NAME GRADING KEY

10–DIGIT PUID ____________________

RECITATION INSTRUCTOR ____________________

RECITATION TIME ____________________

Page 1	/18
Page 2	/24
Page 3	/28
Page 4	/30
TOTAL	/100

DIRECTIONS

1. Write your name, 10–digit PUID, recitation instructor's name and recitation time in the space provided above. Also write your name at the top of pages 2, 3, and 4.
2. The test has four (4) pages, including this one.
3. Write your answers in the boxes provided.
4. You must show sufficient work to justify all answers unless otherwise stated in the problem. Correct answers with inconsistent work may not be given credit.
5. Credit for each problem is given in parentheses in the left hand margin.
6. No books, notes or calculators may be used on this test.

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(10) 1. Let $\vec{a}$, $\vec{b}$, $\vec{c}$ be three–dimensional vectors. For each statement below, circle T if the statement is always true, or F if it is not always true.

2 pts each

(i) $(2\vec{a})\cdot(3\vec{b}) = 6\vec{a}\cdot\vec{b}$ — (T) F

(ii) $(\vec{a}\cdot\vec{b})\vec{c} = (\vec{b}\cdot\vec{c})\vec{a}$ — T (F)

(iii) $(\vec{a}-\vec{b})\cdot(\vec{a}+\vec{b}) = |\vec{a}|^2 - |\vec{b}|^2$ — (T) F

(iv) $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\times\vec{c}$ — $\vec{i}\times(\vec{i}\times\vec{j}) = \vec{i}\times\vec{k} = -\vec{j}$, $(\vec{i}\times\vec{i})\times\vec{j} = \vec{0}\times\vec{j} = \vec{0}$ — T (F)

(v) If $\vec{a}\times\vec{b} = \vec{0}$ then $\vec{a}$ and $\vec{b}$ are parallel — (T) F

10

(4) 2. If $\vec{a} = 2\vec{i} - \vec{j} + 2\vec{k}$, find a unit vector in the direction opposite to $\vec{a}$.

NPC

$|\vec{a}| = \sqrt{4+1+4} = \sqrt{9} = 3$

Answer: $-\frac{1}{3}(2\vec{i} - \vec{j} + 2\vec{k})$

4

(4) 3. Find $\vec{a}\cdot\vec{b}$ if $|\vec{a}| = 12$, $|\vec{b}| = 15$ and the angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{6}$ radians.

$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta = 12\cdot 15\cos\frac{\pi}{6} = 12\cdot 15\cdot\frac{\sqrt{3}}{2} = 90\sqrt{3}$

-1 pt for $180\cos\frac{\pi}{6}$

Answer: $\vec{a}\cdot\vec{b} = 90\sqrt{3}$

4

(6) 4. Find the values of t for which the vectors $\langle 3t, -t, -3\rangle$ and $\langle -1, t^2, -4t\rangle$ are orthogonal.

$\langle 3t, -t, -3\rangle \cdot \langle -1, t^2, -4t\rangle = 0$ ③

$-3t - t^3 + 12t = 0$

$9t - t^3 = 0 \rightarrow t(9-t^2) = 0$

③ $t = 0, 3, -3$

[6]

(4) 5. A constant force $\vec{F} = 3\vec{i} + 5\vec{j} + 10\vec{k}$ moves an object along the line segment from $(1,0,2)$ to $(5,3,8)$. Find the work done if the distance is measured in meters and the force in newtons.

Displacement vector $\vec{D} = 4\vec{i} + 3\vec{j} + 6\vec{k}$ ②

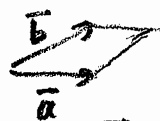
$W = \vec{F}\cdot\vec{D} = (3\vec{i} + 5\vec{j} + 10\vec{k})\cdot(4\vec{i} + 3\vec{j} + 6\vec{k})$

$= 12 + 15 + 60$

② $W = 87$ J

[4]

(6) 6. Find the area of the parallelogram determined by the vectors $\vec{a} = \vec{i} + 5\vec{j} + \vec{k}$ and $\vec{b} = -2\vec{i} + \vec{j} + 3\vec{k}$.

Area of  $= |\vec{a}\times\vec{b}|$

$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 5 & 1 \\ -2 & 1 & 3 \end{vmatrix} = 14\vec{i} - 5\vec{j} + 11\vec{k}$ ③

② Grade consistently with ①

$|\vec{a}\times\vec{b}| = \sqrt{14^2 + (-5)^2 + (11)^2}$

$= \sqrt{196 + 25 + 121} = \sqrt{342}$

$\sqrt{342}$

[6]

(8) 7. Find the center and radius of the sphere

$$x^2 + y^2 + z^2 + 2x - 10y = -1.$$

$x^2 + 2x \quad + y^2 - 10y \quad + z^2 = -1$

$x^2 + 2x + 1 + y^2 - 10y + 25 + z^2 = -1 + 1 + 25$

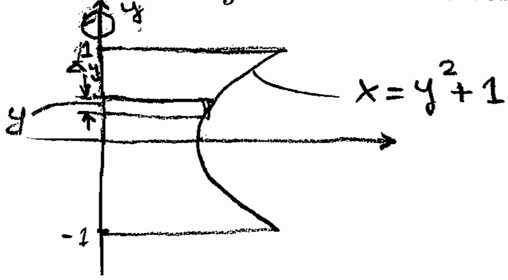
$(x+1)^2 + (y-5)^2 + z^2 = 25$

center: $(-1, 5, 0)$ ④

radius: 5 ④

[8]

(10) 8. The region bounded by the curves $x = y^2 + 1$, $y = -1$, $y = 1$, and $x = 0$ is rotated about the y–axis. Find the volume of the resulting solid.



Volume of typical approximating disk

$$\Delta V = \pi (y^2+1)^2 \Delta y$$

$$V = \int_{-1}^{1} \pi (y^2+1)^2 \, dy$$

(1) (1) 5 (1)

Rule*: 0 pts for problem if more than 1 item is wrong. Limits count as 1 item in this rule.

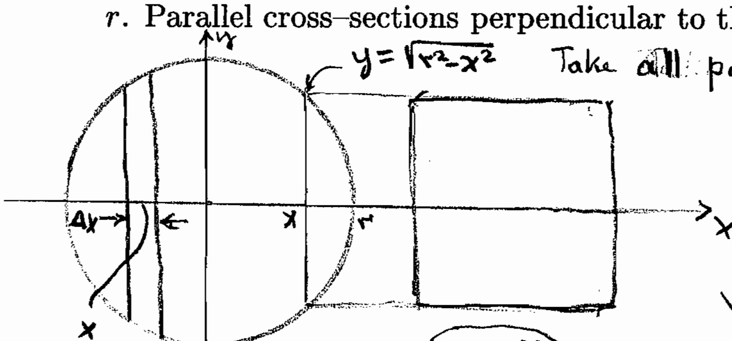
$$V = 2\pi \int_0^1 (y^4 + 2y^2 + 1)\, dy$$

$$= 2\pi \left[\frac{y^5}{5} + \frac{2y^3}{3} + y \right]_0^1 = 2\pi \left[\frac{1}{5} + \frac{2}{3} + 1 \right]$$

$$= 2\pi \, \frac{3+10+15}{15} = \boxed{\frac{56\pi}{15}}$$ (2)

10

(10) 9. Find the volume of the following solid S : The base of S is a circular disk with radius r. Parallel cross–sections perpendicular to the base are squares.



Take all parallel cross-sections perpendicular to the x-axis

Area of cross-section at x

$$A(x) = (2\sqrt{r^2 - x^2})^2$$

Volume of typical slice:

$$\Delta V = 4(r^2 - x^2)\Delta x$$

(1) (5) Rule* (1) (1)

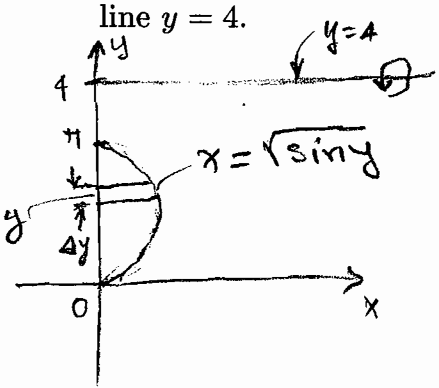
$$V = \int_{-r}^{r} 4(r^2 - x^2)\, dx = 8\int_0^r (r^2 - x^2)\, dx$$

$$= 8\left[r^2 x - \frac{x^3}{3} \right]_0^r = 8\left(r^3 - \frac{r^3}{3} \right) = \frac{16r^3}{3}$$

$$\boxed{\frac{16r^3}{3}}$$ (2)

10

(8) 10. Set up, but do not evaluate, an integral for the volume of the solid obtained by rotating the region bounded by the curves $x = \sqrt{\sin y}$ with $0 \le y \le \pi$, and $x = 0$, about the line $y = 4$.



Volume of typical shell

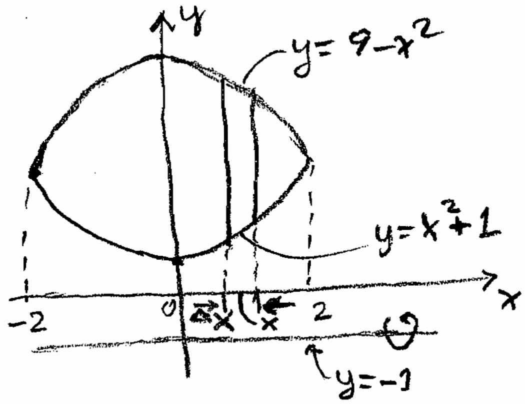
$$\Delta V = 2\pi(4-y)\sqrt{\sin y}\, \Delta y$$

Rule* (1) (5) (1) (1)

$$\boxed{V = \int_0^{\pi} 2\pi(4-y)\sqrt{\sin y}\, dy}$$

8

(8) 11. Set up, but do not evaluate, an integral for the volume of the solid obtained by rotating the region bounded by the curves $y = x^2 + 1$, $y = 9 - x^2$ about the line $y = -1$.



$x^2 + 1 = 9 - x^2 \rightarrow 2x^2 = 8 \rightarrow x^2 = 4 \rightarrow x = \pm 2$

Volume of typical washer

$\Delta V = \left[\pi\left(9 - x^2 - (-1)\right)^2 - \pi\left(x^2 + 1 - (-1)\right)^2\right]\Delta x$

Rule*

$V = \int_{-2}^{2} \left[\pi(10 - x^2)^2 - \pi(x^2+2)^2\right] dx$ (1) (5) (1) (1) [8]

(6) 12. The natural length of a spring is 1m and a force of 10N is required to hold the spring stretched to a total length of 2m. How much work is done in stretching the spring from its natural length to a length of 1.5m?

$F = kx \longrightarrow 10 = k1 \longrightarrow k = 10$ (2)

$W = \int_0^{0.5} 10x\,dx = 5x^2 \Big|_0^{\frac{1}{2}} = \frac{5}{4}$

(4)

$W = \frac{5}{4}$ J [6]

(16) 13. Evaluate the integrals.

(a) $\int \tan^{-1} x\,dx$ $= x\tan^{-1}x - \int \frac{x}{x^2+1}\,dx$ (2) (2)

$u = \tan^{-1}x \quad dv = dx$

$du = \frac{1}{x^2+1}dx \quad v = x$

$= x\tan^{-1}x - \frac{1}{2}\ln(x^2+1) + C$

(1) (3) (1)

$x\tan^{-1}x - \frac{1}{2}\ln(x^2+1) + C$ [9]

(b) $\int_1^2 \frac{\ln x}{x^2}\,dx$ $= -\frac{1}{x}\ln x \Big|_1^2 - \int_1^2 \left(-\frac{1}{x}\right)\frac{1}{x}\,dx$ (2) (2)

$u = \ln x \quad dv = \frac{1}{x^2}dx$

$du = \frac{1}{x}dx \quad v = -\frac{1}{x}$

$= -\frac{1}{2}\ln 2 + \int_1^2 \frac{1}{x^2}\,dx$

$= -\frac{1}{2}\ln 2 + \left[-\frac{1}{x}\right]_1^2$

$= -\frac{1}{2}\ln 2 - \frac{1}{2} + 1$

$= -\frac{1}{2}\ln 2 + \frac{1}{2}$

(3)

$\frac{1 - \ln 2}{2}$ [7]