NAME GRADING KEY

STUDENT ID ______

RECITATION INSTRUCTOR ______

RECITATION TIME ______

Page 1	/16
Page 2	/32
Page 3	/28
Page 4	/24
TOTAL	/100

DIRECTIONS

1. Write your name, student ID number, recitation instructor's name and recitation time in the space provided above. Also write your name at the top of pages 2, 3, and 4.
2. The test has four (4) pages, including this one.
3. Write your answers in the boxes provided.
4. You must show sufficient work to justify all answers. Correct answers with inconsistent work may not be given credit.
5. Credit for each problem is given in parentheses in the left hand margin.
6. No books, notes or calculators may be used on this test.

Evaluate the integrals in problems 1–5.

(8) 1. $\int \cot^3 x \sin^2 x\, dx = \int \frac{\cos^3 x}{\sin^3 x} \sin^2 x\, dx$ (2) $= \int \frac{\cos^3 x}{\sin x} dx =$

$$= \int \frac{(1-\sin^2 x)}{\sin x} \cos x\, dx = \int \frac{1-u^2}{u} du$$

$u = \sin x \quad du = \cos x\, dx$

$$= \int \left(\frac{1}{u} - u\right) du = \ln|u| - \frac{1}{2}u^2 + C = \ln|\sin x| - \frac{1}{2}\sin^2 x + C$$

(3) (3)

$\boxed{\ln|\sin x| - \frac{1}{2}\sin^2 x + C}$ $\boxed{8}$

-1 pt for missing $+C$ (one time only for test)

-1pt for missing dx, du, etc (one time only for each problem)

(8) 2. $\int_0^{\frac{\pi}{4}} \sec^4 x \tan x dx = \int_0^{\frac{\pi}{4}} \sec^3 x \sec x \tan x\, dx = \frac{1}{4}\sec^4 x \Big|_0^{\frac{\pi}{4}} =$ (4)

$u = \sec x \quad du = \sec x \tan x\, dx$

$x = 0 \to u = 1, \quad x = \frac{\pi}{4} \to u = \sqrt{2}$

or

$$= \int_1^{\sqrt{2}} u^3 du = \frac{u^4}{4}\Big|_1^{\sqrt{2}} = 1 - \frac{1}{4} = \frac{3}{4}$$ (4) (1) (1) (2)

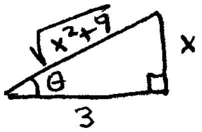
$= \frac{1}{4}\sec^4\left(\frac{\pi}{4}\right) - \frac{1}{4}\sec^4 0$ (1)

$= \frac{1}{4}(\sqrt{2})^4 - \frac{1}{4}1$ (1)

$= 1 - \frac{1}{4} = \frac{3}{4}$ (2)

$\boxed{\frac{3}{4}}$ $\boxed{8}$

(10) 3. $\displaystyle\int \frac{dx}{(x^2+9)^{\frac{3}{2}}}$ $\displaystyle = \int \frac{3\sec^2\theta}{(3\sec\theta)^3}\,d\theta$ (6) ← 0 pts for problem if this is wrong $\displaystyle = \frac{1}{9}\int \frac{1}{\sec\theta}\,d\theta = \frac{1}{9}\int \cos\theta\, d\theta$



$x = 3\tan\theta \quad dx = 3\sec^2\theta\, d\theta.$

$\sqrt{x^2+9} = 3\sec\theta$

$\displaystyle = \frac{1}{9}\sin\theta + C$

−2 pts for coefficient other than $\frac{1}{9}$

$\displaystyle = \frac{1}{9}\frac{x}{\sqrt{x^2+9}} + C$

(4)

$\boxed{\displaystyle \frac{1}{9}\frac{x}{\sqrt{x^2+9}} + C}$ $\boxed{10}$

−2 pts if limits are not changed

(6) 4. $\displaystyle\int_0^3 x\sqrt{9-x^2}\,dx$ $\displaystyle = -\frac{1}{2}\int_9^0 u^{1/2}\,du = -\frac{1}{2}\left.\frac{u^{3/2}}{\frac{3}{2}}\right|_9^0 = -\frac{1}{3}u^{3/2}\Big|_9^0$ (4)

$u = 9-x^2 \quad du = -2x\,dx$

$x=0 \rightarrow u=9,\ x=3 \rightarrow u=0$

$\displaystyle = -\frac{1}{3}0 + \frac{1}{3}9^{3/2} = 9$ (2)

or $\displaystyle\int x\sqrt{9-x^2}\,dx = -\frac{1}{2}\int u^{1/2}du = -\frac{1}{2}\frac{u^{3/2}}{\frac{3}{2}} + C = -\frac{1}{3}(9-x^2)^{3/2} + C$ (4)

$u = 9-x^2 \quad du = -2x\,dx$

$\displaystyle\int_0^3 x\sqrt{9-x^2}\,dx = -\frac{1}{3}(9-x^2)^{3/2}\Big|_0^3 = 0 + \frac{1}{3}9^{3/2} = 9$ (2) $\boxed{9}$ $\boxed{6}$

(10) 5. $\displaystyle\int_0^1 \frac{2x+3}{(x+1)^2}\,dx$

(4) $\displaystyle \frac{2x+3}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2}$ ← 0 credit for problem if anything is wrong here.

$2x+3 = A(x+1) + B$

$\boxed{2 = A}$ $\quad A+B=3 \rightarrow \boxed{B=1}$

$\displaystyle\int_0^1 \frac{2x+3}{(x+1)^2}\,dx = \int_0^1\left(\frac{2}{x+1} + \frac{1}{(x+1)^2}\right)dx$ (2)

$\displaystyle = \left(2\ln(x+1) - \frac{1}{x+1}\right)\Big|_0^1 = 2\ln 2 - \frac{1}{2} + 1 = 2\ln 2 + \frac{1}{2}$ (2) (2) $\boxed{2\ln 2 + \frac{1}{2}}$ $\boxed{10}$

(6) 6. Write out the form of the partial fraction decomposition of the function below. Do not determine the numerical values of the coefficients.

$\displaystyle \frac{1}{(x^2-1)(x^2+1)^2} = \frac{1}{(x+1)(x-1)(x^2+1)^2}$

$\displaystyle = \underbrace{\frac{A}{x+1} + \frac{B}{x-1}}_{3} + \underbrace{\frac{Cx+D}{x^2+1} + \frac{Ex+F}{(x^2+1)^2}}_{3}$

$\boxed{6}$

(12) 7. Determine whether each integral is convergent or divergent. Find its value if it is convergent. <u>Important</u>: Show clearly how limits are involved.

(a) $\int_1^\infty \frac{\ln x}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{\ln x}{x}\,dx$ ③

$= \lim_{t\to\infty} \frac{1}{2}(\ln x)^2 \Big|_1^t$

$= \lim_{t\to\infty} \frac{1}{2}(\ln t)^2$ ②

$= \infty$

① $\boxed{\text{divergent}}$ [6]

(b) $\int_1^2 \frac{1}{\sqrt{x-1}}\,dx = \lim_{t\to 1^+} \int_t^2 \frac{1}{\sqrt{x-1}}\,dx$ ③

$= \lim_{t\to 1^+} \left(2\sqrt{x-1}\right)\Big|_t^2$

$= \lim_{t\to 1^+} \left(2 - 2\sqrt{t-1}\right)$ ②

$= 2$ ①

$\boxed{2}$ [6]

(8) 8. Find the length of the curve $y = f(x)$, $\frac{\pi}{4} \le x \le \frac{\pi}{3}$, given that $f'(x) = \sqrt{\tan^2 x - 1}$.

$L = \int_{\pi/4}^{\pi/3} \sqrt{1 + \tan^2 x - 1}\,dx$ ④

$= \int_{\pi/4}^{\pi/3} \tan x\,dx = \left[-\ln|\cos x|\right]_{\pi/4}^{\pi/3}$ ②

If + grade rest with consistency.

Any form of answer is ok

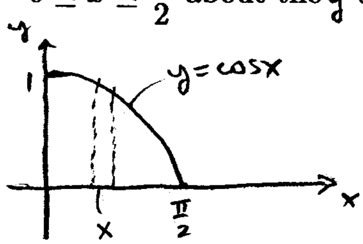
−1pt for error in simplifying

$= -\ln\left(\frac{1}{2}\right) + \ln\frac{1}{\sqrt{2}} = \ln 2 - \frac{1}{2}\ln 2$

$= \frac{1}{2}\ln 2$

② $\boxed{\ln\sqrt{2}}$

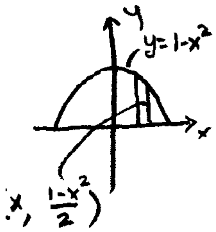
(8) 9. Set up an integral for the area S of the surface obtained by rotating the curve $y = \cos x$, $0 \le x \le \frac{\pi}{2}$ about the y-axis. Do not evaluate the integral.



$dS = 2\pi x\,ds = 2\pi x\sqrt{1+\sin^2 x}\,dx$

$\boxed{S = \int_{0}^{\pi/2} \underbrace{2\pi x\sqrt{1+\sin^2 x}}_{⑤}\,dx}$ ① ① ① [8]

(12) 10. Consider the lamina bounded by the curves $y = 1 - x^2$ and $y = 0$, and with density $\rho = 1$. Find the following:



(a) The mass m of the lamina

$$m = 1A = \int_{-1}^{1} (1-x^2)\,dx = 2\int_0^1 (1-x^2)\,dx = 2\left(x - \frac{x^3}{3}\right)\Big|_0^1 = 2\left(1-\frac{1}{3}\right)$$

$\boxed{m = \frac{4}{3}}$ ② NPC

(b) The moment M_y of the lamina about the y–axis

$$M_y = \int_{-1}^{1} \underbrace{x(1-x^2)}_{\text{odd}}\,dx = 0 \quad \text{(or from symmetry)}$$

$\boxed{M_y = 0}$ ② NPC

(c) The moment M_x of the lamina about the x–axis

$$M_x = \int_{-1}^{1} \frac{1-x^2}{2}(1-x^2)\,dx = \int_0^1 (1-2x^2+x^4)\,dx$$
$$= \left(x - \frac{2}{3}x^3 + \frac{1}{5}x^5\right)\Big|_0^1 = 1 - \frac{2}{3} + \frac{1}{5} = \frac{8}{15}$$

② ③

$\boxed{M_x = \frac{8}{15}}$ ③ NPC

(d) The centroid $(\bar{x}, \bar{y})$ of the lamina

$m\bar{x} = M_y \rightarrow \bar{x} = 0$ or from symmetry

$m\bar{y} = M_x \rightarrow \frac{4}{3}\bar{y} = \frac{8}{15} \quad \bar{y} = \frac{2}{5}$

$\boxed{(\bar{x}, \bar{y}) = \left(0, \frac{2}{5}\right)}$ ok if consistent with above

(12) 11. Determine whether the sequence converges or diverges. If it converges, find the limit. (You need not show work for this problem).

2pts each NPC

(a) $a_n = \dfrac{2^n}{3^{n+1}}$

$$\lim_{n\to\infty} \frac{2^n}{3^{n+1}} = \frac{1}{3}\lim_{n\to\infty}\left(\frac{2}{3}\right)^n = 0$$

$\boxed{0}$ ②

(b) $a_n = \dfrac{(-1)^n}{n}$

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{1}{n} = 0 \quad \therefore \lim_{n\to\infty} a_n = 0$$

$\boxed{0}$ ②

(c) $a_n = \dfrac{n^2}{e^n}$

$$\lim_{x\to\infty} \frac{x^2}{e^x} \overset{L'H}{=} \lim_{x\to\infty} \frac{2x}{e^x} \overset{L'H}{=} \lim_{x\to\infty} \frac{2}{e^x} = 0$$

$\boxed{0}$ ②

(d) $a_n = (-1)^n n$

$-1, 2, -3, 4, -5, \ldots$

$\boxed{\text{divergent.}}$ ②

(e) $a_n = \dfrac{\sqrt{n^2+3}}{n}$

$$\lim_{n\to\infty} \frac{\sqrt{n^2+3}}{n} = \lim_{n\to\infty} \frac{n\sqrt{1+\frac{3}{n}}}{n} = 1$$

$\boxed{1}$ ②

(f) $a_n = \dfrac{n\cos n}{n^2+3}$

$$-\frac{n}{n^2+3} \le \frac{n\cos n}{n^2+3} \le \frac{n}{n^2+3}$$

squeeze theorem as $n\to\infty$: $\downarrow 0$. $\therefore \downarrow 0$ $\downarrow 0$

$\boxed{0}$ ②