NAME GRADING KEY

STUDENT ID ______________________

RECITATION INSTRUCTOR ______________________

RECITATION TIME ______________________

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DIRECTIONS

1. Write your name, student ID number, recitation instructor's name and recitation time in the space provided above. Also write your name at the top of pages 2, 3, and 4.
2. The test has four (4) pages, including this one.
3. Write your answers in the boxes provided.
4. You must show sufficient work to justify all answers. Correct answers with inconsistent work may not be given credit.
5. Credit for each problem is given in parentheses in the left hand margin.
6. No books, notes or calculators may be used on this test.

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(14) 1. Circle the letter of the correct response. (You need not show work for this problem).

(a) Which of the following statements are true for any series $\sum_{n=1}^{\infty} a_n$? NPC

(I) If $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$ as $n \to \infty$. True by theorem

(II) If $0 \leq a_n \leq 1$ for all $n \geq 1$, then $\sum_{n=1}^{\infty}$ is convergent. Not true: ex $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent

(III) If $\sum_{n=1}^{\infty} a_n$ is convergent, then $\sum_{n=1}^{\infty} |a_n|$ is convergent. Not true: $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ is conv. but $\sum_{n=1}^{\infty} |(-1)^n \frac{1}{n}|$ is div.

A. (I) and (II) only B. (I) only C. (II) only D. (III) only E. none $\boxed{7}$

(b) Which of the following series converge?

(I) $1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \frac{1}{125} + \dots = \sum_{n=1}^{\infty} \frac{1}{n^3}$ converges, p-series $p = 3 > 1$.

(II) $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} + \dots = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ diverges, p-series $p = \frac{1}{2} \leq 1$

(III) $\sum_{n=1}^{\infty} \frac{\cos(n^2)}{n^2}$ converges because it converges absolutely: Compare $\sum_{n=1}^{\infty} \frac{|\cos n^2|}{n^2}$ with $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

A. (I) and (III) only B. (I) only C. (II) only D. (III) only E. none $\boxed{7}$

(27) 2. Determine whether each series is convergent or divergent. You must verify that the conditions of the test are satisfied and write your conclusion in the small box.

(a) $\sum_{n=1}^{\infty} \frac{1}{1+\sqrt{n}}$.

For problems 2(a), 2(b) and 2(c) look first for conv. or div.
If wrong → 0 pts for problem
If right → check work and test

Show all necessary work here :

$\sum_{n=1}^{\infty} \frac{1}{1+\sqrt{n}}$ compare with 2 $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ which 1 diverges (p-series $p=\frac{1}{2}<1$)

2 $\lim_{n\to\infty} \frac{\frac{1}{1+\sqrt{n}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{\sqrt{n}}{1+\sqrt{n}}$

$= \lim_{n\to\infty} \frac{1}{\frac{1}{\sqrt{n}}+1} = 1 > 0$ 1 1

2

By the limit comparison test, the series is divergent

9

(b) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2+1}$.

Show all necessary work here :

Alternating series test $b_n = \frac{n}{n^2+1}$ 1

(i) b_n decreasing? Let $f(x) = \frac{x}{x^2+1}$

$f'(x) = \frac{(x^2+1)1 - x2x}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}$ 2, $\underline{f'(x)<0 \text{ if } x>1}$

$\therefore b_n$ are decreasing for all $n \geq 1$ or 2

(ii) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n}{n^2+1} = 0$ 2

2

By the alternating test, the series is convergent

9

(c) $\sum_{n=1}^{\infty} \frac{10^n}{n!}$.

Show all necessary work here :

Ratio test

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{10^{n+1}}{(n+1)!}}{\frac{10^n}{n!}}\right| = \frac{10^{n+1} n!}{(n+1)!\,10^n} = \frac{10}{n+1} \longrightarrow 0 < 1$$

3 2 1 1

as $n \to \infty$

2

By the ratio test, the series is convergent

9

(8) 3. Find the sum of the series if it is convergent or write "divergent" in the box. No partial credit.

(a) $\sum_{n=1}^{\infty} \frac{10}{3^n} = \frac{10}{3} + \frac{10}{3^2} + \frac{10}{3^3} + \cdots = \frac{10}{3}\left[1 + \frac{1}{3} + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 + \cdots\right.$

geometric series with $r = \frac{1}{3}$ $= \frac{10}{3} \frac{1}{1-\frac{1}{3}} = 5$

5

4

(b) $\sum_{n=0}^{\infty} \frac{3^{n+1}}{2^{n-1}} = \sum_{n=0}^{\infty} 3 \cdot 2 \left(\frac{3}{2}\right)^n$

geometric series with $r = \frac{3}{2}$

divergent ; $|r| \geq 1$

divergent

4

(8) 4. Consider the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3}$.

(a) Write out the first six terms of the series.

$1 - \frac{1}{2^3} + \frac{1}{3^3} - \frac{1}{4^3} + \frac{1}{5^3} - \frac{1}{6^3} + \cdots$

$1 - \frac{1}{8} + \frac{1}{27} - \frac{1}{64} + \frac{1}{125} - \frac{1}{216} + \cdots$ or 4

−1 pt for only one wrong term or for all signs reversed

(b) Find the smallest number of terms that we need to add in order to estimate the sum of the series with error < 0.01.

4 NPC

$\frac{1}{125} < 0.01$

4

8

(7) 5. Circle the letter of the correct response. The series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ is

conv (alt ser. test) but not abs. conv. (p-series $p = \frac{1}{2} < 1$)

NPC

A. absolutely convergent B. conditionally convergent C. divergent

7

(16) 6. For the power series $\sum_{n=1}^{\infty} \frac{x^n}{n3^n}$, find the following, showing all work.

(a) The radius of convergence R.

Ratio test

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{x^{n+1}}{(n+1)3^{n+1}}}{\frac{x^n}{n3^n}}\right| = \left|\frac{x^{n+1}\, n3^n}{(n+1)3^{n+1}\, x^n}\right| = \underbrace{\frac{n}{n+1}\,\frac{1}{3}\,|x|}_{4} \longrightarrow \frac{1}{3}|x|, \text{ as } n \to \infty \quad 2$$

$\therefore$ series converges if $\frac{1}{3}|x| < 1$ or $|x| < 3$ $\therefore R = 3$

$R =$ 3 (2)

8

(b) The interval of convergence. (Don't forget to check the end points).

Series converges if $-3 < x < 3$ (2)

When $x = -3$: $\sum_{n=1}^{\infty} \frac{(-3)^n}{n3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges (alt. ser. test) (2)

When $x = 3$: $\sum_{n=1}^{\infty} \frac{3^n}{n3^n} = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges (harmonic series) (2)

Interval of convergence
$[-3, 3)$ (1) (1)
or $-3 \le x < 3$

8

(10) 7. Find the power series representation of $\frac{x^2}{1+2x}$ (about $a = 0$) and give its interval of convergence.

$$\frac{1}{1+2x} = \frac{1}{1-(-2x)} = \sum_{n=0}^{\infty} (-2x)^n = \sum_{n=0}^{\infty} (-1)^n 2^n x^n,$$

$|-2x| < 1$ or $-\frac{1}{2} < x < \frac{1}{2}$

$$\frac{x^2}{1+2x} = x^2 \sum_{n=0}^{\infty} (-1)^n 2^n x^n = \sum_{n=0}^{\infty} (-1)^n 2^n x^{n+2}$$

$\frac{x^2}{1+2x} = \sum_{n=0}^{\infty} (-1)^n 2^n x^{n+2}$ (1) (5) (1)
interval of convergence $\left(-\frac{1}{2}, \frac{1}{2}\right)$ or $-\frac{1}{2} < x < \frac{1}{2}$ (3)

10

(10) 8. Find the first three nonzero terms of the Taylor series of $f(x) = \ln x$ centered at $a = 1$.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n = f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \cdots$$

or (4)

$f(x) = \ln x$ $\quad f(1) = 0$

$f'(x) = x^{-1}$ $\quad f'(1) = 1$

$f''(x) = -x^{-2}$ $\quad f''(1) = -1$

$f'''(x) = 2x^{-3}$ $\quad f'''(1) = 2$

$$\ln x = 0 + \frac{1}{1}(x-1) + \frac{-1}{2}(x-1)^2 + \frac{2}{6}(x-1)^3$$

(2) (2) (2)

$(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$

10