NAME GRADING KEY

STUDENT ID

RECITATION INSTRUCTOR

RECITATION TIME

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Page 5	/18
TOTAL	/100

DIRECTIONS

1. Write your name, student ID number, recitation instructor's name and recitation time in the space provided above. Also write your name at the top of pages 2, 3, 4 and 5.
2. The test has five (5) pages, including this one.
3. Write your answers in the boxes provided.
4. You must show sufficient work to justify all answers. Correct answers with inconsistent work may not be given credit.
5. Credit for each problem is given in parentheses in the left hand margin.
6. No books, notes or calculators may be used on this test.

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(12) 1. Find the integral by means of the substitution $u = \sqrt{x}$.

$$\int \frac{\sqrt{x}+1}{x+1}\,dx = \int \frac{u+1}{u^2+1}\,2u\,du \quad ④$$

$u = \sqrt{x} \quad x = u^2$

$dx = 2u\,du$

$$= 2\int \frac{u^2+u}{u^2+1}\,du = 2\int\left(1 + \overset{②}{\frac{u-1}{u^2+1}}\right)du$$

$$\begin{array}{r l} & 1 \\ u^2+1 & \overline{)\,u^2+u} \\ & \underline{u^2+1} \\ & u-1 \end{array}$$

$$= 2\int\left[1 + \frac{u}{u^2+1} - \frac{1}{u^2+1}\right]du$$

$$= 2\left[u + \tfrac{1}{2}\ln(u^2+1) - \tan^{-1}u\right] + C$$

$$= 2\sqrt{x} + \ln(x+1) - 2\tan^{-1}\sqrt{x} + C$$

-1 pt for missing dx or du (one time only for this problem)

-1 pt for missing $+C$ (one time only for test)

$$\boxed{\overset{②}{2\sqrt{x}} + \overset{②}{\ln(x+1)} - \overset{②}{2\tan^{-1}\sqrt{x}} + C} \quad \boxed{12}$$

(10) 2. Find the integral $\int \frac{3x}{x^2-4x+4}\,dx$.

$$\frac{3x}{x^2-4x+4} = \underset{\textcircled{2}}{\frac{3x}{(x-2)^2}} = \underset{\textcircled{2}}{\frac{A}{x-2}} + \underset{\textcircled{2}}{\frac{B}{(x-2)^2}}$$

← No additional credit beyond this point if anything is wrong here

$3x = Ax - 2A + B$

$3 = A \quad 0 = -2A + B$

$A = 3 \quad B = 6$

$$\int \frac{3x}{x^2-4x+4}\,dx = \int \frac{3}{x-2}\,dx + \int \frac{6}{(x-2)^2}\,dx$$

$$= \underset{\textcircled{2}}{3\ln|x-2|} - \underset{\textcircled{2}}{\frac{6}{(x-2)}} + C$$

$$\boxed{3\ln|x-2| - \frac{6}{(x-2)} + C} \qquad \boxed{10}$$

(18) 3. Determine whether the improper integral converges or diverges. If it converges find its value. Important: Show clearly how limits are involved.

−1 pt each time "=" and "lim" are inconsistent (at most −2 pts for this)

(a) $\int_1^\infty \frac{1}{x^{3/2}}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^{3/2}}\,dx$ ④

$$= \lim_{b\to\infty} \left[\frac{x^{-1/2}}{-\frac{1}{2}}\right]_1^b \quad \textcircled{3}$$

$$= \lim_{b\to\infty} \left[-\frac{2}{\sqrt{x}}\right]_1^b = \lim_{b\to\infty} \left[-\frac{2}{\sqrt{b}} + 2\right] = 2 \quad \textcircled{1}$$

$$\boxed{2} \qquad \boxed{8}$$

(b) $\int_0^{\frac{\pi}{2}} \tan x\,dx = \lim_{b\to\frac{\pi}{2}^-} \int_0^b \tan x\,dx$ ⑤

$$= \lim_{b\to\frac{\pi}{2}^-} \int_0^b \frac{\sin x}{\cos x}\,dx = \lim_{b\to\frac{\pi}{2}^-} \Big[-\ln|\cos x|\Big]_0^b$$

$$= \underbrace{\lim_{b\to\frac{\pi}{2}^-} \big[-\ln(\cos b) + \ln 1\big]}_{\textcircled{3}} = +\infty \quad \textcircled{1}$$

①

$$\boxed{\text{diverges}} \qquad \boxed{10}$$

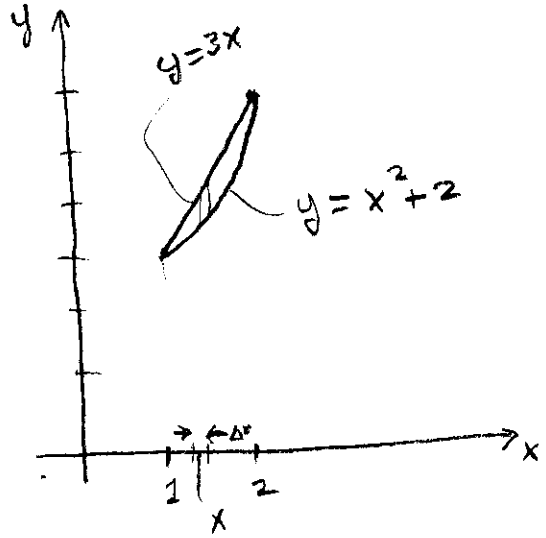
(10) 4. Let R be the region between the graphs of the equations $y = 3x$ and $y = x^2 + 2$. Use the washer method to set up an integral for the volume V of the solid generated by revolving the region R about the x-axis. Do not evaluate the integral.

$y = 3x$, $y = x^2 + 2$ $\longrightarrow$ $x^2 + 2 = 3x$

$x^2 - 3x + 2 = 0$

$(x-1)(x-2) = 0$

$x = 1$, $x = 2$



$$\Delta V = \pi\left[(3x)^2 - (x^2+2)^2\right]\Delta x$$

$$V = \int_1^2 \pi\left[9x^2 - x^4 - 4x^2 - 4\right]dx$$

$$= \int_1^2 \pi\left[5x^2 - x^4 - 4\right]dx$$

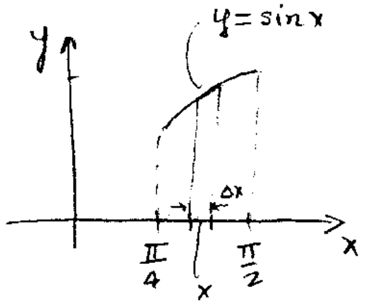
-1pt for algebra mistake here

* 0 credit for problem if more than 1 item is wrong (limits count as 1 item in this rule)

$$\boxed{V = \int_{1\,②}^{2\,②} \overbrace{\pi\left[(3x)^2 - (x^2+2)^2\right]}^{⑤\text{ or}} \overset{①}{dx}}$$

10

(12) 5. Let R be the region between the graph of $f(x) = \sin x$ and the x axis on the interval $[\frac{\pi}{4}, \frac{\pi}{2}]$. Find the volume of the solid generated by revolving R about the y axis.



$$\Delta V = 2\pi x \sin x \,\Delta x$$

$$V = \int_{\frac{\pi}{4}\,①}^{\frac{\pi}{2}\,①} \overbrace{2\pi x \sin x}^{③}\, \overset{①}{dx}$$

* see prob. 4

$$= 2\pi \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} x \sin x\, dx = 2\pi\left[-x\cos x\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos x\, dx\right]$$

$u = x$ $dv = \sin x\,dx$

$du = dx$ $v = -\cos x$

$$= 2\pi\left[\underset{②}{-x\cos x} + \underset{②}{\sin x}\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= 2\pi\left[1 - \left(-\frac{\pi}{4}\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right)\right] \text{ or}$$

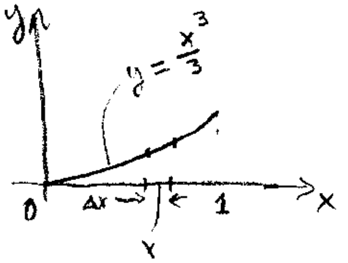
$$= 2\pi\left[1 - \frac{\sqrt{2}}{2}\left(1 - \frac{\pi}{4}\right)\right] \text{ or}$$

②

$$\boxed{\pi\left[2 + \frac{\pi\sqrt{2}}{4} - \sqrt{2}\right]}$$

12

(10) 6. Find the area S of the surface generated by revolving about the x axis the graph of $y = \frac{x^3}{3}$ on the interval $[0, 1]$.



$$\Delta S = 2\pi \frac{x^3}{3} \sqrt{1 + (x^2)^2}\, \Delta x$$

$$S = \int_{0\,\text{①}}^{1\,\text{①}} \underbrace{\frac{2\pi}{3} x^3 \sqrt{1 + x^4}}_{\text{⑤}}\, dx \;\text{①}$$

* see prob. 4

$$\int x^3 \sqrt{1 + x^4}\, dx = \frac{1}{4} \int u^{1/2} du = \frac{1}{4} \frac{u^{3/2}}{\frac{3}{2}} = \frac{1}{6} (1 + x^4)^{3/2}$$

$u = 1 + x^4$
$du = 4x^3 dx$

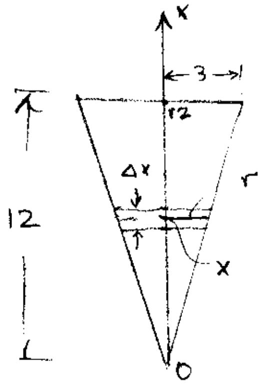
$$S = \frac{2\pi}{3} \frac{1}{6} (1 + x^4)^{3/2} \Big|_0^1 = \frac{\pi}{9} (2\sqrt{2} - 1)$$

②

$$\boxed{S = \frac{\pi}{9} (2\sqrt{2} - 1)}$$

$\boxed{10}$

(10) 7. A tank in the shape of an inverted cone 12 feet tall and 3 feet in radius is full of water. Set up an integral for the work W required to pump all the water over the edge of the tank. Do not evaluate the integral. (Water weighs 62.5 lbs/ft^3).



$$\frac{r}{3} = \frac{x}{12} \qquad r = \frac{x}{4}$$

$$\Delta W = (62.5)(12 - x)\, \pi \left(\frac{x}{4}\right)^2 \Delta x$$

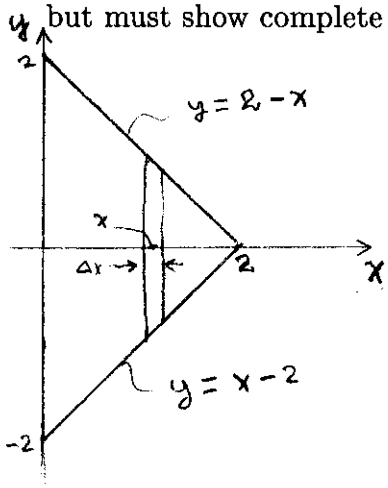
$$W = \int_0^{12} (62.5)(12 - x) \frac{\pi x^2}{16}\, dx$$

0 credit for problem if more than 2 items are wrong (limits count as 1 item in this rule).

$$\boxed{W = \int_{0\,\text{①}}^{12\,\text{①}} \overset{\text{①}}{(62.5)}\, \overset{\text{③}}{(12 - x)}\, \overset{\text{③}}{\frac{\pi x^2}{16}}\, \overset{\text{①}}{dx}}$$

$\boxed{10}$

(10) 8. Find the center of gravity $(\bar{x}, \bar{y})$ of the region R between the graphs of $y = 2 - x$ and $y = x - 2$ on the interval $[0, 2]$. You may use symmetry for one of the coordinates, but must show complete work for the other.



From symmetry $\bar{y} = 0$ ②

$\bar{x} A = M_y$ ①

$A = \frac{1}{2} 4 \cdot 2 = 4$ ①

$\Delta M_y = x \, [(2-x) - (x-2)] \Delta x$

① ① ②

$M_y = \int_0^2 x\,(4 - 2x)\,dx$

$= \left[4\frac{x^2}{2} - 2\frac{x^3}{3}\right]_0^2 = 4 \cdot \frac{4}{2} - 2\frac{8}{3} = \frac{8}{3}$ ①

$\bar{x} = \frac{M_y}{A} = \frac{\frac{8}{3}}{4} = \frac{2}{3}$ ①

$\bar{x} = \frac{2}{3}$ $\bar{y} = 0$

10

(8) 9. Find the third Taylor polynomial $p_3(x)$ of $f(x) = \tan^{-1} x$.

$P_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$

$f(x) = \tan^{-1} x$ $f(0) = 0$

$f'(x) = \frac{1}{1+x^2}$ $f'(0) = 1$

$f''(x) = \frac{-2x}{(1+x^2)^2}$ $f''(0) = 0$

$f'''(x) = \frac{(1+x^2)^2(-2) + 2x \cdot 2(1+x^2)2x}{(1+x^2)^4}$ $f'''(0) = -2$

−2pts for higher order terms or $P_n(x)$.

② ② ② ②

✓

$p_3(x) = x - \frac{x^3}{3}$

8