MA 166 EXAM 1 Spring 2000

NAME GRADING KEY

STUDENT ID __________

RECITATION INSTRUCTOR __________

RECITATION TIME __________

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DIRECTIONS

1. Write your name, student ID number, recitation instructor's name and recitation time in the space provided above. Also write your name at the top of pages 2, 3, 4 and 5.
2. The test has five (5) pages, including this one.
3. Write your answers in the boxes provided.
4. You must show sufficient work to justify all answers. Correct answers with inconsistent work may not be given credit.
5. Credit for each problem is given in parentheses in the left hand margin.
6. No books, notes or calculators may be used on this test.

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(5) 1. Let $\vec{a} = \vec{i} + \vec{j}$ and $\vec{b} = -\vec{i} + x\vec{j}$. Find x so that $\vec{a}$ is perpendicular to $\vec{a} - \vec{b}$.

$$\vec{a} - \vec{b} = 2\vec{i} + (1-x)\vec{j}$$

$$\vec{a} \cdot (\vec{a} - \vec{b}) = 0 \longrightarrow 2 + 1 - x = 0$$

$$x = 3$$

NPC

$x = 3$

$\boxed{5}$

(10) 2. If $P = (1,2,3)$, $Q = (-1,0,1)$ and $R = (1,1,0)$, find $pr_{\overrightarrow{PQ}}\overrightarrow{PR}$.

$$\overrightarrow{PQ} = -2\vec{i} - 2\vec{j} - 2\vec{k} \text{ ②}$$

$$\overrightarrow{PR} = -\vec{j} - 3\vec{k} \text{ ②}$$

$$pr_{\overrightarrow{PQ}}\overrightarrow{PR} = \frac{\overrightarrow{PQ} \cdot \overrightarrow{PR}}{\text{③ } \|\overrightarrow{PQ}\|^2}\overrightarrow{PQ} = \frac{2+6}{12}(-2\vec{i} - 2\vec{j} - 2\vec{k})$$

$$= -\frac{16}{12}(\vec{i} + \vec{j} + \vec{k}) \quad \text{③} \quad \text{or}$$

$$-\frac{4}{3}(\vec{i} + \vec{j} + \vec{k})$$

$\boxed{10}$

(5) 3. Let $\vec{a} = 3\vec{i} - 2\vec{j}$ and $\vec{b} = (s-t)\vec{i} + t\vec{j}$. Find s and t so that $\vec{a} + \vec{b} = \vec{i} + \vec{j}$.

$$\vec{a}+\vec{b} = (3+s-t)\vec{i} + (t-2)\vec{j}$$

$$\vec{a}+\vec{b} = \vec{i}+\vec{j} \longrightarrow 3+s-t=1 \ , \quad t-2=1$$

NPC

$\boxed{s = 1 \ , t = 3}$ $\boxed{5}$

(4) 4. Let $\vec{a}$ and $\vec{b}$ be unit vectors and let θ be the angle between $\vec{a}$ and $\vec{b}$. For what value of θ in $[0, \pi]$ is $\vec{a} \cdot \vec{b}$ maximum?

$$\vec{a} \cdot \vec{b} = \cos\theta$$

max of $\cos\theta = 1 \longrightarrow \theta = 0$

NPC

$\boxed{0}$ $\boxed{4}$

(10) 5. Let $\vec{a} = \vec{i} - 3\vec{j} + 2\vec{k}$ and $\vec{b} = -2\vec{i} + \vec{j} - 5\vec{k}$.

(a) Find a vector perpendicular to both $\vec{a}$ and $\vec{b}$.

$$\vec{a} \times \vec{b} = ③ \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -3 & 2 \\ -2 & 1 & -5 \end{vmatrix} = 13\vec{i} + \vec{j} - 5\vec{k}$$

③

$\boxed{13\vec{i} + \vec{j} - 5\vec{k}}$ $\boxed{6}$

(b) Find a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ and having positive $\vec{k}$ component.

$$\| 13\vec{i} + \vec{j} - 5\vec{k} \| = \sqrt{195}$$

ok if consistent with wrong answer in (a).

②

② $\longrightarrow$ $\boxed{\frac{1}{\sqrt{195}} \left(-13\vec{i} - \vec{j} + 5\vec{k}\right)}$ $\boxed{4}$

(10) 6. The points $Q = (1, 0, 0)$ and $R = (-1, 0, 0)$ lie on the unit sphere

$$x^2 + y^2 + z^2 = 1.$$

If $P = (x, y, z)$ is any other point on this sphere, prove that the vectors $\overrightarrow{QP}$ and $\overrightarrow{RP}$ are perpendicular.

$\overrightarrow{QP} = (x-1)\vec{i} + y\vec{j} + z\vec{k}$ ②

$\overrightarrow{RP} = (x+1)\vec{i} + y\vec{j} + z\vec{k}$ ②

$\overrightarrow{QP} \cdot \overrightarrow{RP} = (x^2-1) + y^2 + z^2$ ②

and since $x^2+y^2+z^2=1$ ② $\overrightarrow{QP} \cdot \overrightarrow{RP} = 0$ ②

$\therefore \overrightarrow{QP} \perp \overrightarrow{RP}$

$\boxed{10}$

(14) 7. Find the following limits:

(a) $$\underset{\frac{0}{0}}{\lim_{x\to0}\frac{\cos(3x)-1}{x^2}} \overset{L'H}{=} \overbrace{\lim_{x\to0}\underset{\frac{0}{0}}{\frac{-3\sin 3x}{2x}}}^{\text{③}} \overset{L'H}{=} \overbrace{\lim_{x\to0}\frac{-9\cos 3x}{2}}^{\text{②}} = -\frac{9}{2}$$ ①

-1 pt for missing lim (one time only for test)

$\boxed{-\frac{9}{2}}$ $\boxed{6}$

(b) $$\underset{1^\infty}{\lim_{x\to\infty}\left(1-\frac{1}{x}\right)^x} = \lim_{x\to\infty} e^{x\ln\left(1-\frac{1}{x}\right)} = \underbrace{e^{\lim_{x\to\infty} x\ln\left(1-\frac{1}{x}\right)}}_{\text{③}}$$

$$\lim_{x\to\infty} \underset{\infty\cdot0}{x\ln\left(1-\frac{1}{x}\right)} = \lim_{x\to\infty} \underset{\frac{0}{0}}{\frac{\ln\left(1-\frac{1}{x}\right)}{\frac{1}{x}}} \text{①} \overset{L'H}{=} \lim_{x\to\infty} \frac{\frac{1}{1-\frac{1}{x}}\left(\frac{1}{x^2}\right)}{-\frac{1}{x^2}} \text{②}$$

$$= \lim_{x\to\infty}\left(-\frac{1}{1-\frac{1}{x}}\right) = -1$$ ①

$\therefore \lim_{x\to\infty}\left(1-\frac{1}{x}\right)^x = e^{-1}$ ①

$\boxed{e^{-1}}$ $\boxed{8}$

-1 pt for missing $+C$ (one time only for test)
-1 pt for missing dx or du ″

(24) 8. Evaluate the integrals:

(a) $\int x\sin(3x)dx$ $$= \overbrace{-\frac{1}{3}x\cos(3x)}^{\textcircled{2}} + \overbrace{\frac{1}{3}\int\cos(3x)\,dx}^{\textcircled{2}} =$$

$u=x$ $dv=\sin(3x)dx$

$du=dx$ $v=-\frac{1}{3}\cos 3x$

$$= \underbrace{-\frac{1}{3}x\cos(3x)+\frac{1}{9}\sin(3x)+C}_{\textcircled{2}}$$

$$\boxed{-\frac{1}{3}x\cos(3x)+\frac{1}{9}\sin(3x)+C} \quad \boxed{6}$$

(b) $\int x\ln x dx$ $$= \overbrace{\frac{x^2}{2}\ln x}^{\textcircled{2}} \overbrace{-\frac{1}{2}\int x^2\frac{1}{x}\,dx}^{\textcircled{2}}$$

$u=\ln x$ $dv=x\,dx$

$du=\frac{1}{x}dx$ $v=\frac{x^2}{2}$

$$= \frac{x^2}{2}\ln x - \frac{1}{2}\int x\,dx$$

$$= \underbrace{\frac{x^2}{2}\ln x - \frac{x^2}{4} + C}_{\textcircled{2}}$$

$$\boxed{\frac{x^2}{2}\ln x - \frac{x^2}{4} + C} \quad \boxed{6}$$

(c) $\int \cos^3 x\sin^2 x dx$ $$= \overbrace{\int(1-\sin^2 x)\sin^2 x\cos x\,dx}^{\textcircled{4}} =$$

or ④ $u=\sin x$ $du=\cos x\,dx$

$$= \int(1-u^2)u^2\,du = \frac{u^3}{3}-\frac{u^5}{5}+C$$

$$= \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C$$

②

$$\boxed{\frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C} \quad \boxed{6}$$

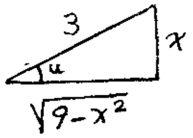
(d) $\int_0^{\frac{\pi}{4}} \cos^2 x dx$ $$= \int_0^{\frac{\pi}{4}} \frac{\overbrace{1+\cos 2x}^{\textcircled{2}}}{2}\,dx = \left(\overbrace{\frac{1}{2}x+\frac{1}{4}\sin 2x}^{\textcircled{2}}\right)\Big|_0^{\frac{\pi}{4}}$$

$$= \frac{\pi}{8} + \frac{1}{4}\sin\frac{\pi}{2} = \frac{\pi}{8}+\frac{1}{4} \quad \textcircled{2}$$

$$\boxed{\frac{\pi}{8}+\frac{1}{4}} \quad \boxed{6}$$

(18) 9. Evaluate the integrals. PARTIAL CREDIT will not be given unless steps are clearly shown.

(a) $\int \frac{1}{(9-x^2)^{\frac{3}{2}}}\,dx = \int \frac{1}{(\sqrt{9-x^2})^3}\,dx = \underbrace{\int \frac{1}{27\cos^3 u}\,3\cos u\,du}_{④} =$



$x = 3\sin u \quad dx = 3\cos u\,du$

$\sqrt{9-x^2} = 3\cos u$

$= \frac{1}{9}\int \sec^2 u\,du = \overbrace{\frac{1}{9}\tan u + C}^{③}$

$= \frac{1}{9}\,\frac{x}{\sqrt{9-x^2}} + C$ ②

$\int \frac{1}{\sqrt{1+x^2}}\,dx = \underbrace{\int \frac{1}{\sec u}\sec^2 u\,du}_{④} = \int \sec u\,du$

$= \ln|\sec u + \tan u| + C$ ①

$= \ln|\sqrt{1+x^2} + x| + C$ ②

$\int_1^{\sqrt{3}} \frac{1}{\sqrt{1+x^2}}\,dx = \ln|\sqrt{1+x^2} + x|\Big|_1^{\sqrt{3}}$

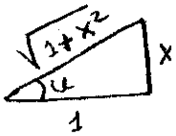
$= \ln|\sqrt{4} + \sqrt{3}| - \ln|\sqrt{2} + 1|$ ②

$\boxed{\frac{1}{9}\,\frac{x}{\sqrt{9-x^2}} + C}$ 9

or ↓

(b) $\int_1^{\sqrt{3}} \frac{1}{\sqrt{1+x^2}}\,dx$. [You may need: $\int \sec x\,dx = \ln|\sec x + \tan x| + C$].

$\int_1^{\sqrt{3}} \frac{1}{\sqrt{1+x^2}}\,dx = \int_{\pi/4\,①}^{\pi/3\,①} \overbrace{\frac{1}{\sec u}\sec^2 u\,du}^{④} =$



$x = \tan u \quad dx = \sec^2 u\,du$

$\sqrt{1+x^2} = \sec u$

$x = 1 \longrightarrow u = \frac{\pi}{4}$

$x = \sqrt{3} \longrightarrow u = \frac{\pi}{3}$

$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sec u\,du = \ln|\underset{①}{\sec u} + \tan u|\Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}}$

$= \ln|\sec\frac{\pi}{3} + \tan\frac{\pi}{3}| - \ln|\sec\frac{\pi}{4} + \tan\frac{\pi}{4}|$

$= \ln|2 + \sqrt{3}| - \ln|\sqrt{2} + 1|$

↑ or

$= \ln\frac{2+\sqrt{3}}{\sqrt{2}+1}$ ②

$\boxed{\ln\frac{2+\sqrt{3}}{\sqrt{2}+1}}$ 9