NAME GRADING KEY

STUDENT ID

RECITATION INSTRUCTOR

RECITATION TIME

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Page 2	/30
Page 3	/34
Page 4	/22
TOTAL	/100

DIRECTIONS

1. Write your name, student ID number, recitation instructor's name and recitation time in the space provided above. Also write your name at the top of pages 2, 3, and 4 .
2. The test has four (4) pages, including this one.
3. Write your answers in the boxes provided.
4. You must show sufficient work to justify all answers. Correct answers with inconsistent work may not be given credit.
5. Credit for each problem is given in parentheses in the left hand margin.
6. No books, notes or calculators may be used on this test.

Find the integrals in problems 1-5.

−1 pt for missing +C (1 time for test)

−1 pt for missing dx, du etc (1 time only for each problem.

(6) 1. $\int_0^{\pi/4} \sin^2 x \, dx$

$$= \int_0^{\frac{\pi}{4}} \frac{1}{2}(1-\cos 2x)\,dx$$

$$= \left[\frac{x}{2} - \frac{\sin 2x}{4}\right]_0^{\frac{\pi}{4}} = \frac{\pi}{8} - \frac{1}{4}$$

② ②

②

$$\boxed{\frac{\pi}{8} - \frac{1}{4}}$$

$\boxed{6}$

(8) 2. $\int \tan^3 x \, dx$ (Hint: $\frac{d}{dx} \ln|\cos x| = -\tan x$)

③

$$= \int \tan x(\sec^2 x - 1)\,dx = \int \tan x \sec^2 x\,dx - \int \tan x\,dx$$

$(u = \tan x, \ du = \sec^2 x\,dx)$

$$= \int u\,du + \ln|\cos x|$$

$$\boxed{\frac{\tan^2 x}{2} + \ln|\cos x| + C}$$

$\boxed{8}$

$$= \frac{\tan^2 x}{2} + \ln|\cos x| + C$$

③ ②

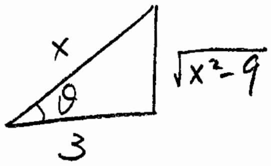
Or $= \frac{\sec^2 x}{2}$ ③ $+ \ln|\cos x|$ ② $+ C$ $\left(\int \tan x \sec^2 x\,dx = \int v\,dv, \ v = \sec x\right)$

(10) 3. $\int \frac{1}{x^2\sqrt{x^2-9}}\,dx = \int \frac{3\sec\theta\tan\theta}{(3\sec\theta)^2(3\tan\theta)}\,d\theta$ ⊛ ④

⊛ 0 pt if integrand is wrong
but -2 pt for wrong
power of 3, and grade
the rest consistently

$(x=3\sec\theta,\ 0\le\theta<\frac{\pi}{2}$ or $\pi\le\theta<\frac{3\pi}{2},\ dx=3\sec\theta\tan\theta\,d\theta)$

$= \int \frac{1}{9}\frac{1}{\sec\theta}\,d\theta = \frac{1}{9}\int\cos\theta\,d\theta$



$= \frac{1}{9}\sin\theta + C$ ③

$= \frac{1}{9}\frac{\sqrt{x^2-9}}{x} + C$

③

$\boxed{\frac{1}{9}\frac{\sqrt{x^2-9}}{x} + C}$ $\boxed{10}$

(10) 4. $\int \frac{1}{(t+4)(t-1)}\,dt$

④

$\frac{1}{(t+4)(t-1)} = \frac{A}{t+4} + \frac{B}{t-1} \Rightarrow 1 = A(t-1) + B(t+4)$

$\begin{cases} 1 = 5B \\ 1 = -5A \end{cases} \Rightarrow \begin{cases} B = \frac{1}{5} \\ A = -\frac{1}{5} \end{cases}$

$\int \frac{1}{(t+4)(t-1)}\,dt = \int \left[-\frac{1}{5}\frac{1}{t+4} + \frac{1}{5}\frac{1}{t-1}\right] dt$

$\boxed{-\frac{1}{5}\ln|t+4| + \frac{1}{5}\ln|t-1| + C}$ $\boxed{10}$

③ ③

(10) 5. $\int \frac{1}{x^2-2x+5}\,dx$

⑤

$= \int \frac{1}{(x-1)^2+4}\,dx = \int \frac{1}{u^2+4}\,du \quad (u=x-1,\ du=dx)$

$= \frac{1}{2}\tan^{-1}\left(\frac{u}{2}\right) + C$ ③

②

$\boxed{\frac{1}{2}\tan^{-1}\left(\frac{x-1}{2}\right) + C}$ $\boxed{10}$

Or $(x-1 = 2\tan\theta,\ dx = 2\sec^2\theta\,d\theta)$

$\int \frac{1}{(x-1)^2+4}\,dx = \int \frac{2\sec^2\theta}{4\tan^2\theta+4}\,d\theta$

$= \int \frac{1}{2}\,d\theta = \frac{\theta}{2} + C$ ③

(12) 6. Consider the partial fraction decomposition:

$$\frac{x^4+x^3-x^2-x+1}{x^3-x} = Ax+B+\frac{C}{x}+\frac{D}{x-1}+\frac{E}{x+1}$$

Find the constants A, B, C, D and E.

$$\begin{array}{r} x+1 \\ x^3-x \,\overline{)\, x^4+x^3-x^2-x+1} \\ \underline{x^4 \quad\quad -x^2 \quad\quad\quad} \\ x^3 \quad\quad -x+1 \\ \underline{x^3 \quad\quad -x \quad} \\ 1 \end{array}$$

$$\therefore \frac{x^4+x^3-x^2-x+1}{x^3-x} = x+1+\frac{1}{x^3-x}$$

$$\frac{1}{x^3-x} = \frac{1}{x(x-1)(x+1)} = \frac{C}{x}+\frac{D}{x-1}+\frac{E}{x+1}$$

$$\Rightarrow 1 = C(x-1)(x+1)+Dx(x+1)+Ex(x-1)$$

$$\Rightarrow \begin{cases} 1 = E(-1)(-2) \\ 1 = C(-1)(1) \\ 1 = D(1)(2) \end{cases} \Rightarrow \begin{cases} E = \frac{1}{2} \\ C = -1 \\ D = \frac{1}{2} \end{cases}$$

$$\boxed{A = 1 \quad , B = 1 \quad , C = -1 \quad , D = \frac{1}{2} \quad , E = \frac{1}{2}}$$

③ ③ ② ② ②

$\boxed{12}$

(12) 7. Determine whether each intergral is convergent or divergent. Find its value if it is convergent. Important: You must show clearly how limits are involved.

a.) $\int_2^\infty \frac{1}{x-1}\,dx = \lim_{t\to\infty}\int_2^t \frac{1}{x-1}\,dx$ ③

$= \lim_{t\to\infty}\left[\ln|x-1|\right]_2^t = \lim_{t\to\infty}\ln|t-1| = \infty$ ②

①

$\boxed{\text{Divergent}}$

$\boxed{6}$

b.) $\int_0^3 \frac{1}{\sqrt{x}}\,dx = \lim_{t\to 0^+}\int_t^3 \frac{1}{\sqrt{x}}\,dx$ ③

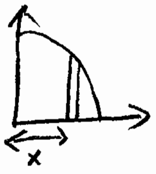
$= \lim_{t\to 0^+}\left[2\sqrt{x}\right]_t^3 = \lim_{t\to 0^+}\left[2\sqrt{3}-2\sqrt{t}\right] = 2\sqrt{3}$ ②

①

$\boxed{2\sqrt{3}}$

$\boxed{6}$

(10) 8. Set up an intergral for the area S of the surface obtained by rotating the curve $y = 1-x^2$, $0 \le x \le 1$ about the y-axis. (DO NOT evaluate the integral.)



$\int_0^1 2\pi x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^1 2\pi x\sqrt{1+4x^2}\,dx$

or $\int_0^1 2\pi x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$

$= \underbrace{\int_0^1}_{①+①} \underbrace{2\pi\sqrt{1-y}}_{③}\,\underbrace{\sqrt{1+\left(\frac{1}{2\sqrt{1-y}}\right)^2}}_{④}\,dy = \int_0^1 2\pi\sqrt{(1-y)+\frac{1}{4}}\,dy$ ①

①

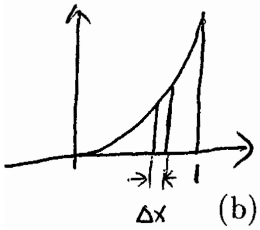
$\boxed{S = \underbrace{\int_0^1}_{①} \underbrace{2\pi x}_{③} \underbrace{\sqrt{1+4x^2}}_{④}\,dx}$ ①

$\boxed{10}$

* 0 credit if more than 1 item is wrong (limits count as one item)

(12) 9. Consider the lamina bounded by the curves $y = x^3$, $y = 0$, $x = 1$ and with density $\rho = 1$. Find the following.

(a) The mass m of the lamina.



$$m = \int_0^1 x^3 dx = \left[\frac{x^4}{4}\right]_0^1 = \frac{1}{4}$$

$m = \frac{1}{4}$ ②

NPC

(b) The moment M_y of the lamina about the y–axis.

$$M_y = \int_0^1 x(x^3 - 0)\,dx = \left[\frac{x^5}{5}\right]_0^1 = \frac{1}{5}$$

$M_y = \frac{1}{5}$ ③

NPC

(c) The moment M_x of the lamina about the x–axis.

$$M_x = \int_0^1 \frac{1}{2}(x^6 - 0)\,dx = \left[\frac{1}{14}x^7\right]_0^1 = \frac{1}{14}$$

$M_x = \frac{1}{14}$ ③

NPC

(d) The center of mass $(\overline{x}, \overline{y})$ of the lamina.

$$\overline{x} = \frac{M_y}{m} = \frac{4}{5}$$

$$\overline{y} = \frac{M_x}{m} = \frac{4}{14}$$

② ②

$(\overline{x}, \overline{y}) = \left(\frac{4}{5}, \frac{2}{7}\right)$

Ok if consistent with abo answer

$\boxed{1}$

(10) 11. Determine whether the sequence converges or diverges. If it converges, find the limit. (You need not show work for this problem.)

NPC

(a) $\left\{\sin\frac{n\pi}{2}\right\}$ $1, 0, -1, 0, 1, 0, \cdots$

diverges ②

(b) $\left\{\cos\frac{\pi}{n}\right\}$ $\lim_{n\to\infty} \cos\frac{\pi}{n} = \cos\left(\lim_{n\to\infty}\frac{\pi}{n}\right)$

1 ②

(c) $a_n = \dfrac{3 + 5n^2}{n + n^2}$ $\lim_{n\to\infty}\dfrac{3+5n^2}{n+n^2} = \lim_{n\to\infty}\dfrac{\frac{3}{n^2}+5}{\frac{1}{n}+1}$

5 ②

(d) $a_n = \dfrac{n + \sin n}{2n}$ $\dfrac{n-1}{2n} \leq \dfrac{n+\sin n}{2n} \leq \dfrac{n+1}{2n}$ as $n\to\infty$; $\dfrac{n-1}{2n} \to \frac{1}{2}$, $\therefore \dfrac{n+\sin n}{2n} \to \frac{1}{2}$, $\dfrac{n+1}{2n} \to \frac{1}{2}$

$\frac{1}{2}$ ②

(e) $a_n = \dfrac{\ln(4 + e^n)}{5n}$

$\frac{1}{5}$ ②

$$\lim_{x\to\infty}\frac{\ln(4+e^x)}{5x} \overset{L'H}{=} \lim_{x\to\infty}\frac{\frac{e^x}{4+e^x}}{5} = \lim_{x\to\infty}\frac{1}{5\left(\frac{4}{e^x}+1\right)} = \frac{1}{5}$$

$\boxed{10}$