NAME GRADING KEY

STUDENT ID # ______

RECITATION INSTRUCTOR ______

RECITATION TIME ______

LECTURER ______

INSTRUCTIONS

1. There are 10 different test pages (including this cover page). Make sure you have a complete test.
2. Fill in the above items in print. I.D.# is your 9 digit ID (probably your social security number). Also write your name at the top of pages 2–10.
3. Do any necessary work for each problem on the space provided or on the back of the pages of this test booklet. Circle your answers in this test booklet. No partial credit will be given, but if you show your work on the test booklet, it may be used in borderline cases.
4. No books, notes or calculators may be used on this exam.
5. Each problem is worth 8 points. The maximum possible score is 200 points.
6. Using a #2 pencil, fill in each of the following items on your answer sheet:
   (a) On the top left side, write your name (last name, first name), and fill in the little circles.
   (b) On the bottom left side, under SECTION, write in your division and section number and fill in the little circles. (For example, for division 9 section 1, write 0901. For example, for division 38 section 2, write 3802).
   (c) On the bottom, under STUDENT IDENTIFICATION NUMBER, write in your student ID number, and fill in the little circles.
   (d) Using a #2 pencil, put your answers to questions 1–25 on your answer sheet by filling in the circle of the letter of your response. Double check that you have filled in the circles you intended. If more than one circle is filled in for any question, your response will be considered incorrect. Use a #2 pencil.
7. After you have finished the exam, hand in your answer sheet and your test booklet to your recitation instructor.

1. Which of the following statements are always true for any three dimensional vectors $\vec{a}$ and $\vec{b}$.

(I) $\vec{a} \cdot \vec{a} = 0$

(II) $(\vec{a} \times \vec{b}) \cdot \vec{a} = 0$

(III) $(\vec{a} \times \vec{b}) \times \vec{a} = \vec{0}$

A. I only
B. II only
C. III only
D. II and III only
E. none

(I) not always true $\vec{a} \cdot \vec{a} = |\vec{a}|^2$
True only for $\vec{a} = \vec{0}$

(II) True: $\vec{a} \times \vec{b}$ is $\perp$ to both $\vec{a}$ and $\vec{b}$

(III) not always true : ex $(\vec{i} \times \vec{j}) \times \vec{i} = \vec{k} \times \vec{i} = \vec{j}$

2. Let $\vec{a} = x\vec{i} - \vec{j} + 10\vec{k}$ and $\vec{b} = x\vec{i} + 7x\vec{j} + \vec{k}$. For which value of x is the vector projection of $\vec{b}$ onto $\vec{a}$ equal to $\vec{0}$? (i.e. $\text{proj}_{\vec{a}}\vec{b} = \vec{0}$)

$$\text{proj}_{\vec{a}} \vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2} \vec{a}$$

$$= \frac{x^2 - 7x + 10}{|\vec{a}|^2} \vec{a}$$

$$\text{proj}_{\vec{a}} \vec{b} = 0 \longrightarrow x^2 - 7x + 10 = 0$$

$$(x-5)(x-2) = 0$$

$$x = 2, 5$$

A. $x = 10$
B. $x = 3$
C. $x = 7$
D. $x = 0$
E. $x = 5$

3. Find the value of a so that the graph of $x^2 + y^2 + z^2 - 2x + 4y - 8z = a$ is a sphere of radius 5.

$$x^2 - 2x + 1 + y^2 + 4y + 4 + z^2 - 8z + 16 = a + 1 + 4 + 16$$

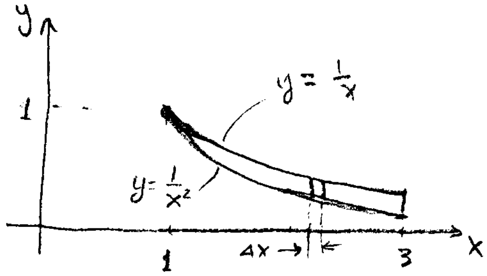
$$(x-1)^2 + (y+2)^2 + (z-4)^2 = a + 21$$

$$a + 21 = 25$$

$$\therefore a = 4$$

A. 5
B. 8
C. 3
D. 10
E. 4

4. Find the area of the region enclosed by the curves $y = \frac{1}{x}$, $y = \frac{1}{x^2}$, and $x = 3$.



$$\Delta A = \left[\frac{1}{x} - \frac{1}{x^2}\right]\Delta x$$

$$A = \int_1^3 \left[\frac{1}{x} - \frac{1}{x^2}\right]dx$$

$$= \left(\ln x + \frac{1}{x}\right)\Big|_1^3$$

$$= \ln 3 + \frac{1}{3} - \ln 1 - 1$$

$$= \ln 3 - \frac{2}{3}$$

A. $\ln 3 - \frac{2}{3}$

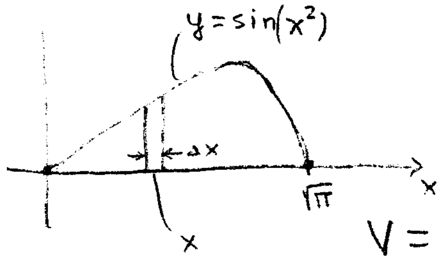
B. $\ln 4 - \frac{1}{2}$

C. $\ln 3 + \frac{5}{2}$

D. $3\ln 4 - 1$

E. 2

5. Let R be the region bounded by the graph of $y = \sin(x^2)$ and the x-axis, for $0 \le x \le \sqrt{\pi}$. The volume of the solid obtained by rotating R about the y-axis is



Volume of typical cylindrical shell:

$$\Delta V = 2\pi x \sin(x^2)\,\Delta x$$

$$V = \int_0^{\sqrt{\pi}} 2\pi x \sin x^2\,dx$$

$$= \left[-\pi\cos(x^2)\right]_0^{\sqrt{\pi}}$$

$$= -\pi\cos\pi + \pi\cos 0 = 2\pi$$

A. $\frac{1}{2}\pi$

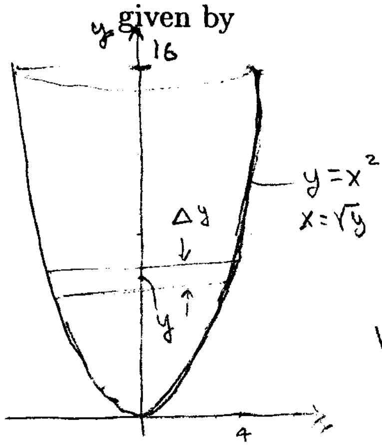
B. π^2

C. $\frac{1}{2}\pi^2$

D. 2π

E. π

6. A tank has the shape of the surface generated by rotating the curve $y = x^2$, $0 \le x \le 4$, about the y-axis. If the tank is full of water at 62.5 lb./ft^3, and the dimensions of the tank are measured in feet, the work required to pump all the water over the top is given by



Weight of layer of water with thickness Δy located at y:

$$62.5(\pi x^2)\Delta y$$

$$= 62.5\,\pi y \Delta y$$

$$W = \int_0^{16} (16-y)\,62.5\pi y\,dy$$

$$= 62.5\pi \int_0^{16} (16-y)y\,dy$$

A. $62.5\pi \int_0^4 (16-y)y^2\,dy$

B. $62.5\pi \int_0^{16} (16-y)y\,dy$

C. $62.5\pi \int_0^{16} (4-y)y^2\,dy$

D. $62.5\pi \int_0^4 y^2\,dy$

E. $62.5\pi \int_0^{16} y^2\,dy$

7. $\int_0^1 x^2 e^x\,dx = x^2e^x\Big|_0^1 - \int_0^1 2xe^x\,dx$

$u = x^2 \quad dv = e^x dx$

$du = 2x \quad v = e^x$

$= e - \int_0^1 2xe^x\,dx = e - \left[2xe^x\Big|_0^1 - \int_0^1 2e^x\,dx\right]$

$u = 2x \quad dv = e^x dx$

$du = 2dx \quad v = e^x$

$= e - 2e + \left[2e^x\right]_0^1 = -e + 2e - 2$

$= e - 2$

A. e^2

B. $2e + 1$

C. $e - 2$

D. $3e - 2$

E. $2e^2 - e$

8. $\int_0^{\frac{\pi}{4}} \tan^4 x \sec^4 x\,dx = \int_0^{\frac{\pi}{4}} \tan^4 x\,(1 + \tan^2 x)\sec^2 x\,dx$

$u = \tan x \quad du = \sec^2 x\,dx$

$x = 0 \to u = 0$

$x = \frac{\pi}{4} \to u = 1$

$= \int_0^1 (u^4 + u^6)\,du$

$= \left[\frac{u^5}{5} + \frac{u^7}{7}\right]_0^1 = \frac{1}{5} + \frac{1}{7} = \frac{12}{35}$

A. $\frac{12}{17}$

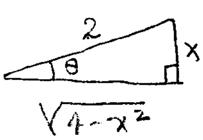
B. $\frac{11}{28}$

C. $\frac{12}{35}$

D. $\frac{41}{11}$

E. $\frac{9}{7}$

9. The integral $\int_0^{\sqrt{2}} \frac{8}{x^2\sqrt{4-x^2}}\,dx$ is transformed by a suitable trigonometric substitution to the integral



$x = 2\sin\theta \quad dx = 2\cos\theta\,d\theta$

$\sqrt{4-x^2} = 2\cos\theta$

$x = 0 \to \theta = 0$

$x = \sqrt{2} \to \theta = \frac{\pi}{4}$

$\int_0^{\frac{\pi}{4}} \frac{8}{4\sin^2\theta \cdot 2\cos\theta}\,2\cos\theta\,d\theta$

$= 2\int_0^{\frac{\pi}{4}} \csc^2\theta\,d\theta$

A. $2\int_0^{\frac{\pi}{2}} \csc^2\theta\sec\theta\,d\theta$

B. $2\int_0^{\frac{\pi}{6}} \csc^2\theta\sec\theta\,d\theta$

C. $2\int_0^{\frac{\pi}{6}} \csc^2\theta\,d\theta$

D. $2\int_0^{\frac{\pi}{4}} \csc^2\theta\,d\theta$

E. $2\int_0^{\sqrt{2}} \csc\theta\sec\theta\,d\theta$

10. $\displaystyle\int_1^2 \frac{1}{x^2+2x}\,dx =$

$$\frac{1}{x^2+2x} = \frac{1}{x(x+2)} = \frac{A}{x} + \frac{B}{x+2}$$

$$1 = Ax + 2A + Bx$$

$$\left.\begin{array}{l} A+B=0 \\ 2A=1 \end{array}\right] \rightarrow \begin{array}{l} A=\frac{1}{2} \\ B=-\frac{1}{2} \end{array}$$

$$\int_1^2 \frac{1}{x^2+2x}\,dx = \int_1^2 \left(\frac{1}{2}x - \frac{1}{2}\frac{1}{x+2}\right)dx$$

$$= \left[\frac{1}{2}\ln x - \frac{1}{2}\ln(x+2)\right]_1^2$$

$$= \frac{1}{2}\ln 2 - \frac{1}{2}\ln 4 - \frac{1}{2}\ln 1 + \frac{1}{2}\ln 3 = \frac{1}{2}\ln\frac{2\cdot 3}{4} = \frac{1}{2}\ln\frac{3}{2}$$

A. $\ln 2 - \ln 3$

B. $\frac{1}{2}\ln\frac{3}{2}$

C. $1 - \frac{1}{2}\ln 2$

D. $\frac{1}{2}\ln 2$

E. $\frac{1}{2}\ln 2 + \ln 3$

11. Find the length of the curve $y = f(x)$, $2 \le x \le 3$, given that $f'(x) = \sqrt{x^2-1}$.

$$L = \int_2^3 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

$$= \int_2^3 \sqrt{1 + x^2 - 1}\,dx$$

$$= \int_2^3 x\,dx = \left.\frac{x^2}{2}\right|_2^3$$

$$= \frac{9}{2} - \frac{4}{2} = \frac{5}{2}$$

A. $\frac{5}{4}$

B. 5

C. 3

D. $\frac{3}{2}$

E. $\frac{5}{2}$

12. The area between the graph of $y = \dfrac{1}{1+x^2}$ and the x-axis, for $-\infty < x < \infty$, is

$$A = \int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$$

$$= 2\int_0^{\infty} \frac{1}{1+x^2}\,dx$$

$$= \lim_{t\to\infty} 2\int_0^t \frac{1}{1+x^2}\,dx = \lim_{t\to\infty}\left[2\tan^{-1}x\right]_0^t$$

$$= \lim_{t\to\infty}(2\tan^{-1}t) = 2\,\frac{\pi}{2} = \pi$$

A. 0

B. 2π

C. $\frac{\pi}{2}$

D. π

E. ∞

13. $\lim_{n\to\infty}\left(\sqrt[n]{n}+\frac{(-1)^n n}{3n^2-2}\right) =$

$$= \lim_{n\to\infty} \sqrt[n]{n} + \lim_{n\to\infty} \frac{(-1)^n n}{3n^2-2}$$

$$= 1 + 0 = 1$$

A. 0

B. 1

C. $\frac{4}{3}$

D. $\frac{1}{3}$

E. does not exist

14. $\sum_{n=1}^{\infty} \frac{2^{n+1}}{3^{n-1}} = \frac{2^2}{3^0} + \frac{2^3}{3^1} + \frac{2^4}{3^2} + \cdots$

$$= 2^2\left[1+\frac{2}{3}+\left(\frac{2}{3}\right)^2+\cdots\right]$$

$$= 2^2 \frac{1}{1-\frac{2}{3}}$$

$$= 2^2 \frac{1}{\frac{1}{3}} = 12$$

A. 3

B. 12

C. $\frac{2}{3}$

D. 1

E. 4

15. For the series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{(n^2+1)^p}$ the only true statement is

$$\frac{n}{(n^2+1)^p} < \frac{n}{n^{2p}} = \frac{1}{n^{2p-1}}$$

for all $n \geq 1$

$\therefore$ the series converges absolutely when $2p-1>1$ or $p>1$, by the comparison test (or limit comparison test)

A. converges absolutely when $p=1$

B. converges conditionally when $p=\frac{1}{2}$

C. converges absolutely when $p>1$

D. diverges when $p=\frac{2}{3}$

E. converges for all $p>0$

16. If $|a_1| = 1$ and $|a_{n+1}| = \dfrac{6n+3}{7n+1}|a_n|$ for all $n \geq 1$, the only true statement is

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{6n+3}{7n+1} = \frac{6}{7} < 1$$

$\therefore \sum_{n=1}^{\infty} a_n$ converges absolutely by the ratio test

A. $\lim_{n\to\infty} a_n = \dfrac{6}{7}$

B. $\sum_{n=1}^{\infty} a_n$ converges absolutely

C. $\sum_{n=1}^{\infty} a_n$ diverges

D. $\sum_{n=1}^{\infty} |a_n|$ diverges

E. $\sum_{n=1}^{\infty} 2^n|a_n|$ converges

17. For the power series $\sum_{n=0}^{\infty} \underbrace{\dfrac{nx^n}{4^n}}_{a_n}$, the radius of convergence R and the interval of convergence are

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)x^{n+1}}{4^{n+1}}\,\frac{4^n}{n\,x^n}\right| = \lim_{n\to\infty}\frac{n+1}{n}\frac{|x|}{4} = \frac{|x|}{4}$$

$\therefore$ series conv. for $|x| < 4$ or $-4 < x < 4$

$\therefore R = 4$

when $x = 4$: $\sum_{n=1}^{\infty} \dfrac{n\,4^n}{4^n} = \sum_{n=1}^{\infty} n$ div.

when $x = -4$: $\sum_{n=1}^{\infty} \dfrac{n(-4)^n}{4^n} = \sum_{n=1}^{\infty} (-1)^n n$ div.

A. $R = 4,\ (-4, 4)$

B. $R = 4,\ [-4, 4)$

C. $R = 4,\ [-4, 4]$

D. $R = 4,\ (-4, 4]$

E. $R = 2,\ [-2, 2]$

18. Use known Maclaurin series to find the coefficient of x^3 in the Maclaurin series for $f(x) = \dfrac{e^{-x}}{1+x^2}$.

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$$

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots$$

$$e^{-x}\frac{1}{1+x^2} = \left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \cdots\right)\left(1 - x^2 + x^4 \cdots\right)$$

$$= 1 - x^2 + x^4 - \cdots \quad - x + x^3 - x^5 + \cdots$$
$$+ \frac{x^2}{2} - \frac{x^4}{2} + \cdots \quad - \frac{x^3}{6} + \frac{x^5}{6} - \cdots$$

$$1 - \frac{1}{6} = \frac{5}{6}$$

A. 1

B. $-\dfrac{1}{2}$

C. $-\dfrac{1}{6}$

D. -1

E. $\dfrac{5}{6}$

19. The coefficient of $\left(x-\frac{\pi}{2}\right)^3$ in the Taylor series for $f(x) = x\sin x$ centered at $a = \frac{\pi}{2}$ is

The coefficient of $\left(x-\frac{\pi}{2}\right)^3$ is $\frac{f^{(3)}(\frac{\pi}{2})}{3!}$

$f(x) = x\sin x$

$f^{(1)}(x) = x\cos x + \sin x$

$f^{(2)}(x) = -x\sin x + \cos x + \cos x$

$f^{(3)}(x) = -x\cos x - \sin x - 2\sin x$

$f^{(3)}\left(\frac{\pi}{2}\right) = -3 \qquad \frac{f^{(3)}(\frac{\pi}{2})}{3!} = -\frac{1}{2}$

A. $-\frac{1}{2}$

B. $-\frac{\pi}{2}$

C. 1

D. $\frac{1}{3!}$

E. 0

20. The curve C is given by the parametric equations

$x = \cos t,\ y = t\sin t;\ 0 \le t \le \pi.$

Find the slope of the tangent line to C at the point $(x, y) = \left(\frac{1}{2}, \frac{\pi}{2\sqrt{3}}\right)$.

$(x,y) = \left(\frac{1}{2}, \frac{\pi}{2\sqrt{3}}\right)$ corresponds to $t = \frac{\pi}{3}$

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{t\cos t + \sin t}{-\sin t}$

When $t = \frac{\pi}{3}$, $\frac{dy}{dx} = \frac{\frac{\pi}{3}\frac{1}{2} + \frac{\sqrt{3}}{2}}{-\frac{\sqrt{3}}{2}}$

$= -\left(\frac{\pi}{3\sqrt{3}} + 1\right)$

A. -1

B. $-\left(1 + \frac{\pi}{3\sqrt{3}}\right)$

C. -2

D. $1 - \frac{\pi}{2\sqrt{3}}$

E. 1

21. The graph of the polar equation $r = 2\sec\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, is a

$r\cos\theta = 2$

$x = 2$

vertical line

A. horizontal line

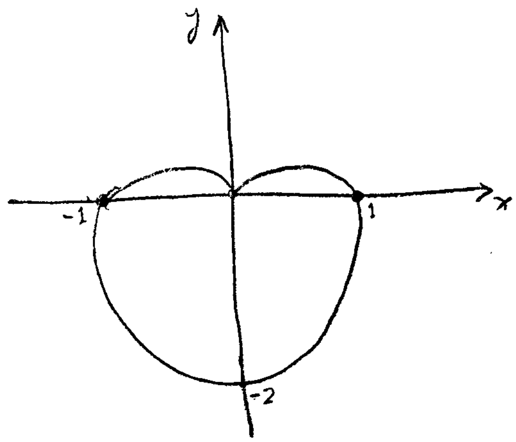
B. circle

C. vertical line

D. cardioid

E. rose

22. The graph of the polar equation $r = 1 - \sin\theta$ is



A. a circle

B. a cardioid symmetric about the line $\theta = 0$

C. a cardioid symmetric about the line $\theta = \frac{\pi}{2}$

D. a cardioid symmetric about the origin

E. a rose

23. The left half ($x \le 0$) of the circle $x^2 + (y-1)^2 = 1$ is described in polar coordinates by

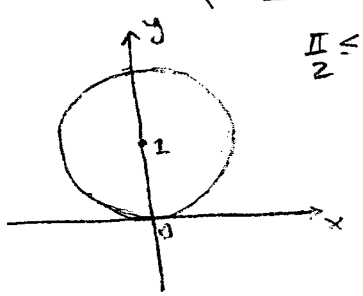
$x^2 + y^2 - 2y + 1 = 1$

$x^2 + y^2 = 2y$

$r^2 = 2r\sin\theta$

$r = 2\sin\theta$

$\frac{\pi}{2} \le \theta \le \pi$



A. $r = \sin\theta,\ \frac{\pi}{2} \le \theta \le \pi$

B. $r = 2\sin\theta,\ \frac{\pi}{2} \le \theta \le \pi$

C. $r = \cos\theta,\ -\frac{\pi}{2} \le \theta \le 0$

D. $r = 2\sin\theta,\ 0 \le \theta \le \pi$

E. $r = -\cos\theta,\ 0 \le \theta \le \frac{\pi}{2}$

24. The integral $\frac{1}{2}\int_0^a \sin^2(4\theta)d\theta$ gives the area of one leaf of the rose $r = \sin 4\theta$, provided $a =$

For 1 leaf:

$4\theta : 0 \to \pi$

$\theta : 0 \to \frac{\pi}{4}$

$\therefore\ a = \frac{\pi}{4}$

A. $\frac{\pi}{4}$

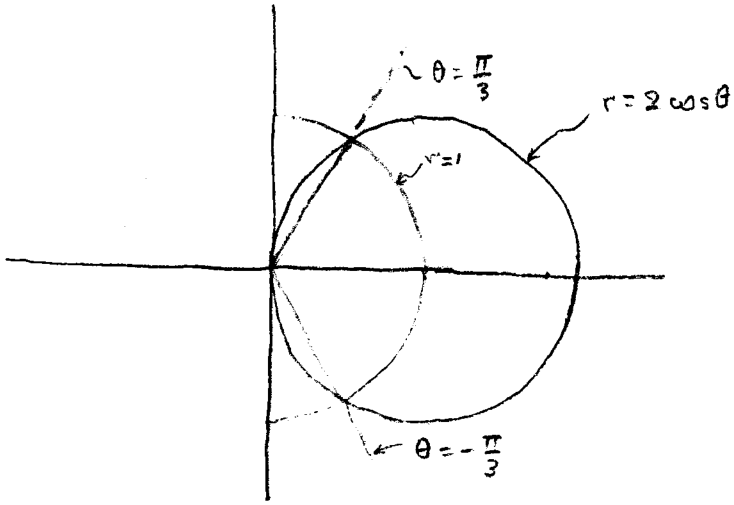
B. $\frac{\pi}{3}$

C. $\frac{\pi}{2}$

D. π

E. $\frac{2\pi}{3}$

25. The area of the region that lies inside the circle $r = 2\cos\theta$ and outside the circle $r = 1$ is



A. $\frac{\pi}{3}$

B. $2\left(\frac{\pi}{3} - \frac{1}{2}\right)$

C. $\frac{\pi}{2} - \frac{\sqrt{3}}{2}$

D. $\frac{\pi}{3} + \frac{\sqrt{3}}{2}$

E. $\frac{\pi}{3} - \frac{\sqrt{3}}{2}$

$r = 1$, $r = 2\cos\theta \longrightarrow 1 = 2\cos\theta \to \cos\theta = \frac{1}{2}$

$\theta = -\frac{\pi}{3}, \frac{\pi}{3}$

$$A = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{1}{2}(2\cos\theta)^2 d\theta - \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{1}{2}\, d\theta$$

$$= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} 2\cos^2\theta\, d\theta - \left[\frac{1}{2}\theta\right]_{-\frac{\pi}{3}}^{\frac{\pi}{3}}$$

$$= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} (1 + \cos 2\theta)\, d\theta - \frac{1}{2}\left[\frac{\pi}{3} - \left(-\frac{\pi}{3}\right)\right]$$

$$= \left[\theta + \frac{\sin 2\theta}{2}\right]_{-\frac{\pi}{3}}^{\frac{\pi}{3}} - \frac{\pi}{3}$$

$$= \frac{\pi}{3} + \frac{\sin\frac{2\pi}{3}}{2} - \left[-\frac{\pi}{3} - \frac{\sin\frac{2\pi}{3}}{2}\right] - \frac{\pi}{3}$$

$$= \frac{\pi}{3} + \sin\frac{2\pi}{3} = \frac{\pi}{3} + \frac{\sqrt{3}}{2}$$