NAME GRADING KEY WITH SOLUTIONS

10-digit PUID # ______

RECITATION INSTRUCTOR ______

RECITATION TIME ______

LECTURER ______

INSTRUCTIONS

1. There are 14 different test pages (including this cover page). Make sure you have a complete test.
2. Fill in the above items in print. Also write your name at the top of pages 2–14.
3. Do any necessary work for each problem on the space provided or on the back of the pages of this test booklet. Circle your answers in this test booklet. No partial credit will be given, but if you show your work on the test booklet, it may be used in borderline cases.
4. No books, notes, calculators, or any electronic devices may be used on this exam.
5. Each problem is worth 8 points. The maximum possible score is 200 points.
6. Using a #2 pencil, fill in each of the following items on your answer sheet:
    (a) On the top left side, write your name (last name, first name), and fill in the little circles.
    (b) On the bottom left side, under SECTION NUMBER, put 0 in the first column and then enter the 3-digit section number. For example, for section 016 write 0016, and fill in the little circles.
    (c) On the bottom, under TEST/QUIZ NUMBER, write 01 and fill in the little circles.
    (d) On the bottom, under STUDENT IDENTIFICATION NUMBER, write in your 10-digit PUID, and fill in the little circles.
    (e) Using a #2 pencil, put your answers to questions 1–25 on your answer sheet by filling in the circle of the letter of your response. Double check that you have filled in the circles you intended. If more than one circle is filled in for any question, your response will be considered incorrect. Use a #2 pencil.
7. After you have finished the exam, hand in your answer sheet and your test booklet to your recitation instructor.

1. The domain of the function $y = \ln(2^{x/3})$ is

[✓ A. All real numbers.]

B. $x > 0$

C. $x > \ln\left(\frac{2}{3}\right)$

D. $x > \ln\left(\frac{3}{2}\right)$

E. $x > -\frac{1}{3}$

$2^{\frac{x}{3}} > 0$ for all real numbers x

$\therefore$ $\ln(2^{\frac{x}{3}})$ is defined for all real numbers

2. Express $\tan(\sin^{-1}(x))$ as an algebraic function of x.

A. $\frac{1}{\sqrt{1-x^2}}$

B. $\frac{1}{\sqrt{x^2-1}}$

[✓ C. $\frac{x}{\sqrt{1-x^2}}$]

D. $\sqrt{1-x^2}$

E. $\frac{\sqrt{1-x^2}}{x}$

Let $y = \sin^{-1} x \iff x = \sin y$, $\underbrace{-\frac{\pi}{2} \le y \le \frac{\pi}{2}}$

$$\tan(\sin^{-1} x) = \tan y = \frac{\sin y}{\cos y}$$

$\sin y = x$ $\quad \cos^2 y = 1 - \sin^2 y = 1 - x^2$

$$\cos y = +\sqrt{1-x^2}$$

$$\therefore \tan(\sin^{-1} x) = \frac{x}{\sqrt{1-x^2}}$$

3. $\lim_{x \to -2} \frac{\frac{1}{2} + \frac{1}{x}}{2 + x} =$

- A. ∞
- B. $-\infty$
- C. 0
- [✓ D. $-\frac{1}{4}$]
- E. $\frac{1}{2}$

$$\lim_{x \to -2} \frac{\frac{1}{2} + \frac{1}{x}}{2 + x} = \lim_{x \to -2} \frac{x+2}{(2+x)2x}$$

$$= \lim_{x \to -2} \frac{1}{2x} = -\frac{1}{4}$$

or

$$\lim_{x \to -2} \underset{\frac{0}{0}}{\frac{\frac{1}{2} + \frac{1}{x}}{2+x}} \overset{L'H}{=} \lim_{x \to -2} \frac{\left(-\frac{1}{x^2}\right)}{1}$$

$$= -\frac{1}{4}$$

4. $\lim_{x \to -\infty} \frac{\sqrt{x^2 + 2}}{x} =$

- A. ∞
- B. $-\infty$
- C. 0
- D. 1
- [✓ E. -1]

$$\lim_{x \to -\infty} \frac{\sqrt{x^2+2}}{x} = \lim_{x \to -\infty} \frac{\sqrt{x^2}\sqrt{1+\frac{2}{x^2}}}{x}$$

$$= \lim_{x \to -\infty} \frac{|x|\sqrt{1+\frac{2}{x^2}}}{x}$$

$$= \lim_{x \to -\infty} \frac{-x\sqrt{1+\frac{2}{x^2}}}{x} = -1$$

5. Find an equation of the tangent line to the curve, $y = \dfrac{x}{x+1}$ at the point $\left(1, \dfrac{1}{2}\right)$.

$$\frac{dy}{dx} = \frac{(x+1)1 - x \cdot 1}{(x+1)^2} = \frac{1}{(x+1)^2}$$

$$\left.\frac{dy}{dx}\right|_{x=1} = \frac{1}{4}$$

eq. of tan. line :

$$y - \frac{1}{2} = \frac{1}{4}(x-1)$$

$$4y - 2 = x - 1$$

$$4y - x = 1$$

A. $y = \dfrac{1}{2}$

B. $x + 2y = 2$

C. $x - 2y = 0$

D. $4y + x = 3$

[✓ E. $4y - x = 1$]

6. The radius of a sphere is increasing at a rate of 2 mm/sec. How fast is the volume increasing when the radius is 10 mm? ($V = \dfrac{4}{3}\,\pi r^3$)

$$\frac{dr}{dt} = 2$$

Find $\dfrac{dV}{dt}$ when $r = 10$

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

When $r = 10$:

$$\frac{dV}{dt} = 4\pi (10)^2 2 = 800\pi$$

A. $640\pi\ mm^3/$ sec

B. $1600\pi\ mm^3/$ sec

C. $1000\pi\ mm^3/$ sec

[✓ D. $800\pi\ mm^3/$ sec]

E. $1200\pi\ mm^3/$ sec

7. Use a linear approximation (or differentials) to estimate $\sqrt[3]{994}$.

- A. 9.95
- B. 9.8
- C. 10.02
- D. 9.99
- [✓ E. 9.98]

$f(x) \approx f(a) + f'(a)(x-a)$, for x near a

$f(x) = \sqrt[3]{x}$ $\quad a = 1000$, $\quad f(1000) = 10$

$f'(x) = \frac{1}{3}\frac{1}{x^{2/3}}$ $\quad f'(1000) = \frac{1}{3}\frac{1}{100} = \frac{1}{300}$

$\sqrt[3]{x} \approx 10 + \frac{1}{300}(x-1000)$, for x near 1000

$$\sqrt[3]{994} \approx 10 + \frac{1}{300}(994-1000)$$

$$= 10 + \frac{1}{300}(-6)$$

$$= 10 - \frac{2}{100} = 9.98$$

8. A population of bacteria doubles every 2 days. How long will it take for the population to triple?

- A. $2\ln(\frac{3}{2})$ days
- [✓ B. $2\,\frac{\ln 3}{\ln 2}$ days]
- C. 3 days
- D. 2.5 days
- E. $2\ln 6$ days

$P(t) = P_0 e^{kt}$ $\quad$ where $P_0 = P(0)$.

$t = 2$: $\quad 2P_0 = P_0 e^{k2}$

$2 = e^{k2}$

$\ln 2 = k2 \to k = \frac{\ln 2}{2}$

$P(t) = P_0 e^{\frac{\ln 2}{2}t}$

Find t for which $P(t) = 3P_0$

$$3P_0 = P_0 e^{\frac{\ln 2}{2}t}$$

$$3 = e^{\frac{\ln 2}{2}t}$$

$$\ln 3 = \frac{\ln 2}{2}t$$

$$t = 2\frac{\ln 3}{\ln 2}$$

9. If $f(x) = \ln(\sin(x^2))$, then $f'(x) =$

$$f(x) = \ln(\sin(x^2))$$

$$f'(x) = \frac{1}{\sin(x^2)} \cos(x^2) \cdot 2x$$

$$= \frac{2x\cos(x^2)}{\sin(x^2)}$$

A. $\dfrac{\sin(x^2)}{2x\cos(x^2)}$

B. $\dfrac{\cos(x^2)}{2x\sin(x^2)}$

[✓ C. $\dfrac{2x\cos(x^2)}{\sin(x^2)}$]

D. $\dfrac{2x\sin(x^2)}{\cos(x^2)}$

E. $\dfrac{2x}{\sin(x^2)\cos(x^2)}$

10. If $f(x) = x^{\sin x}$, then $f'(x) =$

$$f(x) = x^{\sin x} = (e^{\ln x})^{\sin x} = e^{\sin x \ln x}$$

$$f'(x) = e^{\sin x \ln x}\left((\sin x)\frac{1}{x} + \cos x \ln x\right)$$

$$= x^{\sin x}\left(\cos x \ln x + \frac{\sin x}{x}\right)$$

A. $x^{\cos x}$

B. $x^{\sin x}\cos x$

[✓ C. $x^{\sin x}\left(\cos x \ln x + \dfrac{\sin x}{x}\right)$]

D. $x^{\sin x}(\cos x \ln x)$

E. $x^{\sin x}\,\dfrac{\cos x}{x}$

11. Find an equation of the tangent line to the curve $(x^2+y^2)^2 = 4x^2y$ at the point $(x,y) = (1,1)$.

$(x^2+y^2)^2 = 4x^2y$

$2(x^2+y^2)\left(2x+2y\frac{dy}{dx}\right) = 4x^2\frac{dy}{dx} + 8xy$

At $(x,y)=(1,1)$:

$2(1+1)\left(2+2\frac{dy}{dx}\right) = 4\frac{dy}{dx} + 8\cdot 1$

$8 + 8\frac{dy}{dx} = 4\frac{dy}{dx} + 8$

$4\frac{dy}{dx} = 0$

$\left.\frac{dy}{dx}\right|_{(1,1)} = 0$

eq. of tan. line:

$y-1 = 0(x-1)$

$y = 1$

*A. $y = 1$
B. $y = x$
C. $y = 2x-1$
D. $y = -x+2$
E. $y = -2x+3$

12. The absolute maximum value of the function $f(x) = \frac{x}{x^2+1}$ on the interval $[0,2]$ is

$f'(x) = \frac{(x^2+1)1 - x2x}{(x^2+1)^2} = \frac{-x^2+1}{(x^2+1)^2}$

$f'(x)=0$: $x = \pm 1$, $x=-1$ not in $(0,2)$

$x=1$ is the only critical number in $(0,2)$

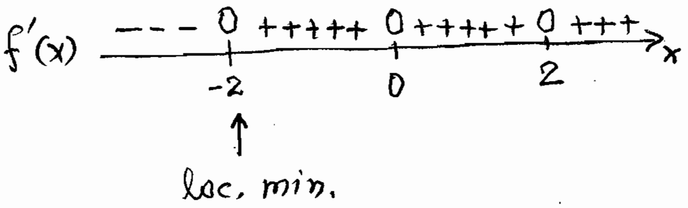
$f(0)=0$

$f(1)=\frac{1}{2}$ ← abs. max. value

$f(2)=\frac{2}{5}$

A. $\frac{2}{5}$
B. 0
C. 1
* D. $\frac{1}{2}$
E. $\frac{3}{4}$

13. Let f be a function whose derivative f' is given by $f'(x) = (x-2)^2 x^4 (x+2)^5$. Then f has a



A. local minimum at $x = 0$

$\ast$ B. local minimum at $x = -2$

C. local maximum at $x = 0$

D. local minimum at $x = 2$

E. local maximum at $x = 2$

14. $\lim\limits_{x\to\frac{\pi}{2}} (\sec x - \tan x) = \lim\limits_{x\to\frac{\pi}{2}} \left(\frac{1}{\cos x} - \frac{\sin x}{\cos x}\right)$

$$= \lim_{x\to\frac{\pi}{2}} \frac{1-\sin x}{\cos x} \overset{L'H}{=} \lim_{x\to\frac{\pi}{2}} \frac{-\cos x}{-\sin x}$$

$$\frac{0}{0} \qquad = \frac{0}{-1} = 0$$

$\ast$ A. 0

B. 1

C. ∞

D. -1

E. $\frac{1}{2}$

15. A box with square base and open top must have a volume of 4000 cm^3. If the cost of the material used is \$1/cm^2, the smallest possible cost of the box is

A. \$500

B. \$600

C. \$1000

* D. \$1200

E. \$2000

Let x be the length of a side of the base and y be the height of the box.

Volume $=4000 \longrightarrow x^2y=4000$

Let S be the area of the material.

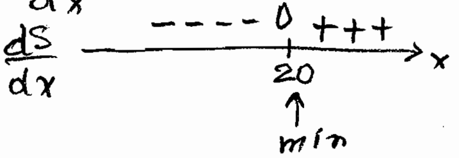
$$S = x^2 + 4xy$$

$$S = x^2 + 4x\,\frac{4000}{x^2}$$

$$S = x^2 + \frac{16000}{x}$$

$$\frac{dS}{dx} = 2x - \frac{16000}{x^2}$$

$$\frac{dS}{dx} = 0 : \quad x^3 = 8000 \longrightarrow x = 20$$



$$\text{min } S = (20)^2 + \frac{16000}{20}$$
$$= 400 + 800 = 1200$$

16. If $\int_{-2}^{2} f(x)dx = 2$ and $\int_{0}^{2} f(x)dx = 3$, then $\int_{-2}^{0} f(x)dx =$

* A. -1

B. 1

C. -5

D. 5

E. -3

$$\int_{-2}^{2} f(x)dx = \int_{-2}^{0} f(x)dx + \int_{0}^{2} f(x)dx$$

$$2 = \int_{-2}^{0} f(x)dx + 3$$

$$\int_{-2}^{0} f(x)\,dx = -1$$

17. $\int_0^1 (x^2 - \sqrt{x} + 1)dx = \int_0^1 (x^2 - x^{1/2} + 1)dx$

$= \left[\frac{x^3}{3} - \frac{x^{3/2}}{\frac{3}{2}} + x\right]_0^1$

$= \frac{1}{3} - \frac{2}{3} + 1 = \frac{2}{3}$

A. $-\frac{1}{6}$

B. $\frac{5}{6}$

[✓ C. $\frac{2}{3}$]

D. $\frac{1}{3}$

E. 1

18. $\int_0^{\frac{\pi}{2}} \sin(2x)dx = \frac{1}{2}\int_0^{\pi} \sin u \, du$

$u = 2x$

$du = 2\,dx$

$x = 0 \rightarrow u = 0$

$x = \frac{\pi}{2} \rightarrow u = \pi$

$= \frac{1}{2}(-\cos u)\Big|_0^{\pi}$

$= \frac{1}{2}(-\cos\pi) - \frac{1}{2}(-\cos 0)$

$= \frac{1}{2} + \frac{1}{2}$

[✓ A. 1]

B. -1

C. 2

D. -2

E. 0

19. $\int_0^1 x^2(x^3+1)^{17}dx = \frac{1}{3}\int_1^2 u^{17}du = \frac{1}{3}\frac{u^{18}}{18}\Big|_1^2$

$u = x^3+1$

$du = 3x^2dx$

$x=0 \to u=1$

$x=1 \to u=2$

$= \frac{2^{18}-1}{54}$

A. $\frac{2^{18}}{18}$

B. $\frac{2^{18}}{54}$

C. $\frac{2^{18}-1}{18}$

[✓ D. $\frac{2^{18}-1}{54}$]

E. $\frac{2^{18}-1}{3}$

20. If $g(x) = \int_0^{2x} e^{t^2}\,dt$, then $g'(x) =$ $e^{(2x)^2}\cdot 2$

A. e^{2x^2}

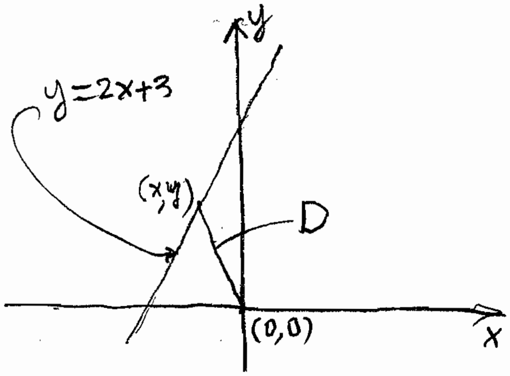
B. $2e^{x^2}$

C. $2e^{2x^2}$

[✓ D. $2e^{4x^2}$]

E. e^{4x^2}

21. Find the x–coordinate of the point on the line $y = 2x + 3$ that is closest to the origin.



$$D = \sqrt{x^2+(2x+3)^2}$$
$$= \sqrt{x^2+4x^2+12x+9}$$
$$= \sqrt{5x^2+12x+9}$$
$$\frac{dD}{dx} = \frac{1}{2\sqrt{5x^2+12x+9}}(10x+12)$$
$$\frac{dD}{dx} = 0 : \quad 10x+12=0$$
$$x = -\frac{6}{5}$$

$\frac{dD}{dx}$ $- - - - 0 + + + + +$ $\rightarrow x$

$-\frac{6}{5}$

↑

min

* A. $-\frac{6}{5}$

B. 3

C. $\frac{3}{4}$

D. $-\frac{2}{7}$

E. $-\frac{4}{3}$

22. Let f be any function that satisfies $1 \le f'(x) \le 3$ for all values of x. Then it follows from the Mean Value Theorem that

The mean value theorem asserts that

$$\frac{f(4)-f(2)}{4-2} = f'(c)$$

for some $c \in (2,4)$

$$1 \le f'(c) \le 3$$
$$1 \le \frac{f(4)-f(2)}{2} \le 3$$
$$2 \le f(4)-f(2) \le 6$$

A. $-6 \le f(4) - f(2) \le -2$

B. $-4 \le f(4) - f(2) \le 0$

C. $-2 \le f(4) - f(2) \le 2$

D. $0 \le f(4) - f(2) \le 4$

* E. $2 \le f(4) - f(2) \le 6$

23. The focus of the parabola $x^2 - 2x - 8y = 23$ is at

A. $(1, -2)$

*B. $(1, -1)$

C. $(-1, 2)$

D. $(-1, 3)$

E. $(1, -3)$

$x^2 - 2x = 8y + 23$

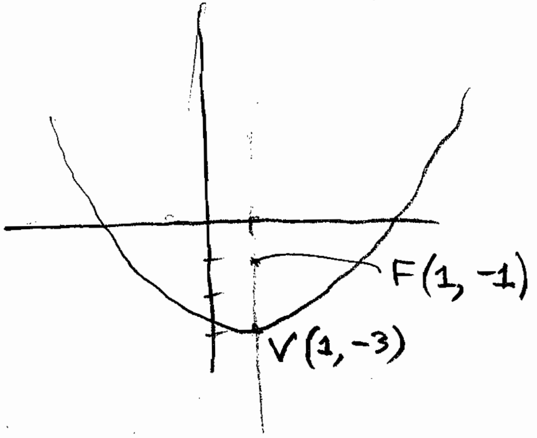
$x^2 - 2x + 1 = 8y + 23 + 1$

$(x-1)^2 = 8(y+3)$

vertex $(1, -3)$

$4p = 8$

$p = 2$



24. Find an equation of the ellipse with center $(1, -2)$, focus $(3, -2)$, and vertex $(4, -2)$.

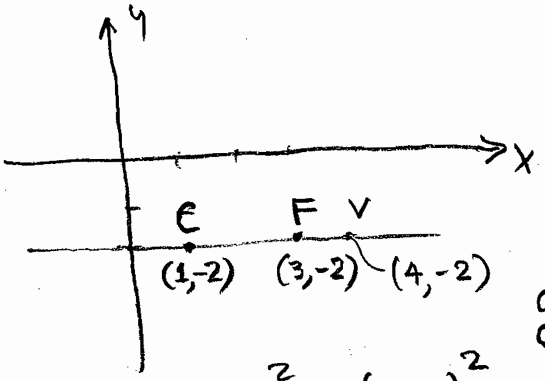
A. $9(x-1)^2 + 5(y+2)^2 = 45$

B. $16(x-1)^2 + 9(y+2)^2 = 144$

C. $16(x+1)^2 + 9(y-2)^2 = 144$

D. $7(x+1)^2 + 16(y-2)^2 = 112$

*E. $5(x-1)^2 + 9(y+2)^2 = 45$



$a = 3$

$c = 2$

$c^2 = a^2 - b^2$

$4 = 9 - b^2 \rightarrow b^2 = 5$

$$\frac{(x-1)^2}{9} + \frac{(y+2)^2}{5} = 1$$

$5(x-1)^2 + 9(y+2)^2 = 45$

25. Find the vertices and foci of the hyperbola $\frac{(y-2)^2}{4} - \frac{(x+3)^2}{3} = 1$.

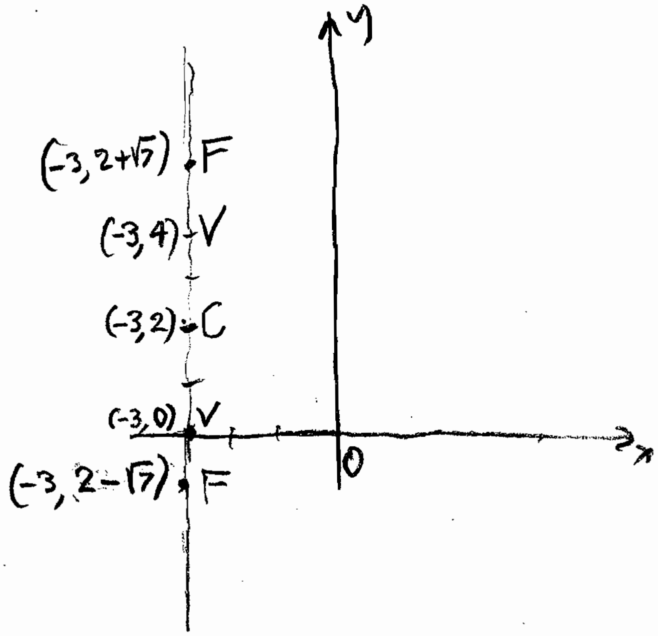
A. vertices $(-3,0)$, $(-3,5)$; foci $(-3,4)$, $(-3,-1)$

B. vertices $(-3,4)$, $(-3,0)$; foci $(-3,5)$, $(-3,-1)$

[✓ C.] vertices $(-3,4)$, $(-3,0)$; foci $(-3,2+\sqrt{7})$, $(-3,2-\sqrt{7})$

D. vertices $(-1,2)$, $(-5,2)$; foci $(-3-\sqrt{7},2)$, $(-3+\sqrt{7},2)$

E. vertices $(-1,2)$, $(-5,2)$; foci $(0,2)$, $(-6,2)$



center $(-3,2)$

$a=2$

$b=\sqrt{3}$

$c^2=a^2+b^2=4+3$

$c=\sqrt{7}$