NAME SOLUTIONS

10-digit PUID # ______

RECITATION INSTRUCTOR ______

RECITATION TIME ______

LECTURER ______

INSTRUCTIONS

1. There are 10 different test pages (including this cover page). Make sure you have a complete test.
2. Fill in the above items in print. Also write your name at the top of pages 2–10.
3. Do any necessary work for each problem on the space provided or on the back of the pages of this test booklet. Circle your answers in this test booklet. No partial credit will be given, but if you show your work on the test booklet, it may be used in borderline cases.
4. No books, notes, calculators, or any electronic devices may be used on this exam.
5. Each problem is worth 8 points. The maximum possible score is 200 points.
6. Using a #2 pencil, fill in each of the following items on your answer sheet:
   - (a) On the top left side, write your name (last name, first name), and fill in the little circles.
   - (b) On the bottom left side, under SECTION NUMBER, put 0 in the first column and then enter the 3-digit section number. For example, for section 016 write 0016, and fill in the little circles.
   - (c) On the bottom, under TEST/QUIZ NUMBER, write 01 and fill in the little circles.
   - (d) On the bottom, under STUDENT IDENTIFICATION NUMBER, write in your 10-digit PUID, and fill in the little circles.
   - (e) Using a #2 pencil, put your answers to questions 1–25 on your answer sheet by filling in the circle of the letter of your response. Double check that you have filled in the circles you intended. If more than one circle is filled in for any question, your response will be considered incorrect. Use a #2 pencil.
7. After you have finished the exam, hand in your answer sheet and your test booklet to your recitation instructor.

1. The domain of $f(x) = \ln(1 - |x|^2)$ is

$1 - |x|^2 > 0$

$|x|^2 < 1$

$-1 < x < 1$

A. $(0, \infty)$

B. $(-1, 1)$

C. $(-\infty, -1)$

D. $(1, \infty)$

E. $(-\infty, \infty)$

2. $\lim\limits_{x \to 0} \dfrac{\sqrt{4+2x} - 2}{x} \overset{L'H}{=} \lim\limits_{x \to 0} \dfrac{\frac{1}{2\sqrt{4+2x}} \cdot 2}{1}$

$\dfrac{0}{0}$

$= \lim\limits_{x \to 0} \dfrac{1}{\sqrt{4+2x}} = \dfrac{1}{2}$

A. $\frac{1}{2}$

B. 0

C. 1

D. ∞

E. Does not exist

3. $\lim\limits_{x \to 1^-} \dfrac{\overbrace{3 - x}^{\to 2}}{\underbrace{(x-1)^3}_{\to 0^-}} = -\infty$

A. ∞

B. 2

C. -2

D. $-\infty$

E. Does not exist

4. If $f(x) = \dfrac{3x-1}{5x+2}$, find the inverse function $f^{-1}(x)$.

$y = f(x) \iff x = f^{-1}(y)$

$y = \dfrac{3x-1}{5x+2}$

$5yx + 2y = 3x - 1$

$(5y-3)x = -2y - 1$

$x = \dfrac{2y+1}{-5y+3}$

$\therefore f^{-1}(y) = \dfrac{2y+1}{-5y+3}$

$f^{-1}(x) = \dfrac{2x+1}{-5x+3}$

A. $f^{-1}(x) = \dfrac{5x+2}{3x-1}$

B. $f^{-1}(x) = \dfrac{3-x}{5+2x}$

C. $f^{-1}(x) = \dfrac{1-3x}{5x+2}$

D. $f^{-1}(x) = \dfrac{2x+1}{-5x+3}$

E. $f^{-1}(x) = \dfrac{3-5x}{2x+1}$

5. Find the values of a for which the function $f(x) = \begin{cases} x^2 + a^2, & \text{if } x < a \\ 2x - a, & \text{if } x \geq a \end{cases}$ is continuous for all x.

f is continuous for all $x \neq a$ (polynomial)

For f to be continuous at a: $\lim_{x \to a} f(x) = f(a)$

$\lim_{x \to a^-} f(x) = \lim_{x \to a^-} (x^2 + a^2) = 2a^2$

$\lim_{x \to a^+} f(x) = \lim_{x \to a^+} (2x - a) = a$

$\lim_{x \to a} f(x)$ exists if $2a^2 = a$

$a(2a-1) = 0$

$a = 0,\ a = \frac{1}{2}$

A. 0 only

B. 0 and 1

C. 0 and $\frac{1}{2}$

D. 1 only

E. 1 and $\frac{1}{2}$

6. Find an equation for the line tangent to the curve $y = \dfrac{4}{1-2x}$ at the point where $x = 0$.

$\dfrac{dy}{dx} = \dfrac{(1-2x)\cdot 0 - 4(-2)}{(1-2x)^2} = \dfrac{8}{(1-2x)^2}$

$\left.\dfrac{dy}{dx}\right|_{x=0} = \dfrac{8}{1} = 8$

$x = 0 \to y = 4$

$y - 4 = 8(x - 0)$

$8x - y + 4 = 0$

A. $2x - y + 2 = 0$

B. $4x - y + 4 = 0$

C. $4x + y - 4 = 0$

D. $x + 4y - 16 = 0$

E. $8x - y + 4 = 0$

7. Which of the following is a horizontal asymptote of the curve $y = \dfrac{\sqrt{2x^2+3}}{5x-2}$?

$$\lim_{x\to\infty} \frac{\sqrt{2x^2+3}}{5x-2} = \lim_{x\to\infty} \frac{\sqrt{\frac{2x^2+3}{x^2}}}{5-\frac{2}{x}}$$

$$= \lim_{x\to\infty} \frac{\sqrt{2+\frac{3}{x^2}}}{5-\frac{2}{x}} = \frac{\sqrt{2}}{5}$$

$(x>0 \to x=\sqrt{x^2})$

A. $x = \frac{2}{5}$

B. $x = 0$

C. $y = \frac{\sqrt{2}}{5}$

D. $y = 0$

E. $y = \frac{3}{2}$

8. Suppose that $F(x) = f(g(x))$ and $g(3) = 1$, $g'(3) = 2$, $f(3) = 2$, $f'(3) = 3$, $f(1) = 4$ and $f'(1) = 5$. Find $F'(3)$.

$F(x) = f(g(x))$

$F'(x) = f'(g(x))\, g'(x)$

$F'(3) = f'(g(3))\, g'(3)$

$= f'(1)\, g'(3)$

$= 5 \cdot 2 = 10$

A. 10

B. 8

C. 6

D. 12

E. 5

9. $\dfrac{d}{dx} x^{\sin x} =$

$\dfrac{d}{dx} x^{\sin x} = \dfrac{d}{dx} \left(e^{\ln x}\right)^{\sin x}$

$= \dfrac{d}{dx} e^{(\ln x)(\sin x)}$

$= e^{(\ln x)(\sin x)} \left[(\ln x)(\cos x) + (\sin x)\frac{1}{x}\right]$

$= x^{\sin x} \left[(\cos x)(\ln x) + \frac{\sin x}{x}\right]$

A. $x^{\sin x - 1}$

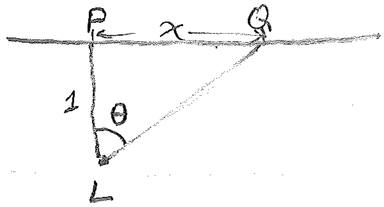
B. $(\ln x)x^{\sin x}$

C. $(\cos x)(\ln x) + \dfrac{\sin x}{x}$

D. $x^{\sin x}[(\cos x)(\ln x) + \dfrac{\sin x}{x}]$

E. $x^{\sin x} \cos x$

10. A lighthouse is located on a small island 1 mile away from the nearest point P on a straight shoreline and its light makes 3 revolutions per minute. How fast is the beam of light moving along the shoreline when it is 3 miles from P?



$\frac{d\theta}{dt} = 6\pi$

Find $\frac{dx}{dt}$ when $x=3$

$\tan\theta = x$

$\sec^2\theta \cdot \frac{d\theta}{dt} = \frac{dx}{dt}$

When $x=3$: $\tan^2\theta + 1 = \sec^2\theta$

$\therefore \sec^2\theta = 3^2 + 1 = 10$

$\frac{dx}{dt} = (10)(6\pi) = 60\pi$

A. 60π miles/min
B. 30π miles/min
C. 60 miles/min
D. 30 miles/min
E. $\frac{20}{3}\pi$ miles/min

11. Using a linear approximation of $f(x) = \sqrt[4]{x}$ at $a = 16$, we find the estimate of $\sqrt[4]{15.68}$ to be

$f(x) \approx f(a) + f'(a)(x-a)$, for x near a

$f(x) = \sqrt[4]{x}$ $\quad f(16) = 2$

$f'(x) = \frac{1}{4}\frac{1}{x^{3/4}}$ $\quad f'(16) = \frac{1}{4}\frac{1}{8} = \frac{1}{32}$

$\therefore \sqrt[4]{x} \approx 2 + \frac{1}{32}(x-16)$, for x near 16

$\sqrt[4]{15.68} \approx 2 + \frac{1}{32}(15.68 - 16)$

$= 2 + \frac{1}{32}(-0.32) = 2 - 0.01 = 1.99$

A. 1.9
B. 1.99
C. 2.01
D. 2
E. 1.96

12. $\lim_{x\to 1^-} [(\ln x)(\tan \frac{\pi x}{2})] = \lim_{x\to 1^-} \frac{\ln x}{\cot \frac{\pi x}{2}} =$

$0 \cdot \infty$ $\quad \frac{0}{0}$

$\overset{L'H}{=} \lim_{x\to 1^-} \frac{\frac{1}{x}}{(-\csc^2 \frac{\pi x}{2})(\frac{\pi}{2})} = -\frac{2}{\pi} \lim_{x\to 1^-} \sin^2 \frac{\pi x}{2}$

$= -\frac{2}{\pi} \cdot 1 = -\frac{2}{\pi}$

A. $\frac{\pi}{2}$
B. $-\frac{\pi}{2}$
C. $-\frac{2}{\pi}$
D. π
E. $-\pi$

13. The absolute maximum and absolute minimum values of the function $f(x) = -x^3 + 3x^2 - 1$ on the closed interval $[-2, 3]$ are:

$f'(x) = -3x^2 + 6x$

$f'(x) = 0$; $-3x^2 + 6x = 0$

$-3x(x-2) = 0$

$x = 0, 2$

$f(-2) = -(-8) + 3 \cdot 4 - 1 = 19 \leftarrow$ max

$f(0) = -1 \leftarrow$ min

$f(2) = -8 + 12 - 1 = 3$

$f(3) = -27 + 3 \cdot 9 - 1 = -1 \leftarrow$ min

A. abs. max=3, abs. min=-16

B. abs. max=3, abs. min=-1

C. abs. max=19, abs. min=-1

D. abs. max=19, abs. min=-16

E. abs. max=1, abs. min=-13

14. Suppose that the first derivative of $f(x)$ is

$$f'(x) = -(x+2)(x+1)^2x^3(x-1)^4$$

Then $f(x)$ attains a

A. local minimum at $x = -2$ and local maximum at $x = 0$

B. local minimum at $x = -1$ and local maximum at $x = 1$

C. local minimum at $x = -2$ and at $x = 0$

D. local maximum at $x = -2$ and at $x = 0$

E. It cannot be determined from the first derivative only.

$f'(x)$ - - - 0 + + + + 0 + + + + 0 - - - - 0 - - - - - -

x: -2, -1, 0, 1

-2 ↑ loc. min.

0 ↑ loc. max

15. Suppose that the second derivative of $f(x)$ is

$$f''(x) = (x+2)^5(x+1)^2(x-1)^3(x-2)^4$$

Then the number of inflection points of the graphs of $y = f(x)$ is

A. 1

B. 2

C. 3

D. 4

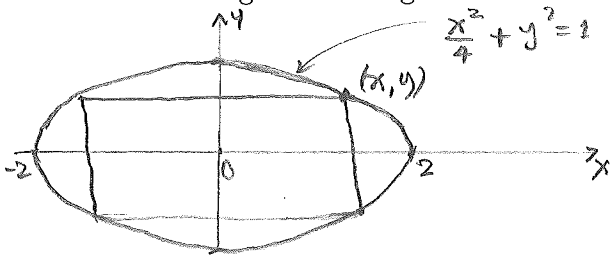
E. It cannot be determined from the second derivative only

$f''(x)$ + + + + 0 - - - - 0 - - - - - - 0 + + + + 0 + + + +

-2   -1   1   2

↑   ↑

16. Consider the ellipse $\frac{x^2}{4} + y^2 = 1$ and rectangles inscribed in the ellipse (with corners on the ellipse and sides parallel to the coordinate axes). Find the horizontal dimension of the rectangle with largest area.



A. 4

B. 2

C. 1

D. $2\sqrt{2}$

E. $4\sqrt{2}$

$A = 4xy$

$y = \sqrt{1 - \frac{x^2}{4}}$

$A = 4x\sqrt{1 - \frac{x^2}{4}}$ $0 \le x \le 2$

$\frac{dA}{dx} = 4x \frac{1}{2\sqrt{1-\frac{x^2}{4}}}\left(\frac{-2x}{4}\right) + 4\sqrt{1 - \frac{x^2}{4}} = \frac{-x^2 + 4\left(1 - \frac{x^2}{4}\right)}{\sqrt{1 - \frac{x^2}{4}}}$

$\frac{dA}{dx} = 0$: $-x^2 + 4 - x^2 = 0 \rightarrow x^2 = 2$

$x = \sqrt{2}$

hor. dim. $2\sqrt{2}$

17. A particle moves in a straight line and has acceleration given by $a(t) = 6t + 4$. Its initial velocity is $v(0) = -6$ and its initial position is $s(0) = 9$. Find its position function $s(t)$.

$a(t) = 6t + 4$

$v(t) = 3t^2 + 4t + C$

$t=0: \quad -6 = 0 + 0 + C \longrightarrow C = -6$

$v(t) = 3t^2 + 4t - 6$

$s(t) = t^3 + 2t^2 - 6t + D$

$t=0: \quad 9 = 0 + 0 - 0 + D \rightarrow D = 9$

$s(t) = t^3 + 2t^2 - 6t + 9$

A. $s(t) = 3t^2 + 4t - 6$

B. $s(t) = t^3 + 2t^2 - 6t$

C. $s(t) = t^3 + 2t^2 - 6t + 9$

D. $s(t) = 6t - 9$

E. $s(t) = t^3 + 6t^2 - 9t$

18. $\int_0^{10} |x - 5|dx = \int_0^5 (5-x)\,dx + \int_5^{10} (x-5)\,dx$

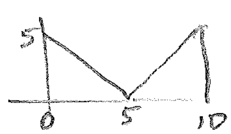
$= \left(5x - \frac{x^2}{2}\right)\Big|_0^5 + \left(\frac{x^2}{2} - 5x\right)\Big|_5^{10}$

$= 5\cdot 5 - \frac{25}{2} + \left(\frac{100}{2} - 5\cdot 10\right) - \left(\frac{25}{2} - 25\right)$

$= 25 - \frac{25}{2} + 50 - 50 - \frac{25}{2} + 25$

$= 50 - 25 = 25$

or



$A = 5\cdot 5 = 25$

A. 0

B. 15

C. 25

D. 45

E. 50

19. $\int_1^2 \frac{(x-1)^2}{x}\,dx = \int_1^2 \frac{x^2 - 2x + 1}{x}\,dx$

$= \int_1^2 \left(x - 2 + \frac{1}{x}\right)dx$

$= \left[\frac{x^2}{2} - 2x + \ln x\right]_1^2$

$= 2 - 4 + \ln 2 - \left(\frac{1}{2} - 2 + 0\right)$

$= -2 + \ln 2 + \frac{3}{2}$

$= \ln 2 - \frac{1}{2}$

A. $\ln 2 + \frac{1}{2}$

B. $\ln 2$

C. $2 \ln 2$

D. $\frac{1}{2}$

E. $\ln 2 - \frac{1}{2}$

20. $\dfrac{d}{dx}\displaystyle\int_x^0 \frac{1}{1+t^2}\,dt = -\frac{d}{dx}\int_0^x \frac{1}{1+t^2}\,dt$

$= -\dfrac{1}{1+x^2}$

(Fund. Th. of Calc.)

A. $-\dfrac{1}{1+x^2}$

B. $\dfrac{1}{1+x^2}$

C. $\tan^{-1} x$

D. $-\tan^{-1} x$

E. $\dfrac{2x}{(1+x^2)^2}$

21. $\displaystyle\int \frac{\cos\theta}{\sin^2\theta}\,d\theta =$

$= \displaystyle\int \cot\theta \csc\theta \, d\theta = -\csc\theta + C$

A. $\cot\theta\csc\theta + C$

B. $\ln|\sin\theta| + C$

C. $\ln|\sin\theta + \cos\theta| + C$

D. $-\csc\theta + C$

E. $\csc\theta + C$

22. $\displaystyle\int_1^e \frac{\ln x}{x}\,dx = \int_0^1 u\,du = \frac{u^2}{2}\Big|_0^1 = \frac{1}{2}$

$u = \ln x$

$du = \frac{1}{x}dx$

$x = 1 \rightarrow u = 0$

$x = e \rightarrow u = 1$

A. 0

B. $\dfrac{1}{2}$

C. 1

D. $\dfrac{3}{2}$

E. 2

23. A culture of a single cell creature Amoeba is found to triple its population in three weeks. Its relative growth rate k (in number/week) is

$P(t) = P(0)\, e^{kt}$

$t = 3$: $3P(0) = P(0)\, e^{k3}$

$3 = e^{k3}$

$\ln 3 = k3$

$k = \dfrac{\ln 3}{3}$

A. $\ln 3$

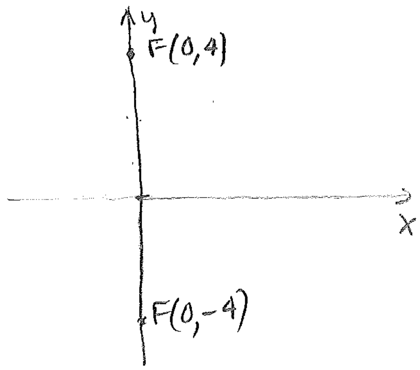
B. $\dfrac{1}{3}$

C. $\dfrac{1}{\ln 3}$

D. $\dfrac{3}{\ln 3}$

E. $\dfrac{\ln 3}{3}$

24. Find an equation of the hyperbola with foci $(0, \pm 4)$ and asymptotes $y = \pm 3x$



$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

Asymptotes: $y = \pm \frac{a}{b} x$

$c = 4$

$\frac{a}{b} = 3 \rightarrow a = 3b$

$c^2 = a^2 + b^2$

$4^2 = (3b)^2 + b^2$

$16 = 10b^2$

$b^2 = \frac{16}{10} = \frac{8}{5}$

$a^2 = 9b^2 = 9 \cdot \frac{8}{5} = \frac{72}{5}$

$$\frac{y^2}{\frac{72}{5}} - \frac{x^2}{\frac{8}{5}} = 1$$

$$\frac{5y^2}{72} - \frac{5x^2}{8} = 1$$

A. $\frac{5x^2}{72} - \frac{3y^2}{8} = 1$

B. $\frac{y^2}{70} - \frac{x^2}{6} = 1$

C. $\frac{y^2}{80} - \frac{x^2}{36} = 1$

D. $\frac{5y^2}{72} - \frac{5x^2}{8} = 1$

E. $\frac{3y^2}{76} - \frac{3x^2}{10} = 1$

25. Find the center of the ellipse $3x^2 + y^2 - 12x + 2y + 10 = 0$.

$3x^2 - 12x \quad + y^2 + 2y \quad = -10$

$3(x^2 - 4x + 4) + (y^2 + 2y + 1) = -10 + 12 + 1$

$3(x-2)^2 + (y+1)^2 = 3$

$(x-2)^2 + \frac{(y+1)^2}{3} = 1$

center $(2, -1)$

A. $(-1, 2)$

B. $(1, -2)$

C. $(2, -1)$

D. $(3, 1)$

E. $(-3, 1)$