MA166 — EXAM I — SPRING 2019
SOLUTIONS

1. Let $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ be vectors such that $|\vec{\mathbf{u}}| = 5$, $|\vec{\mathbf{v}}| = 10$ and the dot product $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = 35$. Which of the alternatives gives the closest approximation for the angle between the vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$? You may need to use that $\sqrt{2} = 1.41...$ and $\sqrt{3} = 1.73...$

A. 0

B. $\boxed{\dfrac{\pi}{4}}$

C. $\dfrac{\pi}{3}$

D. $\dfrac{\pi}{6}$

E. $\dfrac{\pi}{2}$

Solution: If θ is the angle between the vectors, $\cos\theta = \dfrac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|}$. So we conclude that $\cos\theta = \dfrac{35}{50} = 0.7$. Since $\cos\dfrac{\pi}{4} = \dfrac{\sqrt{2}}{2} = \dfrac{1.41..}{2} = 0.705..$, and in $[0, \pi]$, $\cos\theta$ is a continuous invertible function, θ is closest to $\dfrac{\pi}{4}$.

2. Find the center and the radius of the sphere $6x^2 + 6y^2 + 6z^2 - 24x - 12y + 36z = 0$.

A. Center $(1, -2, -3)$ and radius 2

B. Center $(2, 1, 3)$ and radius $\sqrt{14}$

C. $\boxed{\text{Center } (2, 1, -3) \text{ and radius } \sqrt{14}}$

D. Center $(-2, -1, 3)$ and radius $\sqrt{14}$

E. Center $(2, -1, 3)$ and radius 3

Solution: We divide the given equation of the sphere by six, and we obtain $x^2 + y^2 + z^2 - 4x - 2y + 6z = 0$. We complete the squares and find that this equation can be rewritten as $(x - 2)^2 + (y - 1)^2 + (z + 3)^2 = 14$. So the center of the sphere is $(2, 1, -3)$ and its radius is equal to $\sqrt{14}$.

3. Find the maximum volume of the parallelepiped formed by the vectors $\vec{u} = \langle 1, 2, 3\rangle$ $\vec{v} = \langle 1, 4, 5\rangle$ and $\vec{w} = \langle 1, a, 2\rangle$, for a in the interval $[-1, 1]$.

A. 2

B. $\boxed{4}$

C. 5

D. 7

E. 8

<u>**Solution:**</u> The volume of the parallelepiped is given by the absolute value of the determinant

$$\left|\begin{vmatrix} 1 & 2 & 3 \\ 1 & 4 & 5 \\ 1 & a & 2 \end{vmatrix}\right| = |-2a + 2|$$

The maximum of this quantity in the interval $[-1, 1]$ is achieved when $a = -1$ and $|-2a+2| = 4$.

4. The sides of a triangle are given by the vectors $\vec{u} = \langle -1, 2, 1\rangle$, $\vec{v} = \langle 1, 4, 3\rangle$ and $\vec{u} - \vec{v}$. Find the area of the triangle.

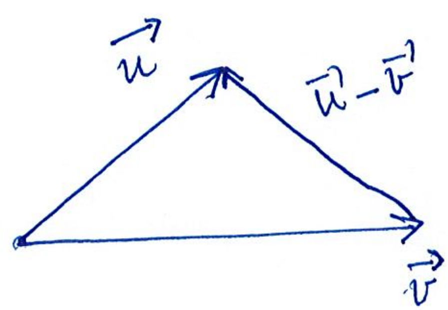
A. $\sqrt{13}$

B. $\sqrt{38}$

C. $\dfrac{\sqrt{54}}{2}$

D. $\dfrac{\sqrt{15}}{2}$

E. $\boxed{\sqrt{14}}$



<u>**Solution:**</u> The area of the triangle is given by $\dfrac{1}{2}|\vec{u} \times \vec{v}|$. But

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 2 & 1 \\ 1 & 4 & 3 \end{vmatrix} = 2\vec{i} + 4\vec{j} - 6\vec{k}.$$

So the area of the triangle is equal to $\dfrac{1}{2}\sqrt{4 + 16 + 36} = \dfrac{1}{2}\sqrt{56} = \sqrt{14}$.

5. Find the area of the region enclosed by the curves $y = x^3$ and $y = -x^3$, $y = 1$ and $y = -1$.

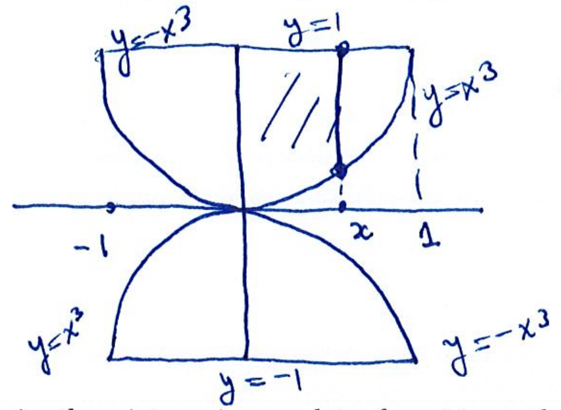
A. 3

B. $\frac{2}{3}$

C. $\frac{5}{3}$

D. 2

E. $\frac{8}{3}$



Solution: The area of the figure shown in the picture is equal to four times the area of the region bounded by the curves $y = x^3$, $x = 0$ and $y = 1$. Therefore the area of the region is given by $4\int_0^1 (1 - x^3)dx = 4(1 - \frac{1}{4}) = 3$.

6. Find the area of the region enclosed by the curves $x = 1 + y^2$ and $x = 2$.

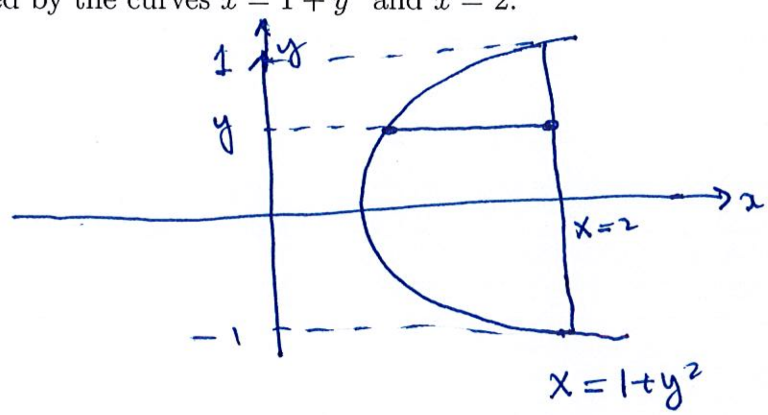
A. 3

B. $\frac{2}{3}$

C. $\boxed{\frac{4}{3}}$

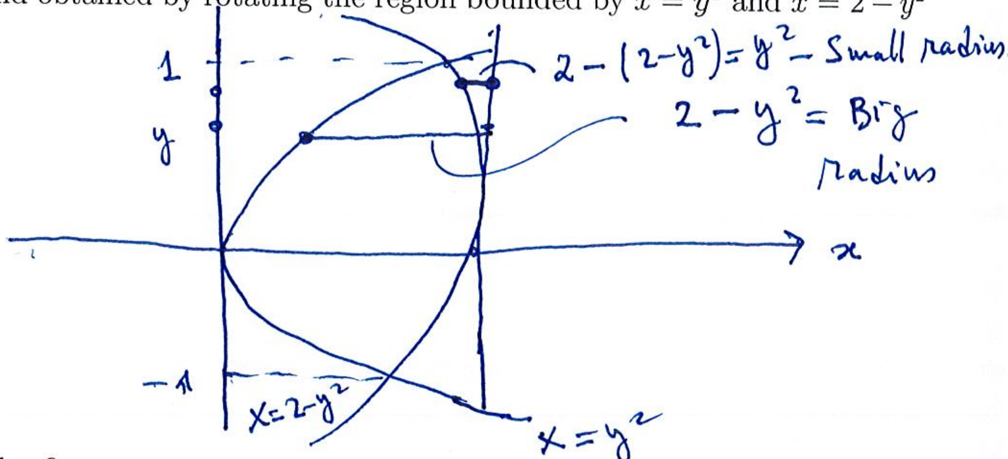
D. 4

E. $\frac{5}{3}$



Solution: The curves $x = 2$ and $x = 1 + y^2$ intersect when $1 + y^2 = 2$ and therefore $y^2 = 1$ so $y = \pm 1$. See the picture above, the area of the regions is given by $\int_{-1}^1 [2 - (1 + y^2)]\, dy =$ $\int_{-1}^1 (1 - y^2)\, dy = 2\int_0^1 (1 - y^2)dy = 2(1 - \frac{1}{3}) = \frac{4}{3}$.

7. Find the volume of the solid obtained by rotating the region bounded by $x = y^2$ and $x = 2 - y^2$ about $x = 2$.



A. 4π

B. $\frac{8}{3}\pi$

C. $\boxed{\frac{16}{3}\pi}$

D. $\frac{20}{3}\pi$

E. 10π

Solution: According to the figure, it is more convenient to use the washers method. Fixed y we see, according to the picture, that the larger radius is $2 - y^2$ and the shorter radius is $2 - (2 - y^2) = y^2$. The curves intersect when $2 - y^2 = y^2$, and so $y = \pm 1$, see figure above. The volume then is given by the integral

$$\pi \int_{-1}^{1} [(2 - y^2)^2 - (y^2)^2]dy = \pi \int_{-1}^{1} [4 - 4y^2]dy =$$

$$4\pi \int_{-1}^{1} (1 - y^2)dy = 8\pi \int_{0}^{1} (1 - y^2)dy = 8\pi(1 - \frac{1}{3}) = \frac{16\pi}{3}.$$

8. Evaluate the integral $\int_0^{\frac{\pi}{4}} x^2 \cos(2x)dx$.

A. $\frac{\pi^2}{32} - \frac{1}{8}$

B. $\boxed{\frac{\pi^2}{32} - \frac{1}{4}}$

C. $\frac{\pi^2}{32} + \frac{1}{4}$

D. $\frac{\pi^2}{32} - 1$

E. $\frac{\pi^2}{32} - 2$

Solution: This can be solved by integration by parts. We begin by setting $u = x^2$ and $dv = \cos 2x dx$, so $du = 2x dx$ and $v = \frac{1}{2}\sin 2x$, and we conclude that

$$\int x^2 \cos 2x dx = \frac{1}{2}x^2 \sin 2x - \int x \sin 2x dx.$$

We integrate the latter integral by parts by setting $u = x$ and $dv = \sin 2x dx$. So $du = dx$ and $v = -\frac{1}{2}\cos 2x$. So we find that

$$\int x \sin 2x dx = -\frac{1}{2}x\cos 2x + \frac{1}{2}\int \cos 2x dx = -\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x + C.$$

We conclude that

$$\int_0^{\frac{\pi}{4}} x^2 \cos 2x dx = \frac{1}{2}x^2 \sin 2x + \frac{1}{2}x\cos 2x - \frac{1}{4}\sin 2x\Big|_0^{\frac{\pi}{4}} = \frac{\pi^2}{32} - \frac{1}{4}.$$

9. Find the volume of the solid generated by rotating the region bounded by $x = 2y^2, x = 1 + y^2$ about $y = 1$.

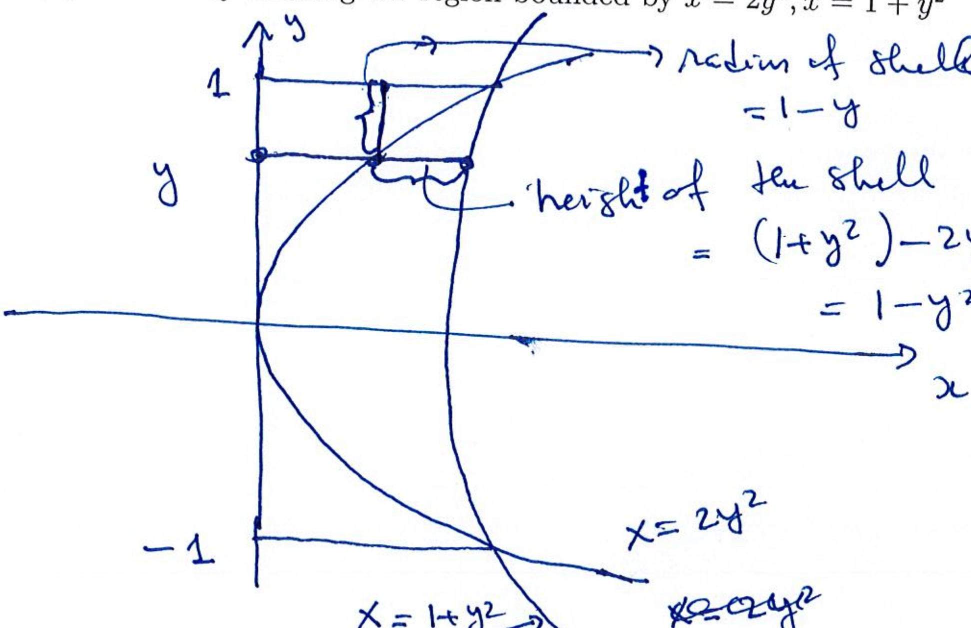
A. $\boxed{\frac{8}{3}\pi}$

B. $\frac{4}{3}\pi$

C. $\frac{2}{3}\pi$

D. $\frac{5}{3}\pi$

E. 2π



__Solution:__ In this case it is better to use the method of cylindrical shells, see the figure above. The curves $x = 2y^2$ and $x = 1 + y^2$ intersect when $2y^2 = 1 + y^2$ and so $y^2 = 1$ or $y = \pm 1$. Since the region is rotated about the axis $y = 1$, the radius of a shell is $1 - y$ and the height of the shell is $1 + y^2 - 2y^2 = 1 - y^2$, see figure above. Then the volume is equal to

$$2\pi \int_{-1}^{1} (1-y)(1-y^2)dy = 2\pi \int_{-1}^{1} (1-y^2)dy - 2\pi \int_{-1}^{1} y(1-y^2)dy.$$

The latter integral is equal to zero, since the function $y(1 - y^2)$ is odd. So the volume is equal to

$$2\pi \int_{-1}^{1} (1-y)(1-y^2)dy = 2\pi \int_{-1}^{1} (1-y^2)dy = 4\pi \int_{0}^{1} (1-y^2)dy = 4\pi(1 - \frac{1}{3}) = \frac{8\pi}{3}.$$

10. An aquarium has the shape of a box with all sides perpendicular to each other. Its sides measure 10 m long, 2 m wide, and 1 m deep and the aquarium is full of water. Find the work in Joules needed to pump all the water out of the aquarium. (Use the fact that the density of water is 1000 kg/m^3 and denote the acceleration of gravity by g.)

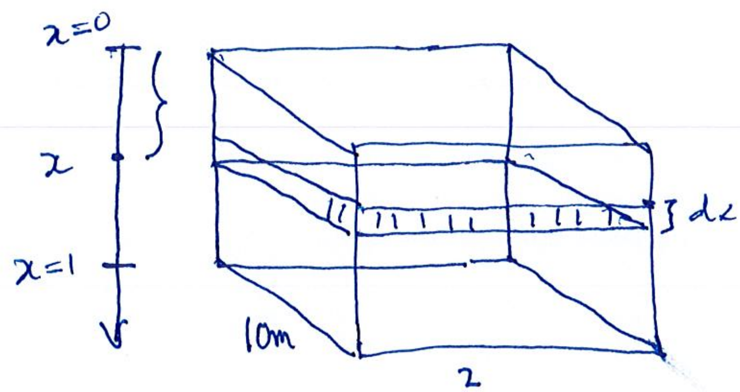
A. $\boxed{10000g}$

B. $1000g$

C. $500g$

D. $2500g$

E. $5000g$



Solution: As one starts by choosing a axis that measures height along the takn. See the figure above for the choice of x. As usual one fitst computes the weight of cross section of the tank with height dx. The volume of such a slab is equal to $20dx$, since the aquarium is 10 m long and 2 m wide. The weight of liquid that occupies this slab is equal to $1000 \times thevolume \times g = 20000gdx$. The work necessary to move this slab to the top of the tank is equal to $20000gxdx$. The total work necessary to empty the tank is equal to $\int_1^2 20000gxdx = 10000g$.

11. Evaluate the integral $\int_0^{\frac{\pi}{4}} \tan^5(x)\sec^4(x)dx$.

A. $\boxed{\frac{7}{24}}$

B. $\frac{15}{4}$

C. $\frac{21}{5}$

D. $\frac{15}{24}$

E. $\frac{17}{8}$

Solution: We start by writing

$$\int_0^{\frac{\pi}{4}} \tan^5(x)\sec^4(x)dx = \int_0^{\frac{\pi}{4}} \tan^5(x)\sec^2(x)\sec^2(x)dx.$$

Then we use that $\sec^2 x = 1 + \tan^2 x$ and we find that

$$\int_0^{\frac{\pi}{4}} \tan^5(x)\sec^4(x)dx = \int_0^{\frac{\pi}{4}} \tan^5(x)\sec^2(x)\sec^2(x)dx =$$

$$\int_0^{\frac{\pi}{4}} \tan^5(x)(1+\tan^2(x))\sec^2(x)dx.$$

Now we set $u = \tan x$, and so $du = \sec^2(x)dx$ and the latter integral is equal to

$$\int_0^1 u^5(1+u^2)du = \int_0^1 (u^5+u^7)du = \frac{1}{6} + \frac{1}{8} = \frac{7}{24}.$$

MA166 — EXAM I — SPRING 2019 — FEBRUARY 5, 2019
TEST NUMBER 22

INSTRUCTIONS:

1. **Do not open the exam booklet until you are instructed to do so.**
2. **Before you open the booklet fill in the information below and use a # 2 pencil to fill in the required information on the scantron.**
3. **MARK YOUR TEST NUMBER ON YOUR SCANTRON**
4. Once you are allowed to open the exam, make sure you have a complete test. There are 7 different test pages (including this cover page).
5. Do any necessary work for each problem on the space provided or on the back of the pages of this test booklet. Circle your answers on this test booklet.
6. The exam has 11 problems and each one is worth 9 points and everyone gets one point. The maximum possible score is 100 points. No partial credit.
7. Do not leave the exam room during the first 20 minutes of the exam.
8. If you do not finish your exam in the first 40 minutes, you must wait until the end of the exam period to leave the room.
9. After you have finished the exam, hand in your scantron and your test booklet to your recitation instructor.

DON'T BE A CHEATER:

1. Do not give, seek or obtain any kind of help from anyone to answer questions on this exam. If you have doubts, consult only your instructor.
2. Do not look at the exam or scantron of another student.
3. Do not allow other students to look at your exam or your scantron.
4. You may not compare answers with anyone else or consult another student until after you have finished your exam, given it to your instructor and left the room.
5. Do not consult notes or books.
6. **Do not handle** phones or cameras, calculators or any electronic device until after you have finished your exam, given it to your instructor and left the room.
7. After time is called, the students have to put down all writing instruments and remain in their seats, while the TAs collect the scantrons and the exams.
8. Anyone who violates these instructions will have committed an act of academic dishonesty. Penalties for academic dishonesty include an F in the course. All cases of academic dishonesty will be reported to the Office of the Dean of Students.

I have read and understand the above statements regarding academic dishonesty:

STUDENT NAME: SOLUTIONS

STUDENT SIGNATURE: ______________________

STUDENT ID NUMBER: ______________________

SECTION NUMBER AND RECITATION INSTRUCTOR: ______________________

1. Let $\vec{u}$ and $\vec{v}$ be vectors such that $|\vec{u}| = 5$, $|\vec{v}| = 10$ and the dot product $\vec{u} \cdot \vec{v} = 40$. Which of the alternatives gives the closest approximation for the angle between the vectors $\vec{u}$ and $\vec{v}$? You may need to use that $\sqrt{2} = 1.41...$ and $\sqrt{3} = 1.73...$

A. 0

B. $\frac{\pi}{4}$

C. $\frac{\pi}{3}$

D. $\boxed{\frac{\pi}{6}}$

E. $\frac{\pi}{2}$

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}| \cdot |\vec{v}|} = \frac{40}{50} = \frac{4}{5} = 0.8$$

$\cos\theta = 0.8$ $\quad \cos\pi/6 = \frac{\sqrt{3}}{2} = \frac{1.73...}{2} = 0.86...$

$\cos 0 = 1, \quad \cos\pi/4 = \frac{\sqrt{2}}{2} = 0.705...$

$\cos\pi/3 = 1/2 = 0.5$

$\cos\pi/2 = 0$.

Since in $[0, \pi]$ $\cos\theta$ is continuous and invertible θ is closest to $\pi/6$

2. Find the center and the radius of the sphere $5x^2 + 5y^2 + 5z^2 - 20x - 10y + 20z = 0$.

A. Center $(1, -2, -3)$ and radius 2

B. Center $(2, 1, 3)$ and radius $\sqrt{14}$

C. Center $(2, 1, -3)$ and radius 2

D. Center $(-2, 1, -2)$ and radius 3

E. $\boxed{\text{Center } (2, 1, -2) \text{ and radius } 3}$

Divide the equation by 5.

$x^2 + y^2 + z^2 - 4x - 2y + 4z = 0$

$(x-2)^2 - 4 + (y-1)^2 - 1 + (z+2)^2 - 4 = 0$

$(x-2)^2 + (y-1)^2 + (z+2)^2 = 9$

Center $(2, 1, -2)$; Radius 3

3. Find the maximum volume of the parallelepiped formed by the vectors $\vec{u} = \langle 1,2,3 \rangle$ $\vec{v} = \langle 1,4,5 \rangle$ and $\vec{w} = \langle 1,2,a \rangle$, for a in the interval $[-1,1]$.

A. 2

B. 4

C. 5

D. 7

$\boxed{\text{E. } 8}$

$$\text{The volume} = \left| \begin{vmatrix} 1 & 2 & 3 \\ 1 & 4 & 5 \\ 1 & 2 & a \end{vmatrix} \right| =$$

$$|(4a-10) - 2(a-5) + 3(-2)|$$

$= |2a-6|$ The maximum of this quantity for a in $[-1,1]$ is achieved when $a=-1$

$\text{Volume} = |-2-6| = |-8| = 8.$

4. The sides of a triangle are given by the vectors $\vec{u} = \langle -1,2,1 \rangle$, $\vec{v} = \langle 1,4,1 \rangle$ and $\vec{u} - \vec{v}$. Find the area of the triangle.

$\boxed{\text{A. } \sqrt{11}}$

B. $\sqrt{38}$

C. $\frac{\sqrt{54}}{2}$

D. $\frac{\sqrt{15}}{2}$

E. $\sqrt{14}$

$\text{Area} = \frac{1}{2}|\vec{u} \times \vec{v}|$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 2 & 1 \\ 1 & 4 & 1 \end{vmatrix} = \vec{i}(-2) - \vec{j}(-2) + \vec{k}(-6)$$

$$\vec{u} \times \vec{v} = -2\vec{i} + 2\vec{j} - 6\vec{k}$$

$$|\vec{u} \times \vec{v}| = \sqrt{4+4+36} = \sqrt{44} = \sqrt{4 \times 11} = 2\sqrt{11}$$

$$\text{Area} = \sqrt{11}$$

5. Find the area of the region enclosed by the curves $y = 2x^3$ and $y = -2x^3$, $y = 2$ and $y = -2$.

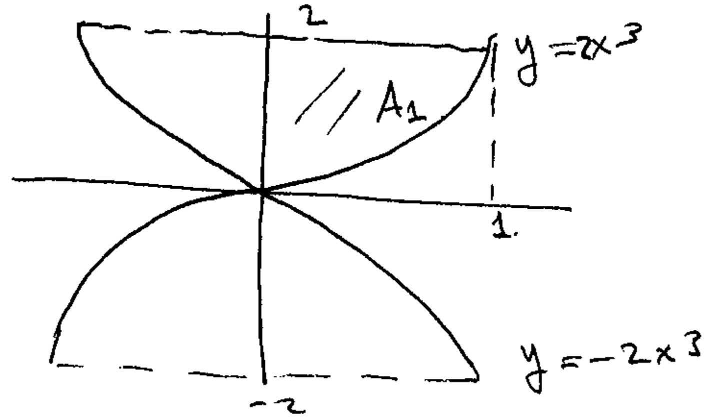
A. 3

B. $\frac{2}{3}$

C. $\frac{5}{3}$

D. 2

$\boxed{\text{E. } 6}$



Total area = $4A_1$. But

$$A_1 = \int_0^1 (2 - 2x^3)\,dx = 2\int_0^1 (1 - x^3)\,dx = 2\left(1 - \tfrac{1}{4}\right) = 6/4.$$

Total area = 6.

6. Find the area of the region enclosed by the curves $x = 1 + y^2$ and $x = 5$.

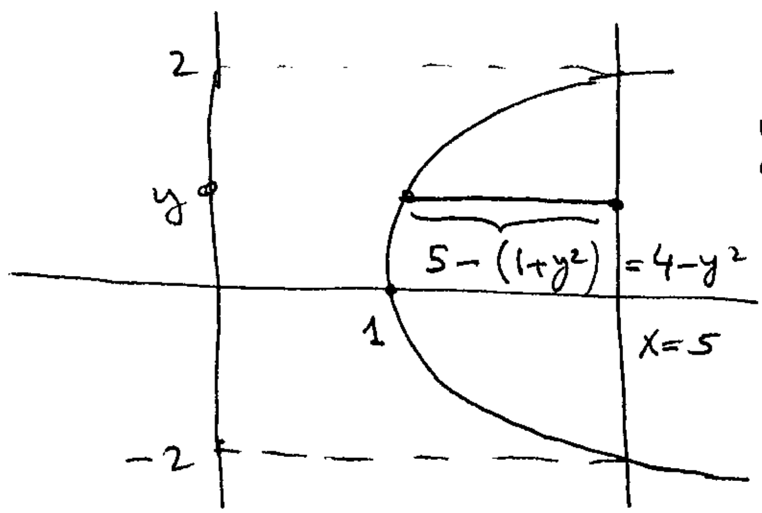
A. 3

B. $\frac{2}{3}$

C. $\frac{4}{3}$

D. 4

$\boxed{\text{E. } \frac{32}{3}}$



$y^2 + 1 = 5$

$y^2 = 4$

$y = \pm 2$

$$A = \int_{-2}^{2} \left(5 - (1+y^2)\right) dy = \int_{-2}^{2} (4 - y^2)\,dy$$

$$= 2\int_0^2 (4 - y^2)\,dy = 2 \cdot \left[8 - \frac{8}{3}\right] = \frac{32}{3}$$

7. Find the volume of the solid obtained by rotating the region bounded by $x = 2y^2, x = 4 - 2y^2$ about $x = 4$.

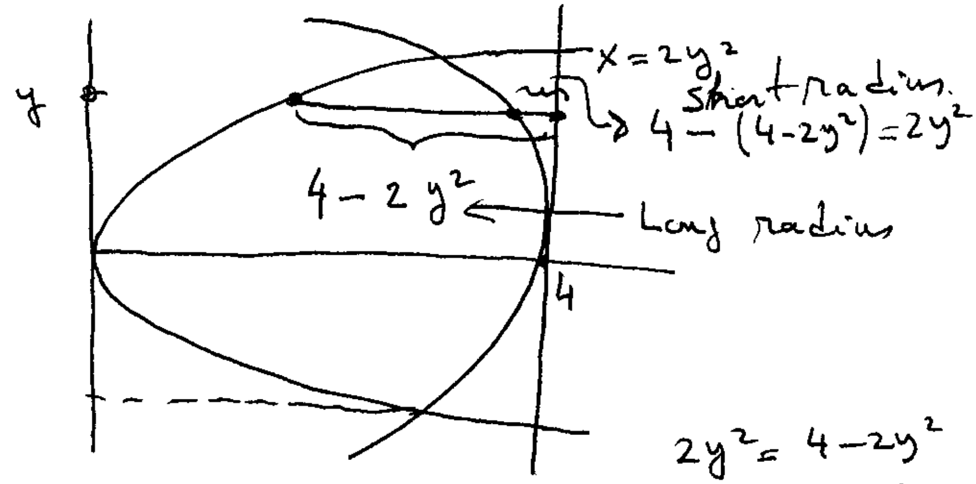
A. $\frac{32}{3}\pi$

B. $\frac{16}{3}\pi$

C. $\frac{28}{3}\pi$

D. $\frac{58}{3}\pi$

E. $\boxed{\frac{64}{3}\pi}$



$2y^2 = 4 - 2y^2$

$4 = 4y^2$

$y = \pm 1$

$$V = \pi \int_{-1}^{1} \left[(4-2y^2)^2 - (2y^2)^2 \right] dy$$

$$= \pi \int_{-1}^{1} [16 - 16y^2]\, dy = 16\pi \int_{-1}^{1} (1-y^2)\, dy = 32\pi \int_0^1 (1-y^2)\, dy$$

$$= 32\pi \cdot \left(1 - \frac{1}{3}\right) = \frac{64\pi}{3}$$

8. Evaluate the integral $\int_0^{\frac{\pi}{4}} x^2 \sin(2x)\, dx$.

A. $\boxed{\frac{\pi}{8} - \frac{1}{4}}$

B. $\frac{\pi}{8} + \frac{1}{4}$

C. $\frac{\pi}{8} - \frac{1}{2}$

D. $\frac{\pi}{8} + \frac{1}{2}$

E. $\frac{\pi}{8} + \frac{3}{2}$

Set $u = x^2$; $dV = \sin 2x\, dx$

$du = 2x\, dx$, $V = -\frac{1}{2} \cos 2x$.

$$\int_0^{\pi/4} x^2 \sin 2x\, dx = \underbrace{-\frac{x^2}{2} \cos 2x \Big|_0^{\pi/4}}_{= 0} + \int_0^{\pi/4} x \cos 2x\, dx$$

$$= \int_0^{\pi/4} x \cos 2x\, dx$$, Now set $u = x$

$dV = \cos 2x\, dx$

$V = +\frac{1}{2} \sin 2x$

$$\int_0^{\pi/4} x \cos 2x\, dx = \frac{x}{2} \sin 2x \Big|_0^{\pi/4} - \int_0^{\pi/4} \frac{1}{2} \sin 2x\, dx$$

$$= \pi/8 + \frac{1}{4} \cos 2x \Big|_0^{\pi/4} = \pi/8 - 1/4.$$

9. Find the volume of the solid generated by rotating the region bounded by $x = 2y^2, x = 1 + y^2$ about $y = 2$.

A. $\frac{4}{3}\pi$

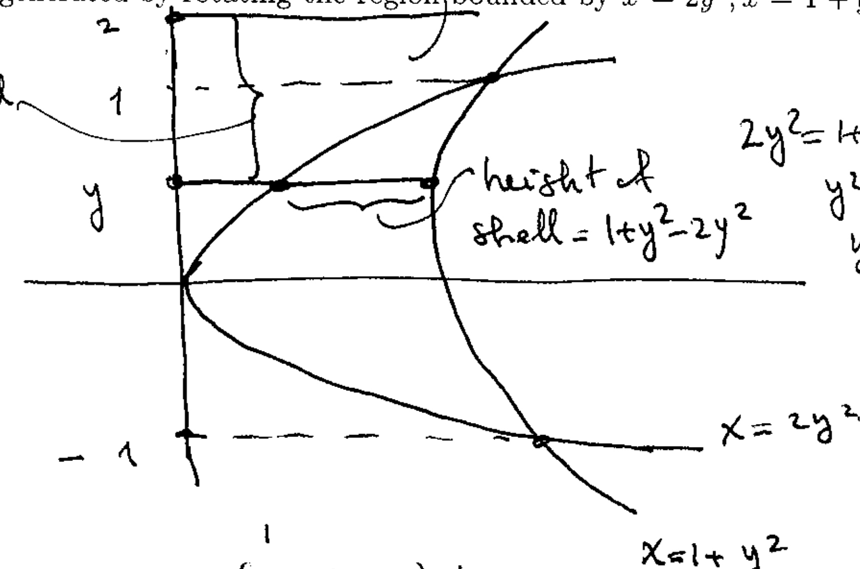
B. $\boxed{\frac{16}{3}\pi}$

C. $\frac{2}{3}\pi$

D. $\frac{5}{3}\pi$

E. 2π

Radius of shell $2-y$



$2y^2 = 1+y^2$

$y^2 = 1$

$y = \pm 1$

$$V = 2\pi \int_{-1}^{1} (2-y)(1-y^2)\,dy = 2\pi \int_{-1}^{1} 2(1-y^2)\,dy$$

$$- \underbrace{2\pi \int_{-1}^{1} y(1-y^2)\,dy}_{=0 \text{ because the function is odd}} = 4\pi \int_{-1}^{1} (1-y^2)\,dy = 8\pi \int_0^1 1-y^2\,dy$$

$$= 8\pi\left(1 - \frac{1}{3}\right) = \frac{16\pi}{3}$$

10. An aquarium has the shape of a box with all sides perpendicular to each other. Its sides measure 2 m long, 5 m wide, and 1 m deep and the aquarium is full of water. Find the work in Joules needed to pump all the water out of the aquarium. (Use the fact that the density of water is 1000 kg/m^3 and denote the acceleration of gravity by g.)

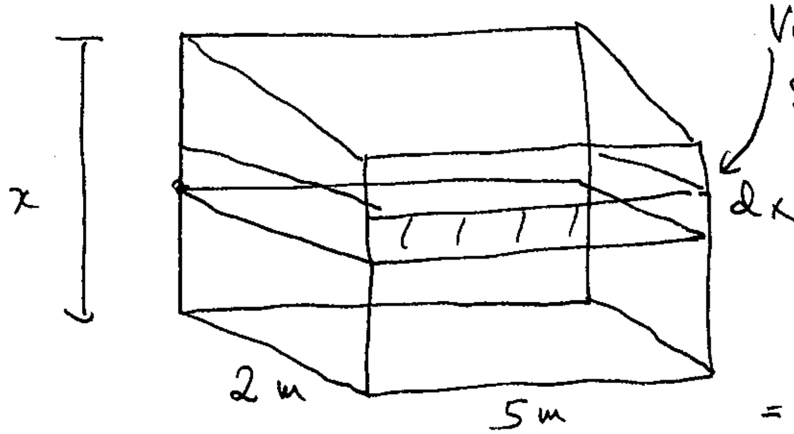
A. $10000g$

B. $1000g$

C. $\boxed{5000g}$

D. $2500g$

E. $500g$



Volume of slab $= 10\,dx$

Weight of slab $= 10\,dx\,1000g = 10000\,g\,dx$

Work to move slab to the top $= 10000\,g\,x\,dx$.

$$\text{Total work} = 10000g \int_0^1 x\,dx = 5000g$$

11. Evaluate the integral $\int_0^{\frac{\pi}{4}} \tan^7(x)\sec^4(x)dx.$ $= \int_0^{\pi/4} \tan^7 x \cdot \sec^2 x \cdot \sec^2 x \, dx$

A. $\frac{8}{45}$

B. $\boxed{\frac{9}{40}}$

C. $\frac{9}{44}$

D. $\frac{7}{54}$

E. $\frac{17}{36}$

$$= \int_0^{\pi/4} \tan^7 x \cdot (1+\tan^2 x) \cdot \sec^2 x \, dx$$

$$u = \tan x$$

$$= \int_0^1 u^7(1+u^2)\, du$$

$$= \int_0^1 (u^7 + u^9)\, du = \frac{1}{8} + \frac{1}{10} = \frac{9}{40}$$