NAME SOLUTIONS

10-digit PUID # ______________________

RECITATION INSTRUCTOR ______________________

RECITATION TIME ______________________

LECTURER ______________________

INSTRUCTIONS

1. There are 9 different test pages (including this cover page). Make sure you have a complete test.
2. Fill in the above items in print. Also write your name at the top of pages 2–9.
3. Do any necessary work for each problem on the space provided or on the back of the pages of this test booklet. Circle your answers in this test booklet.
4. No books, notes, calculators, or any electronic devices may be used on this exam.
5. Each problem is worth 8 points. The maximum possible score is 200 points.
6. Using a #2 pencil, fill in each of the following items on your answer sheet:
   (a) On the top left side, write your name (last name, first name), and fill in the little circles.
   (b) On the bottom left side, under SECTION, write in your division and section number and fill in the little circles. (For example, for division 9 section 1, write 0901. For example, for division 38 section 2, write 3802).
   (c) On the bottom, under STUDENT IDENTIFICATION NUMBER, write in your 10-digit PUID number, and fill in the little circles.
   (d) Using a #2 pencil, put your answers to questions 1–25 on your answer sheet by filling in the circle of the letter of your response. Double check that you have filled in the circles you intended. If more than one circle is filled in for any question, your response will be considered incorrect. Use a #2 pencil.
7. After you have finished the exam, hand in your answer sheet and your test booklet to your recitation instructor.

1. $\lim_{x\to 0} \frac{x - x^2}{|x|} =$

$$\lim_{x\to 0^+} \frac{x-x^2}{|x|} = \lim_{x\to 0^+} \frac{x-x^2}{x} = \lim_{x\to 0^+}(1-x) = 1$$

$$\lim_{x\to 0^-} \frac{x-x^2}{|x|} = \lim_{x\to 0^+} \frac{x-x^2}{-x} = \lim_{x\to 0^-}(-1+x) = -1$$

A. 1

B. -1

C. 0

D. -2

E. Does not exist

2. Let $f(x) = \frac{x+5}{x^2-25}$, $x \neq 5$ and $x \neq -5$. Is it possible to define f at $x = -5$ so as to make f continuous at $x = -5$? If yes, find the value of $f(-5)$ that makes f continuous at $x = -5$.

$$\lim_{x\to(-5)} \frac{x+5}{x^2-25} = \lim_{x\to(-5)} \frac{x+5}{(x+5)(x-5)}$$

$$= \lim_{x\to(-5)} \frac{1}{x-5} = -\frac{1}{10}$$

f is continuous at $x=-5$ if

$$f(-5) = \lim_{x\to(-5)} f(x) = -\frac{1}{10}$$

A. $-\frac{1}{10}$

B. $-\frac{1}{\sqrt{5}}$

C. $\frac{1}{10}$

D. $\frac{1}{\sqrt{5}}$

E. Not possible

3. $\lim_{x\to\infty} \frac{\sin 3x + \cos 5x}{\sqrt{x}} =$

$$-\frac{2}{\sqrt{x}} \le \underbrace{\frac{\sin 3x + \cos 5x}{\sqrt{x}}}_{\downarrow\, 0} \le \frac{2}{\sqrt{x}}$$

as $x\to\infty$ $-\frac{2}{\sqrt{x}} \to 0$, $\frac{2}{\sqrt{x}} \to 0$ $\therefore$ $\frac{\sin 3x + \cos 5x}{\sqrt{x}} \to 0$

by the squeeze theorem

A. 8

B. -2

C. 0

D. $+\infty$

E. Does not exist

4. The vertical asymptotes of the graph of $f(x) = \frac{x^2+x-12}{x^2+x-2}$ are

$$f(x) = \frac{x^2+x-12}{(x-1)(x+2)}$$

$\lim_{x\to 1^+} f(x) = -\infty$ $\therefore$ $x=1$ is V.A.

$\lim_{x\to(-2)^+} f(x) = \infty$ $\therefore$ $x=-2$ is V.A.

A. $x = 3$ and $x = -4$

B. $x = -1$ and $x = -4$

C. $x = 1$ and $x = -2$

D. $x = -1$ and $x = 2$

E. $x = 1$ and $x = 3$

5. The equation of the tangent line to the graph of $y = \cos x$ at $\left(\frac{\pi}{4}, \frac{\sqrt{2}}{2}\right)$ is

$\frac{dy}{dx} = -\sin x$

$\left.\frac{dy}{dx}\right|_{x=\frac{\pi}{4}} = -\frac{\sqrt{2}}{2}$

eq. of tangent line at $\left(\frac{\pi}{4}, \frac{\sqrt{2}}{2}\right)$:

$y - \frac{\sqrt{2}}{2} = -\frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right)$

A. $y - \frac{\pi}{4} = -\frac{\sqrt{2}}{2}\left(x - \frac{\sqrt{2}}{2}\right)$

B. $y - \frac{\pi}{4} = \frac{\sqrt{2}}{2}\left(x - \frac{\sqrt{2}}{2}\right)$

C. $y - \frac{\sqrt{2}}{2} = x - \frac{\pi}{4}$

D. $y - \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right)$

E. $y - \frac{\sqrt{2}}{2} = -\frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right)$

6. If $f(x) = \frac{e^x}{\sqrt{x^2+2}}$, then $f'(x) =$

$f'(x) = \frac{\sqrt{x^2+2}\, e^x - e^x \frac{1}{2\sqrt{x^2+2}}(2x)}{x^2+2}$

$= \frac{e^x(x^2+2-x)}{(x^2+2)^{3/2}}$

A. $\frac{e^x(\sqrt{x^2+2} - x)}{(x^2+2)^{\frac{3}{2}}}$

B. $\frac{xe^x}{(x^2+2)^{\frac{3}{2}}}$

C. $\frac{e^x(x^2 - x + 2)}{(x^2+2)^{\frac{3}{2}}}$

D. $\frac{e^x(2x^2 - x + 4)}{(x^2+2)^{\frac{3}{2}}}$

E. $\frac{e^x(x^2 + x + 2)}{(x^2+2)^{\frac{3}{2}}}$

7. If $f(x) = \ln(\sin x^2)$, then $f'(x) =$

$f'(x) = \frac{1}{\sin x^2} \cdot (\cos x^2)\, 2x$

$= 2x \cot x^2$

A. $2x \cot x$

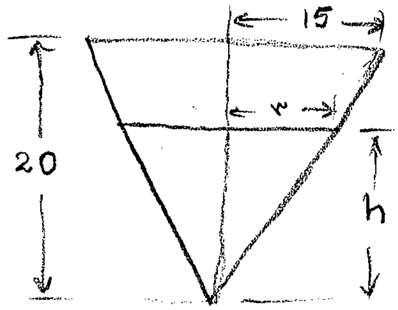
B. $2x \cot x^2$

C. $\frac{2x}{\sin x^2}$

D. $2 \cot x$

E. $2x \cos x^2(\ln(\sin x^2))$

8. A water tank has the shape of an inverted circular cone with base radius 15 ft and height 20 ft. If water is leaking from the tank at the rate of 72 ft^3/min, how fast is the depth of the water changing when the water is 12 ft deep? ($V = \frac{1}{3}\pi r^2 h$).



$\frac{dV}{dt} = -72$

$V = \frac{1}{3}\pi r^2 h$

From similar triangles:

$\frac{r}{h} = \frac{15}{20} \rightarrow r = \frac{3}{4}h$

$V = \frac{1}{3}\pi\left(\frac{3}{4}h\right)^2 h \rightarrow V = \frac{3\pi}{16}h^3$

$\frac{dV}{dt} = \frac{9\pi}{16}h^2\frac{dh}{dt}$. When $h=12$:

$-72 = \frac{9\pi}{16}(12)^2\frac{dh}{dt} \rightarrow \frac{dh}{dt} = -\frac{8}{9\pi}$ ft/min

A. $-\frac{2}{3\pi}$ ft/min

B. $-\frac{24}{25\pi}$ ft/min

C. $-\frac{8}{3\pi}$ ft/min

D. $-\frac{8}{27\pi}$ ft/min

E. $-\frac{8}{9\pi}$ ft/min

9. The function $f(x) = \frac{x^3}{3} + \frac{7}{2}x^2 + 12x + 5$ attains a local minimum when $x =$

$f'(x) = x^2 + 7x + 12$
$= (x+4)(x+3)$

$f'(x)$: $+++\ 0\ ---\ 0\ ++++$ on the x-axis, zeros at -4 and -3; $-3 \uparrow$ loc. min

or

$f''(x) = 2x + 7$
$f''(-4) = -1$
$f''(-3) = 1 \leftarrow$ loc. min.

A. -4

B. -3

C. $5 - \frac{27}{2}$

D. 3

E. 4

10. The second derivative of a function f is given by

$$f''(x) = (x+2)(x-1)^2(x-5).$$

The graph of f has an inflection point when

$f''(x)$: $+++\ 0\ ---\ 0\ ------\ 0\ +++$ on the x-axis, zeros at -2, 1, 5

A. $x = -2$, $x = 1$ and $x = 5$

B. $x = 1$ only

C. $x = 1$ and $x = 5$

D. $x = -2$ and $x = 1$

E. $x = -2$ and $x = 5$

11. If $g(x) = 4x^3 - 3x^4$, which of the following statements are <u>true</u>?

(1) The graph of g is concave downward for all $x < 0$.

(2) g is decreasing on the interval $(1, \infty)$.

(3) g has a local extreme value at $x = 0$.

$g'(x) = 12x^2 - 12x^3 = 12x^2(1-x)$

$g'(x)$: $+ + + + \; 0 \; + + \; 0 \; - - -$; x: 0, 1

$\therefore$ (2) is true

(3) not true

$g''(x) = 24x - 36x^2 = 12x(2-3x)$

$g''(x)$: $- - - - \; 0 \; + + + \; 0 \; - - - - -$; x: 0, $\frac{2}{3}$

$\therefore$ (1) is true

A. (1), (2) and (3)

B. only (3)

C. only (2)

D. (1) and (2)

E. (2) and (3)

12. Find the absolute maximum and absolute minimum values of $f(x) = 2x^3 - 9x^2 + 12x$ on the interval $[0, 3]$.

$f'(x) = 6x^2 - 18x + 12$

$= 6(x^2 - 3x + 2)$

$= 6(x-1)(x-2)$

$\therefore x = 1, 2$ are critical numbers

$f(0) = 0 \leftarrow$ min

$f(1) = 2 - 9 + 12 = 5$

$f(2) = 16 - 36 + 24 = 4$

$f(3) = 54 - 81 + 36 = 9 \leftarrow$ max

A. max 5, min 4

B. max 5, min -6

C. max 9, min 0

D. max 5, min 0

E. max 9, min -6

13. The slope of the tangent line to the graph of $\ln(x^2 - 3y) = x - y - 1$ at the point $(2, 1)$ is

$$\frac{1}{x^2 - 3y}\left(2x - 3\frac{dy}{dx}\right) = 1 - \frac{dy}{dx}$$

At $(x, y) = (2, 1)$:

$$\frac{1}{4-3}\left(4 - 3\frac{dy}{dx}\right) = 1 - \frac{dy}{dx}$$

$$4 - 3\frac{dy}{dx} = 1 - \frac{dy}{dx}$$

$$3 = 2\frac{dy}{dx}, \quad \frac{dy}{dx} = \frac{3}{2}$$

A. $\frac{3}{2}$

B. 0

C. 1

D. $-\frac{1}{2}$

E. $\frac{3}{4}$

14. The mass of a radioactive substance decreases from 12 grams to 4 grams in 1 day. How long will it take for 18 grams of the substance to decay to 2 grams?

$m(t) = m(0)e^{kt}$

$m(0) = 12, \quad m(1) = 4: \quad 4 = 12e^{k1}$

$e^k = \frac{1}{3} \to k = -\ln 3$

$\therefore \; m(t) = m(0)e^{-(\ln 3)t}$

If $m(0) = 18$, find t so that $m(t) = 2$

$2 = 18e^{-(\ln 3)t}$

$\frac{1}{9} = e^{-(\ln 3)t} \longrightarrow -\ln 9 = -(\ln 3)t$

$t = \frac{2\ln 3}{\ln 3} = 2$

A. $\frac{\ln 3}{\ln 2}$ days

B. $\frac{3}{2}$ days

C. $\frac{\ln 2}{\ln 3}$ days

D. 2 days

E. 3 days

15. $\lim_{x\to 0} \frac{\sin x - x}{x^3} \overset{L'H}{=} \lim_{x\to 0} \frac{\cos x - 1}{3x^2} \overset{L'H}{=} \lim_{x\to 0} \frac{-\sin x}{6x}$

$\frac{0}{0}$ $\quad \frac{0}{0}$ $\quad \frac{0}{0}$

$\overset{L'H}{=} \lim_{x\to 0} \frac{-\cos x}{6} = -\frac{1}{6}$

A. $\frac{1}{6}$

B. $-\frac{1}{6}$

C. $\frac{1}{2}$

D. $\frac{1}{3}$

E. 0

16. $\int_0^{\frac{1}{2}\sqrt{\pi}} x\cos(x^2)\,dx = \frac{1}{2}\int_0^{\frac{\pi}{4}} \cos u\,du = \frac{1}{2}\sin u\Big|_0^{\frac{\pi}{4}}$

$= \frac{1}{2}\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{4}$

$u = x^2$

$du = 2x\,dx$

$x = 0 \to u = 0$

$x = \frac{1}{2}\sqrt{\pi} \longrightarrow u = \frac{\pi}{4}$

A. 1

B. $\frac{1}{2}$

C. $\frac{\sqrt{2}}{2}$

D. $\frac{\sqrt{2}}{4}$

E. $\frac{\sqrt{2}}{2} - \frac{1}{2}$

17. $\int_1^e \frac{dx}{x(2+\ln x)} = \int_2^3 \frac{1}{u}\,du = \ln u\Big|_2^3$

$= \ln 3 - \ln 2$

$= \ln\frac{3}{2}$

$u = 2 + \ln x$

$du = \frac{1}{x}dx$

$x = 1 \to u = 2$

$x = e \to u = 3$

A. $\frac{1}{\ln 3} - \frac{1}{\ln 2}$

B. $\ln\frac{1}{3}$

C. $\ln 3$

D. $\frac{1}{\ln 3}$

E. $\ln\frac{3}{2}$

18. $\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos x}\,dx = -\int_2^1 \frac{1}{u}du = \int_1^2 \frac{1}{u}du = \ln u\Big]_1^2$

$= \ln 2$

$u = 1+\cos x$
$du = -\sin x\,dx$
$x=0 \rightarrow u=2$
$x=\frac{\pi}{2} \rightarrow u=1$

A. $-\ln 2$

B. 2

C. $\frac{\pi}{2}$

D. $\ln 2$

E. π

19. If $f(x) = x^{\sqrt{x}}$, then $f'(4) =$

$f(x) = x^{\sqrt{x}} = e^{\ln x^{\sqrt{x}}} = e^{\sqrt{x}\ln x}$

$f'(x) = e^{\sqrt{x}\ln x}\left(\sqrt{x}\,\frac{1}{x} + \frac{1}{2\sqrt{x}}\ln x\right)$

$f'(4) = e^{2\ln 4}\left(2\cdot\frac{1}{4} + \frac{1}{2\cdot 2}\ln 4\right)$

$= 4^2\left(\frac{1}{2} + \frac{1}{4}\ln 4\right)$

$= 4\,(2+\ln 4)$

A. $8\ln 4$

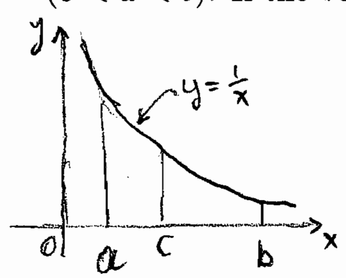
B. 8

C. $4(2+\ln 4)$

D. $8(1+\ln 4)$

E. $16(1+\ln 4)$

20. Let R be the region between the graph of $y = \frac{1}{x}$ and the x-axis, from $x=a$ to $x=b$ $(0<a<b)$. If the vertical line $x=c$ cuts R into two parts of equal area, then $c =$



$\int_a^c \frac{1}{x}dx = \int_c^b \frac{1}{x}dx$

$\ln x\Big|_a^c = \ln x\Big]_c^b$

$\ln c - \ln a = \ln b - \ln c$

$2\ln c = \ln a + \ln b$

$\ln c^2 = \ln(ab)$

$c^2 = ab$

$c = \sqrt{ab}$

A. $\frac{a+b}{2}$

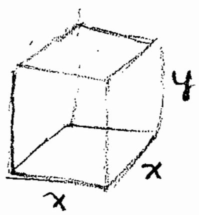
B. $\frac{\ln a + \ln b}{2}$

C. $\ln\left(\frac{a+b}{2}\right)$

D. $\ln\left(\frac{b-a}{2}\right)$

E. $\sqrt{ab}$

21. If 1200 cm^2 of material is available to make a box with a square base and an open top, find the largest possible volume of the box.



$x^2 + 4xy = 1200$

$V = x^2 y$

$V = x^2 \frac{1200 - x^2}{4x}$

$V = 300x - \frac{1}{4}x^3, \quad 0 < x < \sqrt{1200}$

$\frac{dV}{dx} = 300 - \frac{3}{4}x^2$

$\frac{dV}{dx} = 0 \; : \; x^2 = 400 \to x = 20$ $\quad \frac{dV}{dx}$ $++ + 0 - - -$ (x; 20) $\quad$ V is max at $x = 20$

$\max V = 300 \cdot 20 - \frac{1}{4} 20^3 = 6000 - 2000 = 4000 \text{ cm}^3$

A. 5000 cm^3
B. 4000 cm^3
C. 4600 cm^3
D. 6200 cm^3
E. 4800 cm^3

22. The area of the region between the graph of $f(x) = \frac{1}{\sqrt{1-x^2}}$ and the x-axis, from $x = 0$ to $x = \frac{1}{2}$ is

$$A = \int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x \Big|_0^{\frac{1}{2}} =$$
$$= \sin^{-1}\tfrac{1}{2} - \sin^{-1} 0$$
$$= \frac{\pi}{6} - 0$$
$$= \frac{\pi}{6}$$

A. π
B. $\frac{\pi}{2}$
C. $\frac{\pi}{3}$
D. $\frac{\pi}{4}$
E. $\frac{\pi}{6}$

23. If $F(x) = \int_0^{x^3} te^{t^2}\,dt$, then $F'(2) =$

$$F'(x) = x^3 e^{(x^3)^2} \cdot 3x^2 = 3x^5 e^{x^6}$$
$$F'(2) = 3 \cdot 2^5 e^{2^6} = 96 e^{64}$$

A. $96e^{64}$
B. $64e^{32}$
C. $24e^{64}$
D. $38e^{8}$
E. $32e^{64}$

24. The hyperbola $9x^2 - 54x - 4y^2 + 45 = 0$ has vertices at the points

A. $(0, 1)$ and $(0, 5)$

B. $(2, 0)$ and $(6, 0)$

C. $(\frac{3}{2}, 0)$ and $(\frac{5}{2}, 0)$

D. $(1, 0)$ and $(5, 0)$

E. $(0, 2)$ and $(0, 6)$

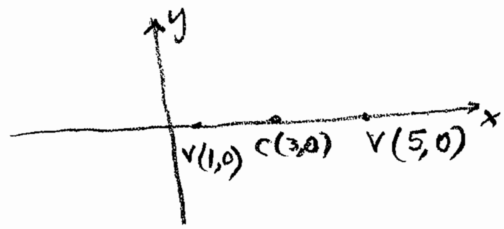
$$9(x^2 - 6x + \quad) - 4y^2 = -45$$

$$9(x^2 - 6x + 9) - 4y^2 = -45 + 81$$

$$9(x-3)^2 - 4y^2 = 36$$

$$\frac{(x-3)^2}{4} - \frac{y^2}{9} = 1$$

center $(3, 0)$, $a = 2$



25. If a parabola with axis parallel to the y-axis and vertex at $(4, -1)$ passes through the point $(7, -2)$, then it crosses the y-axis at $y =$

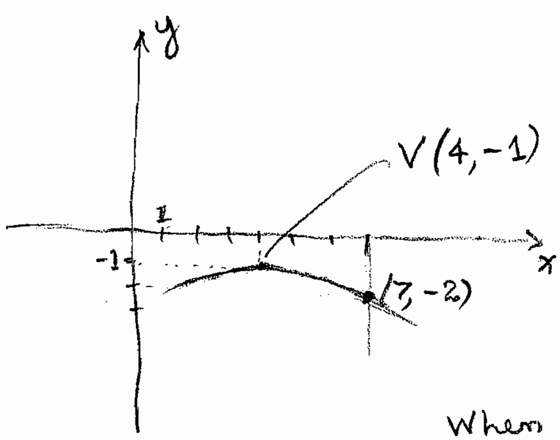
A. $\frac{4}{3}$

B. $-\frac{25}{9}$

C. $\frac{5}{3}$

D. $-\frac{36}{9}$

E. $-\frac{16}{9}$



$$(x-4)^2 = 4p(y+1)$$

$(7, -2)$ on parabola:

$$(7-4)^2 = 4p(-2+1)$$

$$9 = -4p$$

$$(x-4)^2 = -9(y+1)$$

When $x = 0$:

$$16 = -9(y+1)$$

$$16 = -9y - 9$$

$$9y = -25$$

$$y = -\frac{25}{9}$$