NAME GRADING KEY

STUDENT ID

RECITATION INSTRUCTOR

RECITATION TIME

Page 1	/16
Page 2	/25
Page 3	/34
Page 4	/25
TOTAL	/100

DIRECTIONS

1. Write your name, student ID number, recitation instructor's name and recitation time in the space provided above. Also write your name at the top of pages 2, 3, and 4.
2. The test has four (4) pages, including this one.
3. Write your answers in the boxes provided.
4. You must show sufficient work to justify all answers. Correct answers with inconsistent work may not be given credit.
5. Credit for each problem is given in parentheses in the left hand margin.
6. No books, notes or calculators may be used on this test.

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(8) 1. Find the center and radius of the sphere
$x^2 + y^2 + z^2 = 4x - 2y$

$x^2 - 4x + 4 + y^2 + 2y + 1 + z^2 = 4 + 1$

$(x-2)^2 + (y+1)^2 + z^2 = 5$

center $(2, -1, 0)$ 4

radius $\sqrt{5}$ 4

-2 pts if $\sqrt{\ }$ is missing

center: $(2, -1, 0)$

radius: $\sqrt{5}$

8

(8) 2. If $\vec{a} = 2\vec{i} - \vec{j} + 3\vec{k}$, find a vector $\vec{b}$ whose length is 8 and whose direction is opposite to that of $\vec{a}$.

$|\vec{a}| = \sqrt{4 + 1 + 9} = \sqrt{14}$ 2

$\vec{u} = \frac{\vec{a}}{|\vec{a}|} = \frac{1}{\sqrt{14}}(2\vec{i} - \vec{j} + 3\vec{k})$ 2

$\vec{b} = -8\vec{u} = -\frac{8}{\sqrt{14}}(2\vec{i} - \vec{j} + 3\vec{k})$ 2

2

or

$\vec{b} = -\frac{16}{\sqrt{14}}\vec{i} + \frac{8}{\sqrt{14}}\vec{j} - \frac{24}{\sqrt{14}}\vec{k}$

8

-1pt if more than 2 → are missing on this page

(15) 3. Consider the three points $P(0,0,0)$, $Q(1,1,0)$, and $R\left(\frac{1}{2},\frac{1}{2},z\right)$.

(a) Find $\overrightarrow{PQ}\times\overrightarrow{PR}$ (in terms of z).

$$\overrightarrow{PQ} = \vec{i}+\vec{j} \qquad \overrightarrow{PR} = \tfrac{1}{2}\vec{i}+\tfrac{1}{2}\vec{j}+z\vec{k}$$

$$\overrightarrow{PQ}\times\overrightarrow{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & z \end{vmatrix} = \vec{i}\begin{vmatrix} 1 & 0 \\ \frac{1}{2} & z \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & 0 \\ \frac{1}{2} & z \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix}$$

$$= z\vec{i} - z\vec{j}$$

$$\boxed{z\vec{i} - z\vec{j}}$$ 5

(b) Find the area of the triangle PQR (in terms of z).

$$\text{Area of } PQR = \frac{1}{2}\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right| \quad 4$$

$$= \frac{1}{2}\sqrt{z^2+z^2}$$

or 2

$$\boxed{\frac{1}{2}\sqrt{2z^2}}$$ 6

(c) Find all values of z for which the area of the triangle PQR is equal to 1.

$$\frac{1}{2}\sqrt{2z^2} = 1$$

$$\frac{\sqrt{2}}{2}\sqrt{z^2} = 1 \quad [\text{or } z^2 = 2 \to z = \pm\sqrt{2}]$$

$$|z| = \sqrt{2} \to z = \pm\sqrt{2}$$

2 2

$$\boxed{\sqrt{2}, \ -\sqrt{2}}$$ 4

(10) 4. Suppose that the angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{6}$ and that $|\vec{a}| = 2$ and $|\vec{b}| = 15$.

(a) Find $\vec{a}\cdot\vec{b}$

$$\vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|\cos\frac{\pi}{6} = 2\cdot 15\cdot\frac{\sqrt{3}}{2}$$

3

2

$$\boxed{15\sqrt{3}}$$ 5

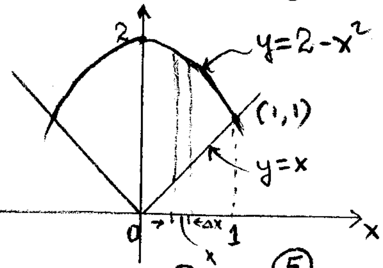
(b) Find $|\vec{a}\times\vec{b}|$ $= |\vec{a}|\,|\vec{b}|\sin\frac{\pi}{6} = 2\cdot 15\cdot\frac{1}{2}$

3

2

$$\boxed{15}$$ 5

(12) 5. Find the area of the region enclosed by the curves $y = |x|$ and $y = 2 - x^2$.



Note symmetry about y-axis

For $x > 0$, curves intersect when

In right half: $2 - x^2 = x \rightarrow x^2 + x - 2 = 0$

$(x+2)(x-1) = 0$

$\boxed{x = 1}$

Area of typical rectangle:

$$\Delta A = \left[(2 - x^2) - x\right]\Delta x$$

5

$$A = 2\int_0^1 (2 - x^2 - x)\,dx = 2\left(2x - \frac{x^3}{3} - \frac{x^2}{2}\right)\Big|_0^1 = 2\left(2 - \frac{1}{3} - \frac{1}{2}\right) = 2\cdot\frac{7}{6}$$

or $A = \int_{-1}^{1} (2 - x^2 - |x|)\,dx$

6

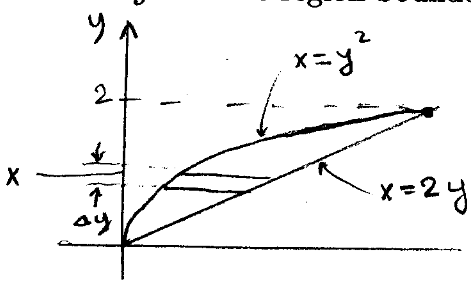
(*) 0 credit for problem if more than 1 item in integral is wrong. (limits count as 1 item in this rule)

3

$\boxed{\dfrac{7}{3}}$

$\boxed{12}$

(12) 6. Use the method of washers to find the volume of the solid obtained by rotating about the y-axis the region bounded by the curves $y^2 = x$ and $x = 2y$.



Curves intersect when $y^2 = 2y$

or $y(y-2) = 0 \rightarrow y = 0, 2$

Volume of typical washer:

$$\Delta V = \left[\pi(2y)^2 - \pi(y^2)^2\right]\Delta y$$

$$V = \int_0^2 \pi(4y^2 - y^4)\,dy \qquad (*)$$

6

$$V = \pi\left[4\frac{y^3}{3} - \frac{y^5}{5}\right]_0^2$$

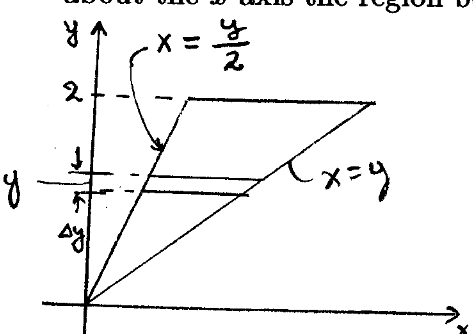
$$= \pi\left[4\cdot\frac{8}{3} - \frac{32}{5}\right] = 32\pi\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{64}{15}\pi$$

3

$\boxed{\dfrac{64}{15}\pi}$

$\boxed{12}$

(10) 7. Use the method of cylindrical shells to find the volume of the solid obtained by rotating about the x-axis the region bounded by the lines $y = x$, $y = 2x$ and $y = 2$.



Volume of typical cylindrical shell:

$$\Delta V = 2\pi y\left(y - \frac{y}{2}\right)\Delta y$$

$$V = \int_0^2 \pi y^2\,dy \qquad (*)$$

5

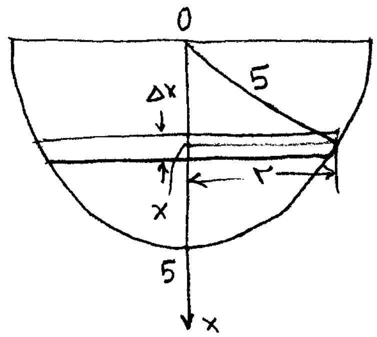
$$V = \pi\frac{y^3}{3}\Big|_0^2 = \frac{8\pi}{3}$$

2

$\boxed{\dfrac{8\pi}{3}}$

$\boxed{10}$

(12) 8. A tank in the shape of the bottom half of a sphere of radius 5 ft has an outlet at the top and is full of water. Set up an integral for the work W required to pump all the water out of the outlet. (Use the fact that water weighs 62.5 lbs/ft^3, and take the axis downwards with the origin at the center of the top of the tank). Do not evaluate the integral.



Volume of typical layer of water:
$= \pi r^2 \Delta x = \pi(25-x^2)\Delta x$

$x^2+r^2=5^2$

Weight of typical layer of water:
$=(62.5)\pi(25-x^2)\Delta x$

Work to lift typical layer to the top of the tank.

$\Delta W = x\, 62.5\, \pi(25-x^2)\Delta x$

0 credit for problem if more than 2 items are wrong (limits count as 1 item in this rule)

$$W = \int_0^5 x\,(62.5)\,\pi\,(25-x^2)\,dx$$

2 2 2 1 4 1

12

(5) 9. The average value of the function $f(x) = \cos x$ on the interval $[0, \pi]$ is

$$f_{ave} = \frac{1}{\pi - 0}\int_0^{\pi} \cos x\, dx = \frac{1}{\pi}\sin x\Big|_0^{\pi} = 0$$ 2

3

$\boxed{0}$

5

(8) 10. Evaluate the integral $\int xe^{-x}dx$.

$\int u\,dv = uv - \int v\,du$

$\int xe^{-x}dx = -xe^{-x} - \int(-e^{-x})dx = -xe^{-x} + \int e^{-x}dx$

$u = x \qquad dv = e^{-x}dx$

$= -xe^{-x} - e^{-x} + C$

$du = dx \qquad v = -e^{-x}$

-1 pt if $+C$ is missing

4 4

$\boxed{-xe^{-x} - e^{-x} + C}$

8