NAME GRADING KEY

STUDENT ID ______

RECITATION INSTRUCTOR ______

RECITATION TIME ______

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DIRECTIONS

1. Write your name, student ID number, recitation instructor's name and recitation time in the space provided above. Also write your name at the top of pages 2, 3, 4 and 5.
2. The test has five (5) pages, including this one.
3. Write your answers in the boxes provided.
4. You must show sufficient work to justify all answers. Correct answers with inconsistent work may not be given credit.
5. Credit for each problem is given in parentheses in the left hand margin.
6. No books, notes or calculators may be used on this test.

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(4) 1. Find a unit vector having the same direction as the vector $\vec{a} = \vec{i} + \vec{j} - \vec{k}$.

$$\|\vec{a}\| = \sqrt{1^2 + 1^2 + (-1)^2} = \sqrt{3}$$

$$\vec{u} = \frac{\vec{a}}{\|\vec{a}\|} = \frac{1}{\sqrt{3}}(\vec{i} + \vec{j} - \vec{k})$$

In 1-7, if arrows indicating vectors are missing, put them in. If more than 2 arrows are missing in 1-7 take 1 pt off

NPC

$\boxed{\frac{1}{\sqrt{3}}(\vec{i} + \vec{j} - \vec{k})}$ 4

(6) 2. If $\vec{a} = 2\vec{i} - \vec{j} + 2\vec{k}$ and $\vec{b} = \vec{i}$, find $pr_{\vec{a}}\vec{b}$.

$$pr_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|^2}\vec{a} \quad 3$$

$$= \frac{2\cdot 1}{4+1+4}(2\vec{i} - \vec{j} + 2\vec{k})$$

$$= \frac{2}{9}(2\vec{i} - \vec{j} + 2\vec{k}) \quad 3$$

-1 pt for minor numerical error

$\boxed{\frac{2}{9}(2\vec{i} - \vec{j} + 2\vec{k})}$ 6

(7) 3. Find the area of the triangle with vertices at $(0,0,0)$, $(3,2,0)$, and $(2,6,0)$.

Let $P=(0,0,0)$, $Q=(3,2,0)$, $R=(2,6,0)$

Area of $\Delta = \frac{1}{2}\|\vec{PQ}\times\vec{PR}\|$ (3)

$\vec{PQ} = 3\vec{i} + 2\vec{j}$, $\vec{PR} = 2\vec{i}+6\vec{j}$

$\vec{PQ}\times\vec{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 2 & 0 \\ 2 & 6 & 0 \end{vmatrix} = 14\vec{k}$ (2)

$\|\vec{PQ}\times\vec{PR}\| = \sqrt{(14)^2} = 14$ (1)

(1)

$\boxed{\frac{1}{2}\cdot 14 = 7}$ 7

(4) 4. Find x so that the vectors

$\vec{a} = \vec{i} + 3\vec{j} - x\vec{k}$ and $\vec{b} = (1+x)\vec{i} + \vec{j} - \vec{k}$

are perpendicular.

$\vec{a}\cdot\vec{b} = 0$ (2)

$(1+x) + 3 + x = 0$

$x = -2$ (2)

$\boxed{-2}$ 4

(8) 5. The points $(1,0,0)$ and $(\frac{7}{3},\frac{2}{3},\frac{2}{3})$ lie on a sphere with center at $(a,-\frac{1}{3},-\frac{1}{3})$ and radius $\sqrt{2}$. Find a.

equation of sphere:

$(x-a)^2 + (y+\frac{1}{3})^2 + (z+\frac{1}{3})^2 = 2$ (4)

$(1-a)^2 + \frac{1}{9} + \frac{1}{9} = 2 \to (1-a)^2 = \frac{16}{9} \to 1-a = \pm\frac{4}{3}$

$a = -\frac{1}{3}$ or $a = \frac{7}{3}$

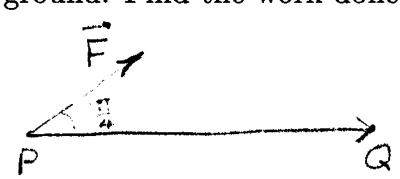
(4)

$(\frac{7}{3}-a)^2 + 1 + 1 = 2 \to (\frac{7}{3}-a)^2 = 0$

$\to a = \frac{7}{3}$

$\boxed{\frac{7}{3}}$ 8

(5) 6. A person pulls a sled 100 feet with a rope that makes an angle of $\frac{\pi}{4}$ with the horizontal ground. Find the work done on the sled if the tension in the rope is 5 pounds.



$W = \vec{F}\cdot\vec{PQ}$ (3)

or

$= \|\vec{F}\|\,\|\vec{PQ}\|\cos\frac{\pi}{4}$

$= 5\cdot 100\cdot\frac{1}{\sqrt{2}}$

(2) or $250\sqrt{2}$

$\boxed{\frac{500}{\sqrt{2}}}$ 5

(4) 7. (a) If $\vec{a} \times \vec{b} = \vec{i} - 2\vec{j} + \vec{k}$, then $\vec{b} \times \vec{a} =$ $\boxed{-\vec{i} + 2\vec{j} - \vec{k}}$ 2

NPC

(b) $\vec{i} \times (\vec{j} \times \vec{k}) =$ $\vec{i} \times \vec{i} = \vec{0}$ $\boxed{\vec{0}}$ 2

(14) 8. Find the following limits

(a) $\lim\limits_{x\to 0^+} \frac{x^2}{x - \sin x} \overset{L'H}{=} \lim\limits_{x\to 0^+} \frac{2x}{1-\cos x} = \lim\limits_{x \to 0^+} \frac{2}{\sin x} = \infty$ 3 2 1

$\frac{0}{0}$ $\frac{0}{0}$

-1pt for omitting lim after =

$\boxed{\infty}$ 6

(b) $\lim\limits_{x\to\infty} \left(1 - \frac{1}{x}\right)^x = \lim\limits_{x\to\infty} e^{x \ln(1-\frac{1}{x})} = \underbrace{e^{\lim\limits_{x\to\infty} x \ln(1-\frac{1}{x})}}_{3},$

1^∞

$\lim\limits_{x\to\infty} x \ln(1 - \tfrac{1}{x}) = \lim\limits_{x\to\infty} \frac{\ln(1-\frac{1}{x})}{\frac{1}{x}} \overset{L'H}{=} \lim\limits_{x\to\infty} \frac{\frac{1}{1-\frac{1}{x}}\left(\frac{1}{x^2}\right)}{-\frac{1}{x^2}}$ 1 2

$\infty \cdot 0$ $\frac{0}{0}$

$= \lim\limits_{x\to\infty} \left(-\frac{1}{1-\frac{1}{x}}\right) = -1$ 1

$\therefore \lim\limits_{x\to\infty} \left(1 - \frac{1}{x}\right)^x = e^{-1}$ 1

$\boxed{e^{-1}}$ 8

(6) 9. $\int_1^2 \ln x\, dx = \underbrace{x \ln x \Big|_1^2 - \int_1^2 x \frac{1}{x} dx}_{4}, = 2\ln 2 - 1$ 1 1

$u = \ln x$

$du = \frac{1}{x} dx$

$dv = dx$

$v = x$

-1 for omitting limits

Or $\int \ln x\, dx = x \ln x - \int x \frac{1}{x} dx$ 3

$= x \ln x - x + C$ 1

$\int_1^2 \ln x\, dx = (x \ln x - x)\Big|_1^2 = 2\ln 2 - 1$ 1 1

$\boxed{2\ln 2 - 1}$ 6

(24) 10. Evaluate the integrals:

(a) $\int x \sin x \, dx$ $= -x\cos x + \int \cos x \, dx$ $= -x\cos x + \sin x + C$

2 2 2

$u = x$ $dv = \sin x \, dx$

$du = dx$ $v = -\cos x$

-1 pt for missing $+C$, only one time for test.

-1 pt for missing dx or du, each time.

$\boxed{-x\cos x + \sin x + C}$ $\boxed{6}$

(b) $\int \tan^2 x \sec^4 x \, dx$ $= \int \tan^2 x \sec^2 x \sec^2 x \, dx$

$= \int \tan^2 x (1 + \tan^2 x) \sec^2 x \, dx$

4 $u = \tan x$ $du = \sec^2 x \, dx$

$= \int u^2 (1 + u^2) \, du = \frac{u^3}{3} + \frac{u^5}{5} + C$

$= \frac{1}{3}\tan^3 x + \frac{1}{5}\tan^5 x + C$ $\boxed{\frac{1}{3}\tan^3 x + \frac{1}{5}\tan^5 x + C}$ $\boxed{6}$

2

(c) $\int \cos^2 x \sin^3 x \, dx$ $= \int \cos^2 x (1 - \cos^2 x) \sin x \, dx$ 4

4 $u = \cos x$ $du = -\sin x \, dx$

$= -\int u^2 (1 - u^2) \, du = -\frac{u^3}{3} + \frac{u^5}{5} + C$

$= -\frac{1}{3}\cos^3 x + \frac{1}{5}\cos^5 x + C$

$\boxed{-\frac{1}{3}\cos^3 x + \frac{1}{5}\cos^5 x + C}$ $\boxed{6}$

2

(d) $\int \tan x \sec^3 x \, dx$ $= \int \sec^2 x \sec x \tan x \, dx$ 4

4 $u = \sec x$ $du = \sec x \tan x \, dx$

$= \int u^2 \, du = \frac{u^3}{3} + C$

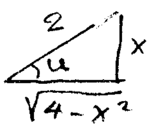
$= \frac{1}{3}\sec^3 x + C$

2

$\boxed{\frac{1}{3}\sec^3 x + C}$ $\boxed{6}$

(18) 11. Evaluate the integrals. PARTIAL CREDIT will not be given unless steps are clearly shown.

(a) $\int_0^2 \sqrt{4-x^2}\,dx$ $= \int_0^{\frac{\pi}{2}} \underbrace{2\cos u \, 2\cos u \, du}_{4}$ 1 1



$x = 2\sin u$

$\sqrt{4-x^2} = 2\cos u$

$dx = 2\cos u\,du$

$x=0 \to u=0$

$x=2 \to u=\frac{\pi}{2}$

$= 4\int_0^{\frac{\pi}{2}} \cos^2 u\,du$

$= 2\int_0^{\frac{\pi}{2}} (1+\cos 2u)\,du$

$= \underline{2\left(u + \frac{1}{2}\sin 2u\right)}\Big|_0^{\frac{\pi}{2}}$ 2

$= 2\left(\frac{\pi}{2}\right) = \pi$ 1

or $\int \sqrt{4-x^2}\,dx = \int 2\cos u\, 2\cos u\,du$ 4

$= 4\int \cos^2 u\,du = 2\int (1+\cos 2u)\,du$

$= 2\left(u + \frac{\sin 2u}{2}\right) = C$

2 $= 2u + 2\sin u\cos u + C$

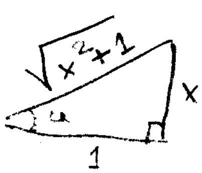
$= \underline{2\sin^{-1}\frac{x}{2} + 2\,\frac{x}{2}\,\frac{\sqrt{4-x^2}}{2}} + C$ 2

$\therefore \int_0^2 \sqrt{4-x^2}\,dx = 2\sin^{-1} 1$

$= 2\,\frac{\pi}{2} = \pi$

$\boxed{\pi}$ 9

(b) $\int \frac{\sqrt{x^2+1}}{x^4}\,dx$ $= \int \frac{\sec u}{\tan^4 u}\sec^2 u\,du$ 3



$x = \tan u$

$\sqrt{x^2+1} = \sec u$

$dx = \sec^2 u\,du$

$= \int \frac{1}{\frac{\sin^4 u}{\cos^4 u}}\,\frac{1}{\cos^3 u}\,du = \int \frac{\cos u}{\sin^4 u}\,du$ 2

$v = \sin u \quad dv = \cos u\,du$

$= \int \frac{dv}{v^4} = -\frac{1}{3v^3} + C$

$= -\frac{1}{3\sin^3 u} + C$ 2

$= -\frac{1}{3(x \quad)^3} + C$ 2

or $\frac{(x^2+1)^{3/2}}{-3x^3} + C$

$\boxed{-\frac{1}{3}\left(\frac{\sqrt{x^2+1}}{x}\right)^3 + C}$ 9