NAME GRADING KEY

STUDENT ID ______

RECITATION INSTRUCTOR ______

RECITATION TIME ______

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Page 2	/32
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TOTAL	/100

DIRECTIONS

−1pt for missing +C (1 time only for test)
−1pt for missing dx, du, etc (1 time only for each problem)

1. Write your name, student ID number, recitation instructor's name and recitation time in the space provided above. Also write your name at the top of pages 2, 3, and 4.
2. The test has four (4) pages, including this one.
3. Write your answers in the boxes provided.
4. You must show sufficient work to justify all answers. Correct answers with inconsistent work may not be given credit.
5. Credit for each problem is given in parentheses in the left hand margin.
6. No books, notes or calculators may be used on this test.

Find the integrals in problems 1–6.

(6) 1. $\int \sec^4 x\,dx$ $= \int \sec^2 x \sec^2 x\,dx = \int (1+\tan^2 x)\sec^2 x\,dx$

$u = \tan x \quad du = \sec^2 x\,dx$

$= \int (1+u^2)\,du = u + \frac{1}{3}u^3 + C$

$= \tan x + \frac{1}{3}\tan^3 x + C$

3 3

$\boxed{\tan x + \frac{1}{3}\tan^3 x + C}$ $\boxed{6}$

(6) 2. $\int_0^{\frac{\pi}{2}} \sin^2(2x)dx$ $= \int_0^{\frac{\pi}{2}} \frac{1-\cos(4x)}{2}\,dx$

$\sin^2(2x) = \frac{1-\cos(4x)}{2}$ 3

$= \left(\frac{x}{2} - \frac{1}{8}\sin 4x\right)\Big|_0^{\frac{\pi}{2}} = \frac{\pi}{4}$

3

$\boxed{\frac{\pi}{4}}$ $\boxed{6}$

(6) 3. $\int_0^1 x\sqrt{x^2+1}\,dx$ (Hint: trigonometric substitution is not necessary).

$u = x^2+1 \quad du = 2x\,dx$

$x=0 \to u=1 \quad x=1 \to u=2$

$= \frac{1}{2}\int_1^2 u^{1/2}\,du = \frac{1}{2}\cdot\frac{2}{3}u^{3/2}\Big|_1^2$ 4

$= \frac{1}{3}(2\sqrt{2}-1)$ 2

−2pts if limits are not changed or $\frac{1}{2}$ is missing

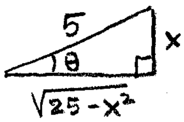
or $\int x\sqrt{x^2+1}\,dx = \frac{1}{2}\int u^{1/2}\,du =$

$= \frac{1}{3}u^{3/2} + C = \frac{1}{3}(x^2+1)^{3/2} + C$ 4 etc. →

$\boxed{\frac{1}{3}(2\sqrt{2}-1)}$ $\boxed{6}$

(12) 4. $\displaystyle\int \frac{1}{x^2\sqrt{25-x^2}}\,dx$ $\displaystyle = \int \frac{1}{25\sin^2\theta \; 5\cos\theta} 5\cos\theta \, d\theta$ 6

0 points for problem if the integrand is wrong, but −2pts for numerical error only and grade rest with consistency



$x = 5\sin\theta \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$dx = 5\cos\theta \, d\theta$

$\sqrt{25-x^2} = 5\cos\theta$

$$= \frac{1}{25}\int \csc^2\theta \, d\theta = -\frac{1}{25}\cot\theta + C = -\frac{1}{25}\frac{\sqrt{25-x^2}}{x} + C$$ 3

3

$$\boxed{-\frac{1}{25}\frac{\sqrt{25-x^2}}{x} + C}$$ $\boxed{12}$

(12) 5. $\displaystyle\int \frac{x^2+3}{x^3+2x}\,dx$

$$\frac{x^2+3}{x^3+2x} = \underset{2}{\frac{x^2+3}{x(x^2+2)}} = \underset{2}{\frac{A}{x}} + \underset{2}{\frac{Bx+C}{x^2+2}}$$

No additional credit beyond this point if anything is wrong here

$x^2+3 = Ax^2 + 2A + Bx^2 + Cx \rightarrow$ $A+B=1$, $C=0$, $2A=3$ $\rightarrow$ $\boxed{A=\frac{3}{2},\ B=-\frac{1}{2},\ C=0}$

$$\int \frac{x^2+3}{x^3+2x}\,dx = \int\left(\frac{3}{2}\frac{1}{x} - \frac{1}{2}\frac{x}{x^2+2}\right)dx = \frac{3}{2}\ln|x| - \frac{1}{4}\ln(x^2+2) + C$$

3 3

$$\boxed{\frac{3}{2}\ln|x| - \frac{1}{4}\ln(x^2+2) + C}$$ $\boxed{12}$

(8) 6. $\displaystyle\int_0^2 \frac{x^2}{x^2+4}\,dx$ $\displaystyle = \int_0^2 \left(1 - \frac{4}{x^2+4}\right)dx$ 3 (for division)

$$= \left[x - 4\cdot\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right]_0^2$$ 3 (for correct antiderivative)

$$= 2 - 2\tan^{-1}1$$

$$= 2 - 2\frac{\pi}{4} = 2 - \frac{\pi}{2}$$ 2

$$\boxed{2 - \frac{\pi}{2}}$$ $\boxed{8}$

(12) 7. Determine whether each integral is convergent or divergent. Find its value if it is convergent. Important: You must show clearly how limits are involved.

(a) $\int_2^\infty \frac{1}{\sqrt{x}}\,dx = \lim_{t\to\infty} \int_2^t \frac{1}{\sqrt{x}}\,dx$ (3)

$= \lim_{t\to\infty} \left[2\sqrt{x}\right]_2^t = \underbrace{\lim_{t\to\infty} \left(2\sqrt{t} - 2\sqrt{2}\right)}_{(2)} = \infty$

(1) $\boxed{\text{divergent}}$

6

(b) $\int_0^3 \frac{1}{\sqrt{x}}\,dx = \lim_{t\to 0^+} \int_t^3 \frac{1}{\sqrt{x}}\,dx$ (3)

$= \lim_{t\to 0^+} \left[2\sqrt{x}\right]_t^3 = \underbrace{\lim_{t\to 0^+} \left(2\sqrt{3} - 2\sqrt{t}\right)}_{(2)} = 2\sqrt{3}$

(1) $\boxed{2\sqrt{3}}$

6

(10) 8. Find the length of the curve $y = f(x)$, $0 \le x \le \frac{\pi}{3}$, given that $f'(x) = 2\sqrt{\cos x + \cos^2 x}$.

$L = \int_0^{\frac{\pi}{3}} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$ (4) $= \int_0^{\frac{\pi}{3}} \sqrt{1 + 4\cos x + 4\cos^2 x}\,dx$

$= \int_0^{\frac{\pi}{3}} \sqrt{(1 + 2\cos x)^2}\,dx = \int_0^{\frac{\pi}{3}} (1 + 2\cos x)\,dx$ (4)

$= \left[x + 2\sin x\right]_0^{\frac{\pi}{3}} = \frac{\pi}{3} + 2\frac{\sqrt{3}}{2}$ ← or (2) or → $\boxed{\frac{\pi}{3} + \sqrt{3}}$

10

(6) 9. Write out the form of the partial fraction decomposition of the function below. Do not determine the numerical values of the coefficients.

$$\frac{x+3}{(x^2-5x+4)(x^2+6)^2} = \frac{x+3}{(x-1)(x-4)(x^2+6)^2} \quad (2)$$

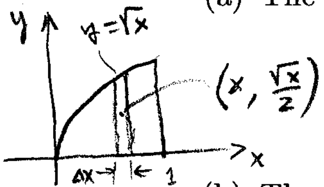
$$= \frac{A}{x-1} + \frac{B}{(x-4)} + \frac{Cx+D}{(x^2+6)} + \frac{Ex+F}{(x^2+6)^2} \quad (4)$$

6

(12) 10. Consider the lamina bounded by the curves $y = \sqrt{x}$, $y = 0$, $x = 1$ and with density $\rho = 1$. Find the following:

For (a),(b),(C) 0pts if answer is correct but no work is shown

(a) The mass m of the lamina



$$m = \int_0^1 1\sqrt{x}\,dx = \frac{2}{3}x^{\frac{3}{2}}\Big|_0^1 = \frac{2}{3}$$

$\boxed{m = \frac{2}{3}}$ 2 NPC

(b) The moment M_y of the lamina about the y–axis

$$M_y = \int_0^1 x\, 1\sqrt{x}\,dx = \int_0^1 x^{3/2}\,dx = \frac{2}{5}\, x^{5/2}\Big|_0^1 = \frac{2}{5}$$

$\boxed{M_y = \frac{2}{5}}$ 3 NPC

(c) The moment M_x of the lamina about the x–axis

$$M_x = \int_0^1 \frac{\sqrt{x}}{2}\, 1\sqrt{x}\,dx = \int_0^1 \frac{x}{2}\,dx = \frac{x^2}{4}\Big|_0^1 = \frac{1}{4}$$

$\boxed{M_x = \frac{1}{4}}$ 3 NPC

(d) The center of mass $(\bar{x}, \bar{y})$ of the lamina

$$\bar{x}\, m = M_y \rightarrow \bar{x}\,\frac{2}{3} = \frac{2}{5} \rightarrow \bar{x} = \frac{3}{5}$$

$$\bar{y}\, m = M_x \rightarrow \bar{y}\,\frac{2}{3} = \frac{1}{4} \rightarrow \bar{y} = \frac{3}{8}$$

2 2

$\boxed{(\bar{x}, \bar{y}) = \left(\frac{3}{5}, \frac{3}{8}\right)}$ ok if consistent with above

(10) 11. Determine whether the sequence converges or diverges. If it converges, find the limit. (You need not show work for this problem). 2 pts each NPC

(a) $a_n = \dfrac{5^{n+1}}{2^n} = 5\left(\frac{5}{2}\right)^n$

$$\lim_{n\to\infty} \frac{5^{n+1}}{2^n} = 5 \lim_{n\to\infty}\left(\frac{5}{2}\right)^n = \infty$$

because $\frac{5}{2} > 1$

$\boxed{\text{diverges}}$ 2

(b) $a_n = \dfrac{n!}{(n+2)!} = \dfrac{1\cdot 2\cdot 3\cdots n}{1\cdot 2\cdot 3\cdots n\,(n+1)(n+2)}$

$$= \frac{1}{(n+1)(n+2)} \longrightarrow 0 \text{ as } n\to\infty$$

$\boxed{0}$ 2

(c) $a_n = \dfrac{1+5\sqrt{n}}{3\sqrt{n}-2}$

$$\lim_{n\to\infty} \frac{1+5\sqrt{n}}{3\sqrt{n}-2} = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n}}+5}{3-\frac{2}{\sqrt{n}}} = \frac{5}{3}$$

$\boxed{\frac{5}{3}}$ 2

(d) $\left\{(-1)^n \sin\left(\dfrac{1}{n}\right)\right\}$

$$\lim_{n\to\infty} \left|(-1)^n \sin\left(\tfrac{1}{n}\right)\right| = \lim_{n\to\infty} \left|\sin\tfrac{1}{n}\right| = 0$$

$$\therefore \lim_{n\to\infty} (-1)^n \sin\left(\tfrac{1}{n}\right) = 0$$

$\boxed{0}$ 2

(e) $a_n = \dfrac{\ln(n^2)}{n} = \dfrac{2\ln n}{n}$

$$\lim_{x\to\infty} \frac{2\ln x}{x} \stackrel{L'H}{=} \lim_{x\to\infty} \frac{\frac{2}{x}}{1} = 0$$

$\frac{\infty}{\infty}$

$$\therefore \lim_{n\to\infty} \frac{\ln(n^2)}{n} = 0$$

$\boxed{0}$ 2