NAME SOLUTIONS

STUDENT ID # ________________

RECITATION INSTRUCTOR ________________

RECITATION TIME ________________

LECTURER ________________

INSTRUCTIONS

1. There are 8 different test pages (including this cover page). Make sure you have a complete test.
2. Fill in the above items in print. I.D.# is your 9 digit ID (probably your social security number). Also write your name at the top of pages 2–8.
3. Do any necessary work for each problem on the space provided or on the back of the pages of this test booklet. Circle your answers in this test booklet.
4. No books, notes or calculators may be used on this exam.
5. Each problem is worth 8 points. The maximum possible score is 200 points.
6. Using a #2 pencil, fill in each of the following items on your answer sheet:
   (a) On the top left side, write your name (last name, first name), and fill in the little circles.
   (b) On the bottom left side, under SECTION, write in your division and section number and fill in the little circles. (For example, for division 9 section 1, write 0901. For example, for division 38 section 2, write 3802).
   (c) On the bottom, under STUDENT IDENTIFICATION NUMBER, write in your student ID number, and fill in the little circles.
   (d) Using a #2 pencil, put your answers to questions 1–25 on your answer sheet by filling in the circle of the letter of your response. Double check that you have filled in the circles you intended. If more than one circle is filled in for any question, your response will be considered incorrect.
7. After you have finished the exam, hand in your answer sheet and your test booklet to your recitation instructor.

1. The domain of $f(x) = \sqrt{1 - \ln x}$ is

$x > 0$ and $1 - \ln x \geq 0$

$\downarrow$

$\ln x \leq 1$

$x \leq e$

$0 < x \leq e$

A. $x \leq e$

B. $0 < x \leq e$

C. $e < x$

D. $0 < x \leq 1$

E. $1 \leq x \leq e$

2. $\lim\limits_{x\to\infty} \underbrace{\frac{1}{x}}_{\downarrow 0} \underbrace{\sin\left(\frac{\pi}{2x}\right)}_{\text{DNE}} =$

$-1 \leq \sin\left(\frac{\pi}{2x}\right) \leq 1$

$x > 0 \rightarrow \quad -\frac{1}{x} \leq \frac{1}{x}\sin\left(\frac{\pi}{2x}\right) \leq \frac{1}{x}$

as $x \to \infty$ $\quad \downarrow 0 \quad \therefore \downarrow 0 \quad \downarrow 0$ by squeeze theorem

A. ∞

B. $\frac{\pi}{2}$

C. 1

D. 0

E. Does not exist

3. Let $f(x)$ be continuous on $[0, 1]$. Which of the following conditions guarantee that there is a solution of $f(x) = 0$ between 0 and 1?

(i) $f(0)f(1) < 0$ $\longrightarrow$ $f(0)$ and $f(1)$ have opposite sign

(ii) $f(0) - f(1) < 0$ $\longrightarrow$ $f(1) > f(0)$

(iii) $f(0)f(1) > 0$ $\longrightarrow$ $f(0)$ and $f(1)$ have same sign

intermediate value theorem

A. (i) only

B. (ii) only

C. (iii) only

D. (i) and (ii)

E. (ii) and (iii)

4. Find the equation of the tangent line to the graph of $y = \tan^2 x$ at the point $\left(\frac{\pi}{4}, 1\right)$.

$\frac{dy}{dx} = 2\tan x \sec^2 x$

$\left.\frac{dy}{dx}\right|_{x=\frac{\pi}{4}} = 2\tan\frac{\pi}{4}\sec^2\left(\frac{\pi}{4}\right)$

$= 2 \cdot 1 \cdot (\sqrt{2})^2 = 4$

$y - 1 = 4\left(x - \frac{\pi}{4}\right)$

$y = 4x - \pi + 1$

A. $y = 4x + 1$

B. $y = 4x - \pi$

C. $y = 4x - \pi + 1$

D. $y = 2x - \frac{\pi}{2} + 1$

E. $y = 2x - \frac{\pi}{2}$

5. Let $f(x) = \cos(e^x)$. $f''(x) =$

$f'(x) = -\sin(e^x)\cdot e^x$
$= -e^x \sin(e^x)$
$f''(x) = -e^x \cos(e^x)\cdot e^x - e^x \sin(e^x)$
$= -e^{2x}\cos(e^x) - e^x \sin(e^x)$

A. $-\cos x(e^x)$
B. $-e^x \cos(e^x)$
C. $-e^x \cos(e^x) - 2e^{2x}\cos(e^x)$
D. $-e^x \sin(e^x) - e^{2x}\cos(e^x)$
E. $-e^x \cos(e^x) - e^x \sin(e^x)$

6. The slope of the tangent line to the curve $x^3 - xy + y^3 = 1$ at $(1,1)$ is

$3x^2 - x\frac{dy}{dx} - y + 3y^2\frac{dy}{dx} = 0$

At $(x,y) = (1,1)$:

$3\cdot 1^2 - 1\frac{dy}{dx} - 1 + 3\cdot 1^2\frac{dy}{dx} = 0$
$2\frac{dy}{dx} = 1 - 3$
$\frac{dy}{dx} = -1$

A. 0
B. $-\frac{1}{2}$
C. $\frac{1}{3}$
D. $\frac{2}{3}$
E. -1

7. $\lim\limits_{x\to 0}\frac{\sqrt{x+5}-\sqrt{5}}{x} = \lim\limits_{h\to 0}\frac{\sqrt{5+h}-\sqrt{5}}{h} = f'(5)$

where $f(x) = \sqrt{x}$
$f'(x) = \frac{1}{2\sqrt{x}}$ $\quad f'(5) = \frac{1}{2\sqrt{5}}$

or L'H. rule

$\lim\limits_{x\to 0}\frac{\sqrt{x+5}-\sqrt{5}}{x} \overset{L'H}{=} \lim\limits_{x\to 0}\frac{\frac{1}{2\sqrt{x+5}} - 0}{1} = \frac{1}{2\sqrt{5}}$

$\frac{0}{0}$

A. ∞
B. $\frac{1}{2\sqrt{5}}$
C. 0
D. $2\sqrt{5}$
E. $\frac{1}{5}$

8. $\lim\limits_{x\to 3}\frac{x-3}{x^2-4x+3} = \lim\limits_{x\to 3}\frac{x-3}{(x-3)(x-1)} = \lim\limits_{x\to 3}\frac{1}{x-1} = \frac{1}{2}$

or L'H. rule

$\lim\limits_{x\to 3}\frac{x-3}{x^2-4x+3} \overset{L'H}{=} \lim\limits_{x\to 3}\frac{1}{2x-4} = \frac{1}{2}$

$\frac{0}{0}$

A. 0
B. ∞
C. $\frac{1}{2}$
D. 3
E. $\frac{1}{4}$

9. Let $f(x)$ be the function given by

$$f(x) = \begin{cases} 1 - x, & \text{for } x < 1 \\ 1, & \text{for } x = 1 \\ -1 + x, & \text{for } x > 1 \end{cases}$$

Which of the following are true?

(i) $\lim_{x \to 1} f(x)$ exists

(ii) f is continuous at $x = 1$

(iii) f is differentiable at $x = 1$

A. (i) and (ii)

B. (i) only

C. (ii) only

D. (1), (ii) and (iii)

E. None of these

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = 0$, but $f(1) = 1$

10. Find the derivative of $f(x) = \ln |x^3 \cos x|$.

$$f'(x) = \frac{1}{x^3 \cos x} \left[x^3(-\sin x) + 3x^2 \cos x \right]$$

$$= -\tan x + \frac{3}{x}$$

A. $\dfrac{1}{x^3 \cos x}$

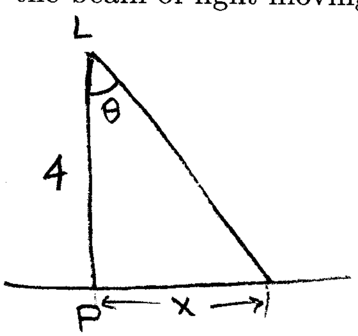
B. $\dfrac{3}{x \cos x}$

C. $\dfrac{3}{x} - \tan x$

D. $\dfrac{3}{x} + \sec x$

E. $\dfrac{1}{x^3} + \sec x$

11. A lighthouse is located on a small island 4 km from the nearest point P on a straight shoreline, and its light makes five rotations per minute (10π rad/min). How fast is the beam of light moving along the shoreline when it is 2 km from P?



$$\frac{d\theta}{dt} = 10\pi \text{ rad/min}$$

$$\tan \theta = \frac{x}{4}$$

$$\sec^2 \theta \frac{d\theta}{dt} = \frac{1}{4} \frac{dx}{dt}$$

A. 50π km/min

B. 200π km/min

C. 20π km/min

D. $20\sqrt{5}\pi$ km/min

E. $40\sqrt{5}\pi$ km/min

When $x = 2$: $\tan \theta = \frac{1}{2}$, $\sec^2 \theta = 1 + \tan^2 \theta = 1 + \frac{1}{4} = \frac{5}{4}$

$$\frac{5}{4} \cdot 10\pi = \frac{1}{4} \frac{dx}{dt} \rightarrow \frac{dx}{dt} = 50\pi$$

12. If $f(x) = \sin x$, use a linear approximation to approximate $\sin\left(\frac{\pi}{6} + \frac{1}{10}\right)$.

$f(x) \approx f(a) + f'(a)(x-a)$, for x near a

Let $f(x) = \sin x$ $a = \frac{\pi}{6}$

$\sin x \approx \sin\left(\frac{\pi}{6}\right) + \cos\left(\frac{\pi}{6}\right)\left(x - \frac{\pi}{6}\right)$, for x near $\frac{\pi}{6}$

$\sin x \approx \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right)$

$\sin\left(\frac{\pi}{6} + \frac{1}{10}\right) \approx \frac{1}{2} + \frac{\sqrt{3}}{2}\left(\frac{\pi}{6} + \frac{1}{10} - \frac{\pi}{6}\right)$

$= \frac{1}{2} + \frac{\sqrt{3}}{2}\frac{1}{10} = \frac{10+\sqrt{3}}{20}$

A. $\frac{5-\sqrt{3}}{10}$

B. $\frac{9}{20}$

C. $\frac{2}{5}$

D. $\frac{10+\sqrt{3}}{20}$

E. $\frac{10-\sqrt{3}}{20}$

13. The minimum and maximum values of $f(x) = \frac{x}{x^2+1}$ on the interval $[0, 2]$ are

$f'(x) = \frac{(x^2+1)\cdot 1 - x \cdot 2x}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}$

$f'(x) = 0$: $\frac{1-x^2}{(x^2+1)^2} = 0 \longrightarrow x = \cancel{-1}, 1.$

$-1 \notin (0,2)$

$f(0) = 0$ ← min

$f(1) = \frac{1}{2}$ ← max

$f(2) = \frac{2}{5}$

A. 0 and $\frac{2}{5}$

B. -1 and $\frac{1}{2}$

C. -1 and $\frac{2}{5}$

D. 0 and $\frac{1}{2}$

E. $\frac{2}{5}$ and $\frac{1}{2}$

14. At 2:00 PM a car's speedometer reads 40 mph. At a quarter of an hour later, it reads 60 mph. The Mean Value Theorem guarantees that at some time between 2:00 and 2:15 the acceleration is exactly

$\frac{v(2.25) - v(2)}{2.25 - 2} = v'(c)$ for some $c \in (2, 2.25)$

$\frac{60-40}{0.25} = a(c) \rightarrow a(c) = 80$

A. 20 mi/hr^2

B. 5 mi/hr^2

C. 80 mi/hr^2

D. 40 mi/hr^2

E. 120 mi/hr^2

15. $\lim_{x\to 0} \frac{1}{x} \tan x = \lim_{x \to 0} \frac{\tan x}{x} \overset{L'H}{=} \lim_{x\to 0} \frac{\sec^2 x}{1} = 1$

$\frac{0}{0}$

A. -1

B. 1

C. 0

D. ∞

E. $-\infty$

16. On the interval $(0, \pi)$, the function $f(x) = \frac{1}{2}x - \sin x$ is

$f'(x) = \frac{1}{2} - \cos x$

$f'(x) = 0$: $\cos x = \frac{1}{2} \rightarrow x = \frac{\pi}{3}$

$f'(x)$: $- - - - + + + + + +$ on 0, $\frac{\pi}{3}$, π

(A.) Decreasing on $(0, \frac{\pi}{3})$ and increasing on $(\frac{\pi}{3}, \pi)$.

B. Increasing on $(0, \frac{\pi}{3})$ and decreasing on $(\frac{\pi}{3}, \pi)$.

C. Decreasing on $(0, \frac{\pi}{6})$ and increasing on $(\frac{\pi}{6}, \pi)$.

D. Increasing on $(0, \frac{\pi}{6})$ and on $(\frac{\pi}{3}, \pi)$, and decreasing on $(\frac{\pi}{6}, \frac{\pi}{3})$.

E. Decreasing on $(0, \frac{\pi}{6})$ and on $(\frac{\pi}{3}, \pi)$, and increasing on $(\frac{\pi}{6}, \frac{\pi}{3})$.

17. The function $f(x) = -x^4 + 6x^2 + 8x$ has $x = -1$ as a critical number. The point $(-1, -3)$ on the graph of $y = f(x)$ is a:

(i) local maximum.

(ii) local minimum.

(iii) point of inflection.

A. (i) is true

B. (ii) is true

(C.) (iii) is true

D. (i), (ii), and (iii) are all true

E. None of these

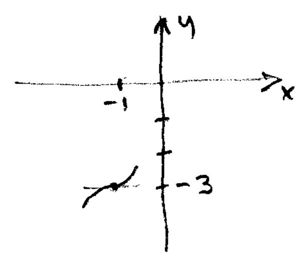
$f'(x) = -4x^3 + 12x + 8$, $f'(-1) = 0$

$f''(x) = -12x^2 + 12 = 12(1 - x^2)$

$f''(-1) = 0$

$f''(x)$: $- - - + + + + + + - -$ on -1, 1, x

$\therefore (-1, -3)$ is an inflection point



18. $\frac{d}{dx} \int_0^{x^3} \frac{t}{\sqrt{1+t^3}}\, dt = \frac{x^3}{\sqrt{1+(x^3)^3}} \, 3x^2$

$= \frac{3x^5}{\sqrt{1+x^9}}$

(A.) $\frac{3x^5}{\sqrt{1+x^9}}$

B. $\frac{x^3}{\sqrt{1+x^9}}$

C. $\frac{3x^3}{\sqrt{1+x^3}}$

D. $\frac{5x^3}{\sqrt{1+x^6}}$

E. $\frac{3x^5}{\sqrt{1+x^3}}$

19. $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos\theta}{\sin^3\theta}\, d\theta =$ $\int_{1/2}^{1} u^{-3}\,du = \left.\frac{u^{-2}}{-2}\right|_{1/2}^{1} = \left.-\frac{1}{2u^2}\right|_{1/2}^{1}$

$u = \sin\theta \quad du = \cos\theta\, d\theta$

$\theta = \frac{\pi}{6} \longrightarrow u = \frac{1}{2}$

$\theta = \frac{\pi}{2} \longrightarrow u = 1$

$= -\frac{1}{2} + \frac{1}{2\cdot\frac{1}{4}}$

$= -\frac{1}{2} + 2 = \frac{3}{2}$

A. $\frac{1}{2}$

B. $\frac{5}{2}$

C. 1

D. 3

E. $\frac{3}{2}$

20. Find the area of the region between the graph of $y = \frac{3}{\sqrt{1-x^2}}$ and the x-axis from $x = -\frac{\sqrt{2}}{2}$ to $x = \frac{\sqrt{2}}{2}$.

$A = \int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} \frac{3}{\sqrt{1-x^2}}\,dx = 3\sin^{-1}x \Big|_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}}$

$= 3\sin^{-1}\left(\frac{\sqrt{2}}{2}\right) - 3\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

$= 3\,\frac{\pi}{4} - 3\left(-\frac{\pi}{4}\right)$

$= \frac{3\pi}{2}$

A. π

B. $\frac{\pi}{2}$

C. $\frac{5\pi}{2}$

D. $\frac{3\pi}{2}$

E. $\frac{2\pi}{3}$

21. $\int_0^1 \frac{e^x}{1+e^x}\,dx =$ $\ln(1+e^x)\Big|_0^1 = \ln(1+e) - \ln 2$

$= \ln\left(\frac{1+e}{2}\right)$

$\int \frac{e^x}{1+e^x}\,du = \int \frac{1}{u}\,du = \ln|u| + C$

$u = 1+e^x \quad du = e^x dx \qquad = \ln(1+e^x) + C$

A. 1

B. $\ln\left(\frac{1+e}{2}\right)$

C. $\ln(1+e)$

D. 2

E. $\ln 2$

22. The half-life of a radioactive substance is 100 years. In how many years will its mass decrease to $\frac{1}{10}$ of its original size?

$m(t) = m_0 e^{kt}$

$\frac{1}{2} m_0 = m_0 e^{k \cdot 100} \rightarrow \frac{1}{2} = e^{100k}$

$-\ln 2 = 100k \longrightarrow k = -\frac{\ln 2}{100}$

$m(t) = m_0 e^{-\frac{\ln 2}{100} t}$

$\frac{1}{10} m_0 = m_0 e^{-\frac{\ln 2}{100} t}$

$-\ln 10 = -\frac{\ln 2}{100} t \rightarrow t = 100 \frac{\ln 10}{\ln 2}$

A. $2\frac{\ln 100}{\ln 2}$

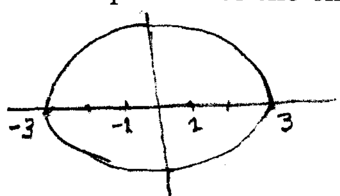
B. $10\frac{\ln 2}{\ln 10}$

C. $50\frac{\ln 10}{\ln 2}$

D. $50\frac{\ln 2}{\ln 10}$

E. $100\frac{\ln 10}{\ln 2}$

23. Find an equation of the ellipse with vertices $(\pm 3, 0)$ and focii $(\pm 1, 0)$.



$a = 3 \quad c = 1$

$c^2 = a^2 - b^2$

$b^2 = a^2 - c^2 = 9 - 1 = 8$

$\frac{x^2}{9} + \frac{y^2}{8} = 1 \qquad 8x^2 + 9y^2 = 72$

A. $4x^2 + 36y^2 = 144$

B. $8x^2 + 9y^2 = 72$

C. $x^2 + 9y^2 = 9$

D. $9x^2 + y^2 = 9$

E. $9x^2 + 8y^2 = 72$

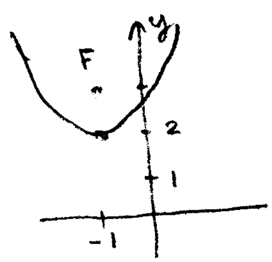
24. The focus of the parabola $x^2 + 2x - 4y + 9 = 0$ is

$x^2 + 2x + 1 = 4y - 9 + 1$

$(x+1)^2 = 4(y-2)$

vertex $(-1, 2)$, $p = 1$

Focus $(-1, 3)$



A. $(-1, 3)$

B. $(-1, 2)$

C. $(1, 3)$

D. $(3, -1)$

E. $(2, 1)$

25. The equation of one of the asymptotes of the hyperbola $x^2 - 9y^2 - 36y = 45$ is

$x^2 - 9(y^2 + 4 + 4) = 45 - 36$

$x^2 - 9(y+2)^2 = 9$

$\frac{x^2}{9} - \frac{(y+2)^2}{1} = 1$

Asymptotes: $y + 2 = \pm \frac{1}{3} x$

$3y + 6 = \pm x$

$\mp x + 3y + 6 = 0$

A. $x + 3y - 1 = 0$

B. $x - 9y - 27 = 0$

C. $x - 3y - 7 = 0$

D. $x + 3y + 6 = 0$

E. $x + 9y + 18 = 0$