NAME GRADING KEY

10–DIGIT PUID ____________

RECITATION INSTRUCTOR ____________

RECITATION TIME ____________

Page 1	/18
Page 2	/32
Page 3	/28
Page 4	/22
TOTAL	/100

DIRECTIONS

1. Write your name, 10–digit PUID, recitation instructor's name and recitation time in the space provided above. Also write your name at the top of pages 2, 3, and 4 .
2. The test has four (4) pages, including this one.
3. Write your answers in the boxes provided.
4. You must show sufficient work to justify all answers unless otherwise stated in the problem. Correct answers with inconsistent work may not be given credit.
5. Credit for each problem is given in parentheses in the left hand margin.
6. No books, notes, calculators nor any electronic devices may be used on this test.

Find the integrals in problems 1-5.

−1 pt for missing $+C$ (one time for test)

−1 pt for missing dx, du, etc (one time for each problem)

(8) 1. $\int \tan^3 x \sec x \, dx = \int \tan^2 x \sec x \tan x \, dx$

$= \int (\sec^2 x - 1) \sec x \tan x \, dx$

$u = \sec x \quad du = \sec x \tan x \, dx$

4

$= \int (u^2 - 1) \, du = \frac{u^3}{3} - u + C$

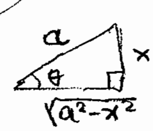
$= \frac{1}{3}\sec^3 x - \sec x + C$

2 2

$$\boxed{\frac{1}{3}\sec^3 x - \sec x + C}$$

8

(10) 2. $\int \frac{x^2}{(a^2 - x^2)^{3/2}} \, dx$



$x = a\sin\theta, \; -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$dx = a\cos\theta \, d\theta$

$\sqrt{a^2 - x^2} = a\cos\theta$

$= \int \frac{a^2 \sin^2\theta}{(a\cos\theta)^3} a\cos\theta \, d\theta = \int \frac{\sin^2\theta}{\cos^2\theta} d\theta = \int \tan^2\theta \, d\theta = \int (\sec^2\theta - 1) d\theta$

4

$= \tan\theta - \theta + C$

$= \frac{x}{\sqrt{a^2 - x^2}} - \sin^{-1}\frac{x}{a} + C$

$$\boxed{\frac{x}{\sqrt{a^2-x^2}} - \sin^{-1}\frac{x}{a} + C}$$

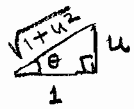
3 3

10

(10) 3. $\int \frac{\cos t}{\sqrt{1+\sin^2 t}}\, dt =$ (Hint: $\frac{d}{dx} \ln|\sec x + \tan x| = \sec x$)

$u = \sin t \quad du = \cos t\, dt$

$= \int \frac{1}{\sqrt{1+u^2}}\, du$ ② $= \int \frac{\sec^2\theta}{\sec\theta}\, d\theta$ ② $= \int \sec\theta\, d\theta$



$u = \tan\theta$

$du = \sec^2\theta\, d\theta$

$\sqrt{1+u^2} = \sec\theta$

$= \ln|\sec\theta + \tan\theta| + C$ ② $= \ln\left|\sqrt{1+u^2} + u\right| + C$ ②

$= \ln\left|\sqrt{1+\sin^2 t} + \sin t\right| + C$ ②

$$\boxed{\ln\left|\sqrt{1+\sin^2 t} + \sin t\right| + C}$$ $\boxed{10}$

(10) 4. $\int \frac{x^2}{x+1}\, dx$

$= \int \left(x - 1 + \frac{1}{x+1} \right) dx$ ④ ④

$= \frac{x^2}{2} - x + \ln|x+1| + C$ ②

$$\begin{array}{r} x - 1 \\ x+1 \,\big)\, \overline{x^2 \qquad} \\ \underline{x^2 + x} \\ -x \\ \underline{-x - 1} \\ 1 \end{array}$$

Or $\int \frac{x^2}{x+1}\, dx = \int \frac{(u-1)^2}{u}\, du$ ⑥

$u = x+1, \ du = dx$

$= \int \frac{u^2 - 2u + 1}{u}\, du = \int (u - 2 + u^{-1})\, du$

$= \frac{u^2}{2} - 2u + \ln|u| + C$ ②

$= \frac{(x+1)^2}{2} - 2(x+1) + \ln|x+1| + C$ ②

$$\boxed{\frac{x^2}{2} - x + \ln|x+1| + C}$$ $\boxed{10}$

(12) 5. $\int \left[\frac{1}{x-1} - \frac{1}{(x-1)^2} - \frac{x}{x^2+1} + \frac{1}{x^2+1} \right] dx$

③ ③ ③ ③

$$\boxed{\ln|x-1| + \frac{1}{x-1} - \frac{1}{2}\ln|x^2+1| + \tan^{-1} x + C}$$ $\boxed{12}$

(9) 6. Determine the constants in the partial fraction decomposition:

$$\frac{3x^2+2x-1}{(x-2)(x^2+2)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+2}$$

$$3x^2+2x-1 = Ax^2+2A+Bx^2+Cx-2Bx-2C$$

$$3x^2+2x-1 = (A+B)x^2+(C-2B)x+2A-2C$$

$$\left\{\begin{array}{l} A+B=3 \longrightarrow B=3-A \\ -2B+C=2 \longrightarrow C=2+2B=2+2(3-A)=8-2A \\ 2A-2C=-1 \longrightarrow 2A-2(8-2A)=-1 \end{array}\right.$$

$$2A-16+4A=-1 \rightarrow A=\frac{15}{6}=\frac{5}{2}$$

$$B=3-\frac{5}{2}=\frac{1}{2}$$

$$C=8-2\cdot\frac{5}{2}=3$$

3 3 3

$A = \frac{5}{2}$, $B = \frac{1}{2}$, $C = 3$

$\boxed{9}$

(19) 7. Determine whether each integral is convergent or divergent. Find its value if it is convergent. Important: You must use the definition of improper intergrals.

a.) $$\int_0^3 \frac{1}{\sqrt{x}}\,dx = \lim_{t\to0^+}\int_t^3 \frac{1}{\sqrt{x}}\,dx \quad 4$$

$$= \lim_{t\to0^+}\left[2\sqrt{x}\right]_t^3 = \lim_{t\to0^+}\left[2\sqrt{3}-2\sqrt{t}\right] \quad 3$$

$$= 2\sqrt{3} \quad 2$$

$2\sqrt{3}$

$\boxed{9}$

b.) $$\int_1^\infty \frac{1}{1+x^2}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{1+x^2}\,dx \quad 4$$

$$= \lim_{t\to\infty}\left[\tan^{-1}x\right]_1^t = \lim_{t\to\infty}\left[\tan^{-1}t-\frac{\pi}{4}\right] \quad 3$$

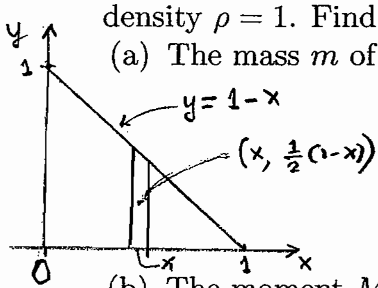
$$= \frac{\pi}{2}-\frac{\pi}{4}=\frac{\pi}{4} \quad 3$$

$\frac{\pi}{4}$

$\boxed{10}$

(12) 8. Consider the lamina bounded by the lines $x + y = 1$, $x = 0$, and $y = 0$ and with density $\rho = 1$. Find the following.

(a) The mass m of the lamina.



NPC

$$m = \int_0^1 1\,(1-x)\,dx = \left[x - \frac{x^2}{2}\right]_0^1 = 1 - \frac{1}{2} = \frac{1}{2}$$

$\boxed{m = \frac{1}{2}}$ (3)

(b) The moment M_y of the lamina about the y–axis.

$$M_y = \int_0^1 x\,1(1-x)\,dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

$\boxed{M_y = \frac{1}{6}}$ (3)

(c) The moment M_x of the lamina about the x–axis.

$$M_x = \int_0^1 \frac{1}{2}(1-x)\,1(1-x)\,dx = \frac{1}{2}\int_0^1 (1 - 2x + x^2)\,dx = \frac{1}{2}\left[x - x^2 + \frac{x^3}{3}\right]_0^1 = \frac{1}{2}\cdot\frac{1}{3} = \frac{1}{6}$$

or $M_x = M_y = \frac{1}{6}$ from symmetry

$\boxed{M_x = \frac{1}{6}}$ (3)

(d) The center of mass $(\overline{x}, \overline{y})$ of the lamina.

$\overline{x}\,m = M_y \longrightarrow \overline{x}\,\frac{1}{2} = \frac{1}{6} \longrightarrow \overline{x} = \frac{1}{3}$

$\overline{y}\,m = M_x \longrightarrow \overline{y}\,\frac{1}{2} = \frac{1}{6} \longrightarrow \overline{y} = \frac{1}{3}$

or $\overline{x} = \overline{y}$ from symmetry

ok if consistent with above

$\boxed{(\overline{x}, \overline{y}) = \left(\frac{1}{3}, \frac{1}{3}\right)}$ (3)

(10) 9. Determine whether the sequence converges or diverges. If it converges, find the limit. (You need not show work for this problem.)

NPC

(a) $a_n = \dfrac{\ln n}{\sqrt{n}}$ $\quad \lim\limits_{n\to\infty} \dfrac{\ln n}{\sqrt{n}} = \lim\limits_{x\to\infty} \dfrac{\ln x}{\sqrt{x}} \overset{L'H}{=} \lim\limits_{x\to\infty} \dfrac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim\limits_{x\to\infty} \dfrac{2\sqrt{x}}{x} = \lim\limits_{x\to\infty} \dfrac{2}{\sqrt{x}}$ $\quad \frac{\infty}{\infty}$

$\boxed{0}$ (2)

(b) $a_n = \dfrac{n \sin n}{n^2+1}$ $\quad -\dfrac{n}{n^2+1} \le \dfrac{n \sin n}{n^2+1} \le \dfrac{n}{n^2+1}$ squeeze th.
$\downarrow 0 \quad \therefore \downarrow 0 \quad \downarrow 0$ as $n\to\infty$

$\boxed{0}$ (2)

(c) $a_n = \ln\left(\dfrac{4n}{2n+1}\right) = \ln\left(\dfrac{4}{2+\frac{1}{n}}\right) \to \ln 2$, as $n\to\infty$

$\boxed{\ln 2}$ (2)

(d) $\left\{\cos\left(\dfrac{2}{n}\right)\right\}$ $\quad \dfrac{2}{n} \to 0$ as $n\to\infty$
$\therefore \cos\dfrac{2}{n} \to \cos 0 = 1$ as $n\to\infty$

$\boxed{1}$ (2)

(e) $a_n = \dfrac{3^n}{2^n(100)} = \dfrac{1}{100}\left(\dfrac{3}{2}\right)^n \longrightarrow \infty$ as $n\to\infty$ since $\frac{3}{2} > 1$.

$\boxed{\text{diverges}}$ (2)