$\lim\limits_{x \to \infty} \ln y = \lim\limits_{x \to \infty} \dfrac{\ln x}{x} = \lim\limits_{x \to \infty} \dfrac{\frac{1}{x}}{1} = 0$ 2

$\lim\limits_{x \to \infty} y = e^{\lim\limits_{x \to \infty} \ln y} = e^0 = 1$ 2

Or $\lim\limits_{x \to \infty} x^{1/x} = \lim\limits_{x \to \infty} e^{\frac{1}{x} \ln x} = e^{\lim\limits_{x \to \infty} \frac{\ln x}{x}}$ 2 2

$\lim\limits_{x \to \infty} \dfrac{\ln x}{x} = \lim\limits_{x \to \infty} \dfrac{\frac{1}{x}}{1} = 0$ 2

$\therefore \lim\limits_{x \to \infty} x^{1/x} = e^0 = 1$

$\boxed{1}$ 2

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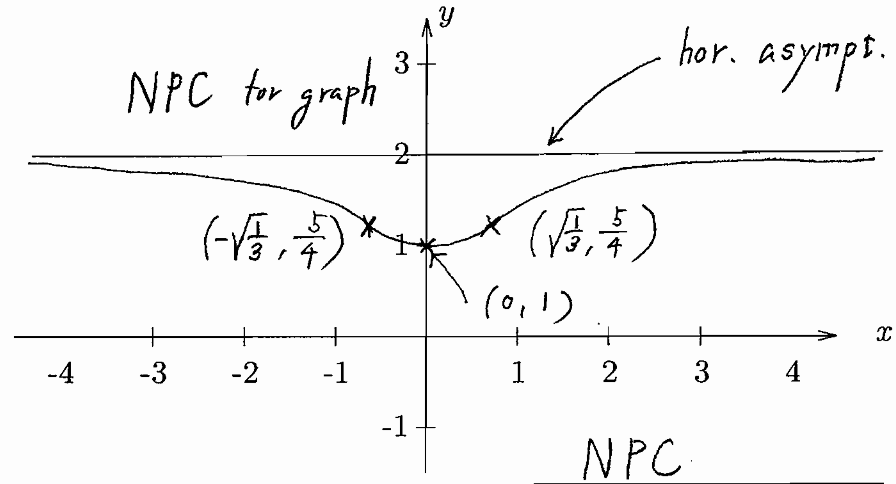
(18) 6. Let $f(x) = \dfrac{x^2}{x^2+1} + 1$. Give all the requested information and sketch the graph of the function on the axes below. Give both coordinates of the intercepts, local extrema and points of inflection, and give an equation for each asymptote. Write NONE where appropriate.



4

NPC

domain	$(-\infty, \infty)$	1
intercepts	$(0, 1)$	1
symmetry	about y-axis (or even)	1
horizontal asymptotes	$y = 2$	2
vertical asymptotes	NONE	1
intervals of increase	$(0, \infty)$	1
intervals of decrease	$(-\infty, 0)$	1
local maxima	NONE	1
local minima	$(0, 1)$	1
intervals of concave down	$(-\infty, -\sqrt{\frac{1}{3}})$ $(\sqrt{\frac{1}{3}}, \infty)$	1
intervals of concave up	$(-\sqrt{\frac{1}{3}}, \sqrt{\frac{1}{3}})$	1
points of inflection	$(-\sqrt{\frac{1}{3}}, \frac{5}{4})$ $(\sqrt{\frac{1}{3}}, \frac{5}{4})$	2

$$f(x) = \frac{x^2}{x^2+1} + 1$$

$$f'(x) = \frac{2x}{(x^2+1)^2}$$

$$f''(x) = \frac{2(x^2+1)^2 - 2x \cdot 2(x^2+1)2x}{(x^2+1)^4} = \frac{-6\left(x^2 - \frac{1}{3}\right)}{(x^2+1)^3}$$

$$\lim_{x\to\infty} f(x) = 2$$

$$\lim_{x\to-\infty} f(x) = 2$$

$$f(0) = 1$$

x		$-\sqrt{\frac{1}{3}}$		0		$\sqrt{\frac{1}{3}}$	
$f'(x)$	−	−	−	0	+	+	+
$f''(x)$	−	0	+	+	+	0	−
$f(x)$							

$$f(-x) = f(x)$$

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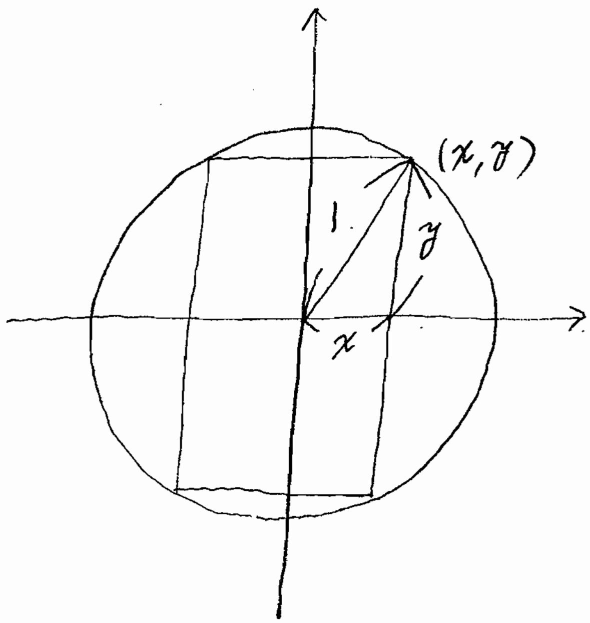
(14) 7. Use calculus to find the largest area of a rectangle inscribed in a circle of radius 1.



$x^2 + y^2 = 1$ 2

Area $A = (2x)(2y)$ 2

$A(x) = 4x\sqrt{1-x^2}$ 2

$(0 \le x \le 1)$

$A'(x) = 4\sqrt{1-x^2} + 4x \dfrac{-2x}{2\sqrt{1-x^2}}$ 2

$= \dfrac{4(1-2x^2)}{\sqrt{1-x^2}}$

$A'(x) = 0 \Rightarrow x = \sqrt{\frac{1}{2}}$ 2

$A\left(\sqrt{\frac{1}{2}}\right) = 4\sqrt{\frac{1}{2}}\sqrt{\frac{1}{2}} = 2.$

Largest area = $2.$ 4

(5) 8. Find $f(x)$ if

$$f'(x) = \frac{3}{1+x^2}, \quad f(\sqrt{3}) = 7.$$

$f(x) = 3\tan^{-1}x + C.$

$f(\sqrt{3}) = 3\tan^{-1}\sqrt{3} + C.$

$= 3\frac{\pi}{3} + C = 7$

$\therefore \quad C = 7 - \pi$

$f(x) = \underbrace{3\tan^{-1}x}_{2} + \underbrace{7 - \pi}_{3}.$

(5) 9. Find the most general antiderivative of

$$f(x) = \frac{\sin x}{\cos^2 x}$$

$f(x) = \dfrac{\sin x}{\cos x} \cdot \dfrac{1}{\cos x}$

$= \tan x \sec x.$

$F(x) = \sec x + C.$

$\left(\text{or} \quad f(x) = -\dfrac{(\cos x)'}{\cos^2 x} \qquad F(x) = \dfrac{1}{\cos x} + C \right)$

$\sec x + C \left(\text{or } \dfrac{1}{\cos x} + C\right)$ 5

-1 pt for missing C.