NAME SOLUTIONS

STUDENT ID # ____________

RECITATION INSTRUCTOR ____________

RECITATION TIME ____________

LECTURER ____________

INSTRUCTIONS

1. There are 9 different test pages (including this cover page). Make sure you have a complete test.
2. Fill in the above items in print. ID # is your 9 digit ID (probably your social security number). Also write your name at the top of pages 2–9.
3. Do any necessary work for each problem on the space provided or on the back of the pages of this test booklet. Circle your answers in this test booklet.
4. No books, notes or calculators may be used on this exam.
5. Each problem is worth 8 points. The maximum possible score is 200 points.
6. Using a #2 pencil, fill in each of the following items on your answer sheet:
   (a) On the top left side, write your name (last name, first name), and fill in the little circles.
   (b) On the bottom left side, under SECTION, write in your division and section number and fill in the little circles. (For example, for division 9 section 1, write 0901. For example, for division 38 section 2, write 3802).
   (c) On the bottom, under STUDENT IDENTIFICATION NUMBER, write in your student ID number, and fill in the little circles.
   (d) Using a #2 pencil, put your answers to questions 1–25 on your answer sheet by filling in the circle of the letter of your response. Double check that you have filled in the circles you intended. If more than one circle is filled in for any question, your response will be considered incorrect. Use a #2 pencil.
7. After you have finished the exam, hand in your answer sheet and your test booklet to your recitation instructor.

1. The domain of $f(x) = \ln(e^4 - e^{2x})$ is

$e^4 - e^{2x} > 0$

$e^{2x} < e^4$

$2x < 4$ since e^x is increasing

$x < 2$

A. $x > 0$

B. $x < 2$

C. $2 < x < 4$

D. $4 \leq x$

E. $x \leq 2$

2. $\lim_{t \to 0} \frac{\sqrt{t^2+25}-5}{3t^2} = \lim_{t \to 0} \frac{(\sqrt{t^2+25}-5)(\sqrt{t^2+25}+5)}{3t^2(\sqrt{t^2+25}+5)}$

$\frac{0}{0}$

$= \lim_{t \to 0} \frac{t^2 + 25 - 25}{3t^2(\sqrt{t^2+25}+5)}$

$= \lim_{t \to 0} \frac{1}{3(\sqrt{t^2+25}+5)} = \frac{1}{30}$

or L'H

$= \lim_{t \to 0} \frac{\frac{2t}{2\sqrt{t^2+25}}}{6t} = \lim_{t \to 0} \frac{1}{6\sqrt{t^2+25}} = \frac{1}{30}$

A. $\frac{25}{3}$

B. $\frac{1}{30}$

C. 0

D. $\frac{1}{15}$

E. Does not exist

3. Let $f(x) = \begin{cases} 8c^3 - x^3 & \text{if } x \leq \pi \\ c \sin x & \text{if } x > \pi \end{cases}$, where c is a constant. Find the value of c so that f is continuous for all x.

f is continuous for all $x \neq \pi$, no matter what c is

$\lim_{x \to \pi^-} f(x) = 8c^3 - \pi^3$ $\quad \lim_{x \to \pi^+} f(x) = 0$

$\lim_{x \to \pi} f(x) = 0 = f(\pi)$, provided $8c^3 - \pi^3 = 0$

or $c^3 = \frac{\pi^3}{8}$

$c = \frac{\pi}{2}$

A. $\frac{\pi}{2}$

B. 0

C. $\frac{\pi}{8}$

D. $\frac{\pi}{3}$

E. π

4. If the tangent line to the curve $y = 4 - 2x^2$ at the point where $x = a$ is parallel to the line $8x + 3y = 4$, then $a =$

$\frac{dy}{dx} = -4x$ $\quad \left.\frac{dy}{dx}\right|_{x=a} = -4a$

Slope of $8x + 3y = 4$ is $-\frac{8}{3}$

$-4a = -\frac{8}{3} \longrightarrow a = \frac{2}{3}$

A. 0

B. $-\frac{2}{3}$

C. $-\frac{1}{3}$

D. $\frac{1}{3}$

E. $\frac{2}{3}$

5. If $y = \ln(\tan x)$, $\dfrac{dy}{dx} = \dfrac{1}{\tan x}\sec^2 x$

$= \dfrac{1}{\frac{\sin x}{\cos x}} \dfrac{1}{\cos^2 x}$

$= \dfrac{1}{\sin x} \dfrac{1}{\cos x}$

$= \csc x \sec x$

A. $\sec x \tan x$

B. $\csc x \cot x$

C. $\sec^2 x$

D. $\csc^2 x$

E. $\sec x \csc x$

6. If $f(x) = \dfrac{x^2}{1-x}$, $f''(0) =$

$f'(x) = \dfrac{(1-x)2x - x^2(-1)}{(1-x)^2} = \dfrac{2x - x^2}{(1-x)^2}$

$f''(x) = \dfrac{(1-x)^2(2-2x) - (2x-x^2)2(1-x)(-1)}{(1-x)^4}$

$f''(0) = 2$

A. 2

B. -2

C. 0

D. 4

E. -4

7. If $y = x^{\sin x}$, $y' =$

$y = x^{\sin x} = e^{\ln x^{\sin x}} = e^{\sin x \ln x}$

$y' = e^{\sin x \ln x}\left(\sin x \cdot \frac{1}{x} + \cos x \ln x\right)$

$= x^{\sin x}\left(\cos x \ln x + \frac{\sin x}{x}\right)$

A. $x^{\sin x}\cos x$

B. $x^{\sin x}\left(\sin x \ln x - \dfrac{\cos x}{x}\right)$

C. $x^{\sin x}\left(\cos x \ln x + \dfrac{\sin x}{x}\right)$

D. $x^{\sin x}\left(\cos x \ln x - \dfrac{\sin x}{x}\right)$

E. $x^{\sin x}\left(\sin x \ln x + \dfrac{\cos x}{x}\right)$

8. If $\sqrt{x+y} + \sqrt{x-y} = 1$, then $y' =$

$\dfrac{1}{2\sqrt{x+y}}(1+y') + \dfrac{1}{2\sqrt{x-y}}(1-y') = 0$

$\sqrt{x-y}\,(1+y') + \sqrt{x+y}\,(1-y') = 0$

$(\sqrt{x-y} - \sqrt{x+y})y' = -\sqrt{x-y} - \sqrt{x+y}$

$y' = \dfrac{\sqrt{x-y} + \sqrt{x+y}}{\sqrt{x+y} - \sqrt{x-y}}$

A. $\dfrac{\sqrt{x+y} + \sqrt{x-y}}{\sqrt{x-y} - \sqrt{x+y}}$

B. $\dfrac{\sqrt{x+y} + \sqrt{x-y}}{\sqrt{x+y} - \sqrt{x-y}}$

C. $1 - \dfrac{\sqrt{x+y}}{\sqrt{x-y}}$

D. $\dfrac{\sqrt{x+y} - \sqrt{x-y}}{\sqrt{x+y} + \sqrt{x-y}}$

E. $1 - \dfrac{\sqrt{x-y}}{\sqrt{x+y}}$

9. $\frac{d}{dx}\sin^{-1}(e^{2x}) = \frac{1}{\sqrt{1-(e^{2x})^2}}\, e^{2x}\cdot 2$

$= \frac{2e^{2x}}{\sqrt{1-e^{4x}}}$

A. $\frac{1}{\sqrt{1-e^{4x}}}$

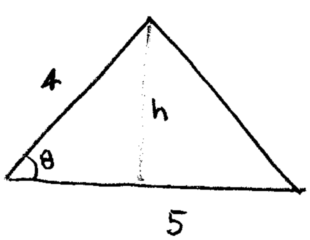
B. $\frac{e^{2x}}{\sqrt{1-e^{4x}}}$

C. $\frac{1}{\sqrt{1-e^{2x}}}$

D. $\frac{2e^{2x}}{\sqrt{1-e^{4x}}}$

E. $\frac{2e^{2x}}{1+e^{4x}}$

10. Two sides of a triangle have fixed lengths of 4 meters and 5 meters, while the angle θ between them is increasing at a rate of 0.06 rad/s. Find the rate at which the area of the triangle is increasing when $\theta = \frac{\pi}{3}$.



$A = \frac{1}{2}\cdot 5h.$ $\quad \frac{d\theta}{dt} = 0.06$

$h = 4\sin\theta$

$A = \frac{1}{2}\cdot 5\cdot 4\sin\theta$

$A = 10\sin\theta$

$\frac{dA}{dt} = 10\cos\theta\,\frac{d\theta}{dt}$

When $\theta = \frac{\pi}{3}$ $\quad \frac{dA}{dt} = 10\cos\frac{\pi}{3}(0.06) = 10\cdot\frac{1}{2}\cdot(0.06) = 0.3$

A. $0.6\ m^2/s$

B. $0.3\ m^2/s$

C. $0.3\sqrt{3}\ m^2/s$

D. $0.2\sqrt{3}\ m^2/s$

E. $0.6\sqrt{3}\ m^2/s$

11. Find the linearization $L(x)$ of $f(x) = \sqrt{1-x}$ at $a = 0$.

$L(x) = f(0) + f'(0)(x-0)$

$f(0) = 1$

$f'(x) = \frac{1}{2\sqrt{1-x}}(-1)$

$f'(0) = -\frac{1}{2}$

$L(x) = 1 - \frac{1}{2}x$

A. $L(x) = \frac{x}{2}$

B. $L(x) = -\frac{x}{2}$

C. $L(x) = 1 + \frac{x}{2}$

D. $L(x) = 1 - \frac{x}{2}$

E. $L(x) = 1 - x$

12. The absolute maximum value of the function $f(x) = e^{x^3-x}$ on the closed interval $[-1,0]$ is

$f'(x) = e^{x^3-x}(3x^2-1)$

$f'(x) = 0$: $e^{x^3-x}(3x^2-1)=0$

$3x^2-1=0 \rightarrow x = -\frac{1}{\sqrt{3}}, \cancel{\frac{1}{\sqrt{3}}}$ not in $[-1,0]$

$f(-1) = e^{-1+1} = e^0 = 1$

$f(0) = e^0 = 1$

$f(-\frac{1}{\sqrt{3}}) = e^{(-\frac{1}{\sqrt{3}})^3 - (-\frac{1}{\sqrt{3}})} = e^{\frac{1}{\sqrt{3}}(1-\frac{1}{3})} = e^{\frac{2}{3\sqrt{3}}} = e^{2\sqrt{3}/9}$

A. $e^{2\sqrt{3}/9}$

B. $e^{-2\sqrt{3}/9}$

C. 1

D. $e^{3/8}$

E. f has no abs. max. value on $[-1,0]$

13. If $f'(x) = g'(x)$ for all x in $(0,\infty)$, and $f(1) - g(1) = 1$, then $f(5) - g(5) =$

Let $F(x) = f(x) - g(x)$

Then $F'(x) = f'(x) - g'(x) = 0$ in $(0,\infty)$

and hence $F(x) =$ constant in $(0,\infty)$

But $F(1) = f(1) - g(1) = 1$

$\therefore$ $F(x) = 1$ for all $x \in (0,\infty)$

and $F(5) = 1$ $\therefore f(5) - g(5) = 1$

A. 5

B. -1

C. 1

D. -5

E. cannot be determined

14. If $f(x) = x^5 - 5x + 3$, which one of the following statements is true?

A. $x = 1$ is the only critical number of f, and f has a local max. at 1

B. $x = -1$ is the only critical number of f, and f has a local min. at -1.

C. $x = 1, -1$ are the only critical numbers of f, and f has a local max. at -1 and a local min. at 1.

D. $x = 1, -1$ are the only critical numbers of f, and f has a local min. at -1 and a local max. at 1.

E. $x = 1, -1$ are the only critical numbers of f, but f has neither a local max. nor a local min. at these critical numbers.

$f'(x) = 5x^4 - 5$ $\quad f'(x) = 0$: $5x^4 - 5 = 0 \rightarrow x^4 = 1 \rightarrow x = 1, -1$

$f''(x) = 20x^3$ $\quad f''(1) = 20 > 0$ $\therefore$ loc. min. at $x = 1$

$f''(-1) = -20 < 0$ $\therefore$ loc. max. at $x = -1$

15. The inflection point(s) of the graph of the function $f(x) = 3x^5 + 5x^4$ are

A. $(0,0), (-1,2)$
B. $(0,0), (1,8)$
C. $(1,8)$
D. $(-1,2)$
E. $(0,0)$

$f'(x) = 15x^4 + 20x^3$

$f''(x) = 60x^3 + 60x^2 = 60x^2(x+1)$

$f''(x)$: $- - - - 0 + + + + 0 + + + +$ on the x-axis, with 0 at $x = -1$ and at $x = 0$

-1: conc. changes

0: conc. does not change

$\therefore$ there is an inflection point when $x = -1$, $f(-1) = 2$

infl. pt. $(-1, 2)$

16. $\lim_{x \to 0+} e^{\frac{\ln x}{x}} =$

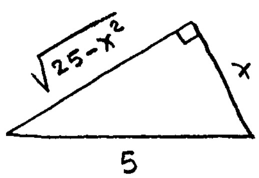
A. $-\infty$
B. 0
C. ∞
D. 1
E. e

$\lim_{x \to 0^+} \frac{\ln x}{x} = -\infty$

$\therefore \lim_{x \to 0^+} e^{\frac{\ln x}{x}} = 0$

17. Among all the right triangles whose hypotenuse has length 5, find the area of the one whose area is maximum.

A. $\frac{25\sqrt{3}}{8}$
B. 6
C. $\frac{750}{169}$
D. $\frac{125}{8}$
E. $\frac{25}{4}$



$A = \frac{1}{2} x \sqrt{25 - x^2}$, $0 < x < 5$

$\frac{dA}{dx} = \frac{1}{2} x \frac{1}{2\sqrt{25-x^2}}(-2x) + \frac{1}{2}\sqrt{25-x^2}$

$= \frac{1}{2}\left(-\frac{x^2}{\sqrt{25-x^2}} + \sqrt{25-x^2}\right)$

$= \frac{1}{2} \frac{-x^2 + 25 - x^2}{\sqrt{25-x^2}} = \frac{25 - 2x^2}{2\sqrt{25-x^2}}$

$\frac{dA}{dx} = 0$ when $x = \frac{5}{\sqrt{2}}$

$\frac{dA}{dx}$: $+ + + + 0 - - - -$ on the x-axis from 0 to 5, with 0 at $\frac{5}{\sqrt{2}}$ (max)

max area: $A = \frac{1}{2} \frac{5}{\sqrt{2}} \sqrt{25 - \frac{25}{2}} = \frac{1}{2} \frac{5}{\sqrt{2}} \frac{5}{\sqrt{2}} = \frac{25}{4}$

18. A particle moves in a straight line and has acceleration given by $a(t) = 6t + 4$. Its initial displacement is $s(0) = 9$ and the displacement at times $t = 1$ is $s(1) = 6$. Find the velocity of the particle at $t = 2$.

$a(t) = 6t + 4$

$v(t) = 3t^2 + 4t + C$

$s(t) = t^3 + 2t^2 + Ct + D$

$t=0: \quad 9 = 0 + 0 + 0 + D \rightarrow D = 9$

$t=1: \quad 6 = 1 + 2 + C + 9 \rightarrow C = -6$

$v(t) = 3t^2 + 4t - 6$

$v(2) = 3 \cdot 4 + 4 \cdot 2 - 6 = 14$

A. $v(2) = 14$
B. $v(2) = 7$
C. $v(2) = 4$
D. $v(2) = 9$
E. $v(2) = 0$

19. $\lim\limits_{x \to 0} \dfrac{\cos x - 1}{x^2} \overset{L'H}{=} \lim\limits_{x \to 0} \dfrac{-\sin x}{2x} \overset{L'H}{=} \lim\limits_{x \to 0} -\dfrac{\cos x}{2} = -\dfrac{1}{2}$

$\frac{0}{0}$ $\quad \frac{0}{0}$

A. 1
B. -1
C. 0
D. $\frac{1}{2}$
E. $-\frac{1}{2}$

20. $\dfrac{d}{dx} \displaystyle\int_2^{x^2} \sqrt{11 + t^3}\, dt = \sqrt{11 + (x^2)^3} \cdot 2x$

$= 2x\sqrt{11 + x^6}$

A. $\sqrt{11 + x^6}$
B. $x^2\sqrt{11 + x^6}$
C. $x^2\sqrt{11 + x^3}$
D. $2x\sqrt{11 + x^3}$
E. $2x\sqrt{11 + x^6}$

21. $\int_0^{\frac{\pi}{4}} \tan^3 x \sec^2 x \, dx = \int_0^1 u^3 du = \frac{u^4}{4}\Big|_0^1 = \frac{1}{4}$

$u = \tan x \quad du = \sec^2 x \, dx$

$x = 0 \rightarrow u = 0$

$x = \frac{\pi}{4} \rightarrow u = 1$

A. $\frac{1}{2}$

B. $\frac{1}{3}$

C. $\frac{1}{4}$

D. $\frac{2}{3}$

E. $\frac{1}{\sqrt{2}}$

22. The area of the region between the graph of $y = 5 - x^2$, and the x-axis, from $x = -1$ to $x = 2$ is

$$A = \int_{-1}^{2} (5 - x^2)\, dx$$

$$= \left(5x - \frac{x^3}{3}\right)\Big|_{-1}^{2}$$

$$= \left(5 \cdot 2 - \frac{2^3}{3}\right) - \left(5(-1) - \frac{(-1)^3}{3}\right)$$

$$= \left(10 - \frac{8}{3}\right) - \left(-5 + \frac{1}{3}\right)$$

$$= 15 - \frac{9}{3} = 12$$

A. 3

B. 12

C. $\frac{14}{3}$

D. $\frac{7}{3}$

E. 18

23. A certain sample of a radioactive substance decays to 30% of its original mass in 9 years. What is its half-life?

$m = m_0 e^{kt}$

$0.30\, m_0 = m_0 e^{k9} \qquad \frac{3}{10} = e^{9k}$

$\ln \frac{3}{10} = 9k \quad \longrightarrow k = \frac{1}{9} \ln \frac{3}{10}$

$m = m_0 e^{\frac{1}{9} \ln \frac{3}{10} \cdot t}$

$\frac{1}{2} m_0 = m_0 e^{\frac{1}{9} \ln \frac{3}{10} \cdot t} \qquad t\,?$

$\ln \frac{1}{2} = \frac{1}{9} \ln \frac{3}{10} \cdot t \longrightarrow t = 9 \frac{\ln \frac{1}{2}}{\ln \frac{3}{10}}$

A. $9\frac{\ln \frac{1}{2}}{\ln \frac{7}{10}}$

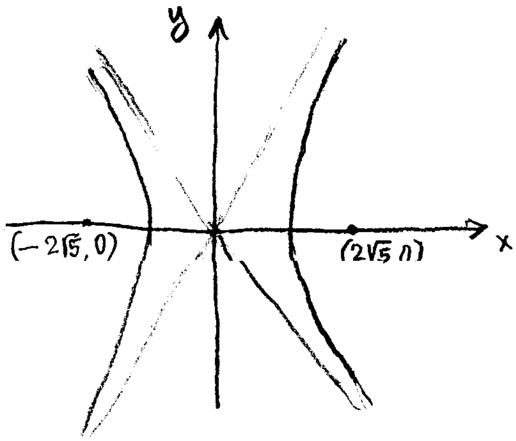
B. $9\frac{\ln \frac{1}{2}}{\ln \frac{3}{10}}$

C. $9\frac{\ln 2}{\ln 3}$

D. $\frac{1}{9}\frac{\ln \frac{1}{2}}{\ln \frac{3}{10}}$

E. $\frac{1}{9}\frac{\ln \frac{3}{10}}{\ln \frac{1}{2}}$

24. Find an equation of the hyperbola with foci $(\pm 2\sqrt{5}, 0)$ and asymptotes $y = \pm 2x$.



$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$c^2 = a^2 + b^2$$

$$c = 2\sqrt{5}$$

$$a^2 + b^2 = 20$$

asymptotes : $y = \pm \frac{b}{a} x$

$\therefore \frac{b}{a} = 2 \rightarrow b = 2a$

$$a^2 + (2a)^2 = 20$$

$$5a^2 = 20 \rightarrow a = 2, \quad b = 2 \cdot 2 = 4$$

$$\frac{x^2}{4} - \frac{y^2}{16} = 1$$

$$4x^2 - y^2 = 16$$

A. $4x^2 - y^2 = 1$

B. $-4x^2 + y^2 = 1$

C. $4x^2 - y^2 = 16$

D. $-4x^2 + y^2 = 16$

E. $4x^2 - y^2 = 20$

25. A focus of the ellipse $\dfrac{(x-2)^2}{25} + \dfrac{(y+3)^2}{9} = 1$ is

center $(2, -3)$

$a = 5, \quad b = 3$

$c^2 = a^2 - b^2 = 25 - 9 = 16$

$c = 4$

A. $(-1, -3)$

B. $(6, -3)$

C. $(-1, 3)$

D. $(-6, -3)$

E. $(2, 3)$