NAME GRADING KEY

10–DIGIT PUID ______

RECITATION INSTRUCTOR ______

RECITATION TIME ______

Page 1	/24
Page 2	/20
Page 3	/26
Page 4	/30
TOTAL	/100

DIRECTIONS

1. Write your name, 10–digit PUID, recitation instructor's name and recitation time in the space provided above. Also write your name at the top of pages 2, 3, and 4.
2. The test has four (4) pages, including this one.
3. Write your answers in the boxes provided.
4. You must show sufficient work to justify all answers unless otherwise stated in the problem. Correct answers with inconsistent work may not be given credit.
5. Credit for each problem is given in parentheses in the left hand margin.
6. No books, notes, calculators, or any electronic devices may be used on this test.

---

(12) 1. Determine whether the following statements are true or false for any series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$. (Circle T or F. You do not need to show work).

4 pts each NPC

(a) If $0 < a_n < b_n$ for all n and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges. comparison test — T F

(b) If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges. $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges — T F

(c) If $\sum_{n=1}^{\infty} |a_n|$ is convergent, then $\sum_{n=1}^{\infty} (-1)^n a_n$ is convergent. — T F

$\sum_{n=1}^{\infty} (-1)^n a_n$ is absolutely convergent and hence convergent $\boxed{12}$

(12) 2. Determine whether each of the following series is convergent or divergent. (You do not need to show work).

4 pts each NPC

(a) $\sum_{n=1}^{\infty} (\frac{2}{n\sqrt{n}} + 3n^{-1.1}) = 2\underbrace{\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}}_{\text{conv. p-ser. } p=\frac{3}{2}>1} + 3\underbrace{\sum_{n=1}^{\infty} \frac{1}{n^{1.1}}}_{\text{conv. p-ser. } p=1.1>1}$ $\boxed{\text{convergent}}$ $\boxed{4}$

(b) $\sum_{n=1}^{\infty} \frac{n}{(n+1)2^n}$ compare with $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ which conv. geom. ser. $r=\frac{1}{2}$, $|\frac{1}{2}|<1$

$\frac{n}{(n+1)2^n} < \left(\frac{1}{2}\right)^n$ for all $n \geq 1$, $\boxed{\text{convergent}}$ $\boxed{4}$

(c) $\sum_{n=1}^{\infty} \frac{n^2-7}{n^3+2n^2-1} = -3 - \frac{1}{5} + \sum_{n=3}^{\infty} \frac{n^2-7}{n^3+2n^2-1}$ $\boxed{\text{divergent}}$ $\boxed{4}$

div. by lim. comp. test

compare with $\sum_{n=3}^{\infty} \frac{1}{n}$

(30) 3. Determine whether each series is convergent or divergent. You must verify that the conditions of the test you are using are satisfied and write your conclusion in the small box.

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$

In problems 3(a)(b)(c) look first for conv. or div.
If wrong → 0 points for problem
If right → check work and test. | If there is no work → 0 points for problem

In problems 3(a)(b)(c) and 6(a): if lim or → notation is wrong, -1pt for that problem

Show all necessary work here: Integral test $\displaystyle\int_2^{\infty} \frac{1}{x\sqrt{\ln x}}\,dx$

$f(x) = \dfrac{1}{x\sqrt{\ln x}}$ is continuous, positive and decresing in $[2, \infty)$

$$\int_2^{\infty} \frac{1}{x\sqrt{\ln x}}\,dx \overset{2}{=} \lim_{t\to\infty} \int_2^{t} \frac{1}{x\sqrt{\ln x}}\,dx \overset{2}{=} \lim_{t\to\infty} \Big[2\sqrt{\ln x}\Big]_2^t$$

$$= \lim_{t\to\infty} \Big[2\sqrt{\ln t} \overset{2}{-} 2\sqrt{\ln 2}\Big] = \overset{2}{\infty}$$

integral is divergent and ∴ series is divergent

By the integral (2) test, the series is divergent

10

(b) $\displaystyle\sum_{n=1}^{\infty} e^{-n} n!$

Show all necessary work here: Ratio test

$$\overset{4}{\left|\frac{a_{n+1}}{a_n}\right|} = \frac{e^{-n-1}(n+1)!}{e^{-n}\, n!} \overset{2}{=} e^{-1}(n+1) \xrightarrow{\ 2\ } \infty, \text{ as } n\to\infty$$

∴ ser. is div.

Or $\quad a_n = \dfrac{n!}{e^n} = \dfrac{1\cdot 2\cdot 3\cdot 4\cdots n}{e\cdot e\cdot e\cdot e\cdots e} \geq \dfrac{2}{e^2}$ for all $n \geq 3$

∴ $\displaystyle\lim_{n\to\infty} a_n \neq 0$

and the series is divergent by the test for divergence

By the ratio (2) test, the series is divergent

10

(c) $\sum_{n=2}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+4}$

Show all necessary work here: Alternating series test

$b_n = \frac{n^2}{n^3+4}$ $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n^2}{n^3+4} = 0$ 2

$b_{n+1} \le b_n$ for all $n \ge 2$?

Let $f(x) = \frac{x^2}{x^3+4}$

$f'(x) = \frac{(x^3+4)2x - x^2 3x^2}{(x^3+4)^2} = \frac{2x^4+8x-3x^4}{(x^3+4)^2}$ 2

$= \frac{8x-x^4}{(x^3+4)^2} = \frac{x}{(x^3+4)^2}(8-x^3) < 0,$ 2 2 for $x > 2$

2

By the alternating series test, the series is convergent

$\boxed{10}$

(4) 4. Find the sum of the series $\sum_{n=1}^{\infty} \frac{2^{2n+1}}{5^n}$.

NPC

$= 2\sum_{n=1}^{\infty}\left(\frac{4}{5}\right)^n = 2\cdot\left(\frac{4}{5}\right)\sum_{n=1}^{\infty}\left(\frac{4}{5}\right)^{n-1} = \frac{8}{5}\,\frac{1}{1-\frac{4}{5}} = 8$

$\boxed{8}$

$\boxed{4}$

(12) 5. For each function f, find the Maclaurin series and its radius of convergence. You may use known series to get your answer.

(a) $f(x) = xe^{2x}$

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \; -\infty < x < \infty$

$e^{2x} = \sum_{n=0}^{\infty} \frac{2^n x^n}{n!}, \; -\infty < x < \infty$

$xe^{2x} = \sum_{n=0}^{\infty} \frac{2^n x^{n+1}}{n!}, \; -\infty < x < \infty$

1 4 1

$xe^{2x} = \sum_{n=0}^{\infty} \frac{2^n x^{n+1}}{n!}$, $R = \infty$

$\boxed{6}$

1 pt if both limits are correct

(b) $f(x) = \sin(x^2)$

$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \; -\infty < x < \infty$

1 4 1

$\sin(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!}$, $R = \infty$

$\boxed{6}$

or

$\sin(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!}, \; -\infty < x < \infty$

(16) 6. For the power series $\sum_{n=0}^{\infty} n^3(x-5)^n$, find the following, showing all work.

(a) The radius of convergence R.

Ratio test: $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)^3(x-5)^{n+1}}{n^3(x-5)^n}\right| = \left(\frac{n+1}{n}\right)^3 |x-5| \longrightarrow |x-5|$, as $n\to\infty$ 4 2

$\therefore$ series converges if $|x-5|<1$ or $4<x<6$ 2

and diverges if $|x-5|>1$.

$R = 1$ 2

(b) The interval of convergence. (Don't forget to check the end points).

Series converges if $4<x<6$

When $x=4$: $\sum_{n=0}^{\infty} n^3(-1)^n$ diverges because $\lim_{n\to\infty} a_n = \lim_{n\to\infty}(-1)^n n^3$ DNE 2

When $x=6$: $\sum_{n=0}^{\infty} n^3$ diverges because $\lim_{n\to\infty} a_n = \infty$ 2

Interval of convergence

1 $(4, 6)$ 1

or $4<x<6$

16

9) 7. Write out all the terms of the Taylor series for $f(x) = 1 + x + x^2$ centered at $a = 2$.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n = \frac{f(2)}{0!} + \frac{f'(2)}{1!}(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \cdots$$

$f(x) = 1+x+x^2$ $\quad f(2) = 7$

$f'(x) = 1+2x$ $\quad f'(2) = 5$

$f''(x) = 2$ $\quad f''(2) = 2$

$f'''(x) = 0$ $\quad f'''(2) = 0$

3 3 3

$7 + 5(x-2) + (x-2)^2$

9

(5) 8. The Taylor series for $f(x) = \ln x$ centered at $a = 2$ is

$$f(x) = \ln x = \ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2^n(n)}(x-2)^n.$$

Find $f^{(166)}(2)$. Leave your answer in terms of powers and factorials.

$$\frac{f^{(n)}(2)}{n!} = \frac{(-1)^{n-1}}{2^n(n)}$$

$$\frac{f^{(166)}(2)}{166!} = \frac{(-1)^{165}}{2^{166}(166)} = -\frac{1}{2^{166}(166)}$$

$$f^{(166)}(2) = -\frac{165!}{2^{166}}$$

NPC

$f^{(166)}(2) = -\frac{165!}{2^{166}}$

5