NAME GRADING KEY + SOLUTIONS

STUDENT ID # ____________________

RECITATION INSTRUCTOR ____________________

RECITATION TIME ____________________

LECTURER ____________________

INSTRUCTIONS

1. There are 10 different test pages (including this cover page). Make sure you have a complete test.
2. Fill in the above items in print. I.D.# is your 9 digit ID (probably your social security number). Also write your name at the top of pages 2–10.
3. Do any necessary work for each problem on the space provided or on the back of the pages of this test booklet. Circle your answers in this test booklet. No partial credit will be given, but if you show your work on the test booklet, it may be used in borderline cases.
4. No books, notes or calculators may be used on this exam.
5. Each problem is worth 8 points. The maximum possible score is 200 points.
6. Using a #2 pencil, fill in each of the following items on your answer sheet:
   (a) On the top left side, write your name (last name, first name), and fill in the little circles.
   (b) On the bottom left side, under SECTION, write in your division and section number and fill in the little circles. (For example, for division 9 section 1, write 0901. For example, for division 38 section 2, write 3802).
   (c) On the bottom, under STUDENT IDENTIFICATION NUMBER, write in your student ID number, and fill in the little circles.
   (d) Using a #2 pencil, put your answers to questions 1–25 on your answer sheet by filling in the circle of the letter of your response. Double check that you have filled in the circles you intended. If more than one circle is filled in for any question, your response will be considered incorrect. Use a #2 pencil.
7. After you have finished the exam, hand in your answer sheet and your test booklet to your recitation instructor.

1. Which of the following statements are always true for any three dimensional <u>unit</u> vectors $\vec{a}$ and $\vec{b}$?

(I) $|\vec{a}\cdot\vec{b}|\leq 1$ $\quad |\vec{a}|=1 \quad |\vec{b}|=1$

(II) $|\vec{a}\times\vec{b}|\leq 1$

(III) $|\vec{a}\times\vec{b}|\neq 0$

(IV) $\vec{a}\cdot\vec{a}=1$

A. (I) and (IV) only
B. (III) and (IV) only
C. (I), (II) and (III) only
D. (I), (II) and (IV) only
E. All

(I) $|\vec{a}\cdot\vec{b}| = |\vec{a}||\vec{b}||\cos\theta| \leq 1$ ✓

(II) $|\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|\sin\theta \leq 1$ ✓

(III) Not true when $\theta=0$ or $\theta=\pi$

(IV) $\vec{a}\cdot\vec{a} = |\vec{a}|^2 = 1$ ✓

2. Find the value of x for which the vectors $\vec{a}=x\vec{i}-\vec{j}+\vec{k}$ and $\vec{b}=4\vec{i}+2x\vec{j}-8\vec{k}$ are perpendicular.

$\vec{a}\cdot\vec{b}=0$

$4x-2x-8=0$

$2x=8$

$x=4$

A. 8
B. 4
C. -2
D. -5
E. 6

3. Suppose that $x^2-2x+y^2+2y+z^2-4z=a$, where a is a constant, is the equation of a sphere passing through the point $(1,-3,2)$. Find the radius of the sphere.

Equation must be satisfied when $(x,y,z)=(1,-3,2)$

$1^2-2\cdot 1+(-3)^2+2(-3)+2^2-4\cdot 2=a$

$1-2+9-6+4-8=a \longrightarrow a=-2$

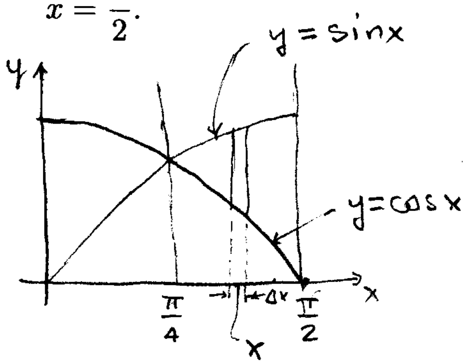
Substitute $a=-2$ in equation and complete squares

$(x^2-2x+1)+(y^2+2y+1)+(z^2-4z+4) \quad =-2+1+1+4$

$(x-1)^2+(y+1)^2+(z-2)^2=4 \quad \therefore$ radius $=2$

A. $\sqrt{2}$
B. 5
C. $\sqrt{6}$
D. 1
E. 2

4. Find the area of the region enclosed by the curves $y = \sin x$, $y = \cos x$, $x = \frac{\pi}{4}$ and $x = \frac{\pi}{2}$.



Area of typical rectangle:

$\Delta A = (\sin x - \cos x)\Delta x$

$A = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\sin x - \cos x)\,dx$

$= (-\cos x - \sin x)\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}}$

$= -1 - \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right) = -1 + \sqrt{2}$

A. 1

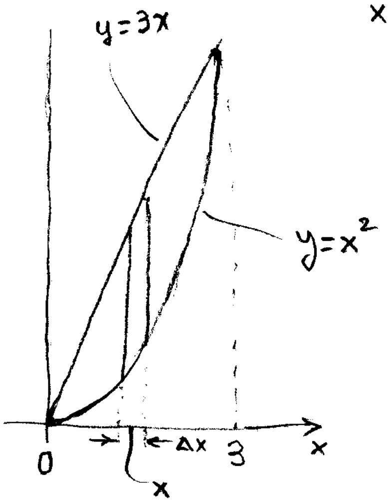
B. $\sqrt{2} - 1$

C. $\sqrt{2} + \pi$

D. $2\sqrt{2}$

E. $\frac{\pi}{4}$

5. Let R be the region enclosed by the curves $y = x^2$ and $y = 3x$. The volume of the solid obtained by rotating R about the x-axis is given by



$x^2 = 3x \longrightarrow x = 0, 3$

Volume of typical washer

$\Delta V = \left[\pi(3x)^2 - \pi(x^2)^2\right]\Delta x$

$V = \int_0^3 \pi\,(9x^2 - x^4)\,dx$

$= \int_0^3 \pi x^2 (9 - x^2)\,dx$

A. $\int_0^3 2\pi x(x-3)dx$

B. $\int_0^1 \pi x(9-x^2)dx$

C. $\int_0^3 \pi x(x-3)dx$

D. $\int_0^1 2\pi x(x-3)dx$

E. $\int_0^3 \pi x^2(9-x^2)dx$

6. The natural length of a spring is 2 ft. A force of 15 lbs is required to stretch the spring from its natural length to a length of 2.5 ft. Find the work required to stretch the spring from its natural length to a length of 4 ft.

$F = kx \longrightarrow 15 = k(0.5) \rightarrow k = 30$

$F = 30x$

$W = \int_0^2 30x\,dx = 30\frac{x^2}{2}\Big|_0^2 = 60$

A. 60 ft-lbs

B. 30 ft-lbs

C. 15 ft-lbs

D. 45 ft-lbs

E. 80 ft-lbs

7. $\displaystyle\int_4^6 \frac{1}{x^2-4x+3}\,dx =$

A. $\ln\frac{3}{2} - 1$

B. $\frac{1}{2}\ln\frac{3}{2}$

C. $\frac{1}{7}\ln\frac{7}{11} + \frac{3}{4}$

D. $\frac{1}{3}\ln\frac{5}{11}$

E. $\frac{1}{2}\ln\frac{9}{5}$

$$\frac{1}{x^2-4x+3} = \frac{1}{(x-3)(x-1)} = \frac{A}{x-3} + \frac{B}{x-1}$$

$$1 = Ax - A + Bx - 3B \qquad \left.\begin{array}{l} A+B=0 \\ -A-3B=1 \end{array}\right\} \to \begin{array}{l} A=\frac{1}{2} \\ B=-\frac{1}{2} \end{array}$$

$$\int_4^6 \frac{1}{x^2-4x+3}\,dx = \int_4^6 \left(\frac{1}{2}\frac{1}{x-3} - \frac{1}{2}\frac{1}{x-1}\right)dx$$

$$= \left[\frac{1}{2}\ln|x-3| - \frac{1}{2}\ln|x-1|\right]_4^6 = \frac{1}{2}\ln 3 - \frac{1}{2}\ln 5 - \frac{1}{2}\ln 1 + \frac{1}{2}\ln 3$$

$$= \frac{1}{2}(2\ln 3 - \ln 5) = \frac{1}{2}\ln\frac{9}{5}$$

8. Use integration by parts to evaluate $\displaystyle\int_0^1 \sin^{-1} x\,dx$

A. $\frac{\pi}{2} - 1$

B. $\frac{3\pi}{4}$

C. $\pi + \frac{1}{2}$

D. 2π

E. $\sin^{-1}\frac{1}{3}$

$$\int \sin^{-1}x\,dx = x\sin^{-1}x - \int \frac{x}{\sqrt{1-x^2}}\,dx = x\sin^{-1}x + \sqrt{1-x^2} + C$$

$u = \sin^{-1}x$, $dv = dx$

$du = \frac{1}{\sqrt{1-x^2}}\,dx$, $v = x$

$$\int_0^1 \sin^{-1}x\,dx = \left(x\sin^{-1}x + \sqrt{1-x^2}\right)\Big|_0^1$$

$$= \sin^{-1}1 - 1 = \frac{\pi}{2} - 1$$

9. The integral $\displaystyle\int_0^{\frac{1}{2}} \frac{x^3}{\sqrt{1-x^2}}\,dx$ is transformed by a suitable trigonometric substitution to the integral

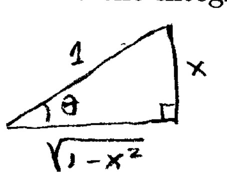
A. $\displaystyle\int_0^{\frac{\pi}{6}} \sin^3\theta\,d\theta$

B. $\displaystyle\int_0^{\frac{\pi}{3}} \sin^3\theta\,d\theta$

C. $\displaystyle\int_0^{\frac{\pi}{6}} \frac{\sin^3\theta}{\cos\theta}\,d\theta$

D. $\displaystyle\int_0^{\frac{\pi}{3}} \sin^2\theta\cos\theta\,d\theta$

E. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin^3\theta}{\cos\theta}\,d\theta$



$x = \sin\theta \qquad dx = \cos\theta\,d\theta$

$\sqrt{1-x^2} = \cos\theta$

$x = 0 \longrightarrow \theta = 0$

$x = \frac{1}{2} \longrightarrow \theta = \frac{\pi}{6}$

$$\int_0^{\frac{1}{2}} \frac{x^3}{\sqrt{1-x^2}}\,dx = \int_0^{\frac{\pi}{6}} \frac{\sin^3\theta}{\cos\theta}\cos\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{6}} \sin^3\theta\,d\theta$$

10. $\int_0^{\frac{\pi}{2}} \sin^3 x \cos^2 x \, dx =$

$$= \int_0^{\frac{\pi}{2}} \sin^2 x \cos^2 x \sin x \, dx$$

$$= \int_0^{\frac{\pi}{2}} (1 - \cos^2 x)\cos^2 x \sin x \, dx = -\int_1^0 (1-u^2)u^2 \, du =$$

$u = \cos x \qquad du = -\sin x \, dx$

$x = 0 \rightarrow u = 1 \qquad x = \frac{\pi}{2} \rightarrow u = 0$

$$= \int_0^1 (u^2 - u^4) \, du = \left(\frac{u^3}{3} - \frac{u^5}{5}\right)\Big|_0^1 = \frac{1}{3} - \frac{1}{5} = \frac{2}{15}$$

A. $\frac{1}{5}$

B. $\frac{3}{5}$

C. $\frac{4}{15}$

D. $\frac{2}{15}$

E. $\frac{7}{5}$

11. Decide whether each integral converges or diverges.

(I) $\int_1^3 \frac{1}{x-2} \, dx$ (II) $\int_1^\infty \frac{1}{x^2} \, dx$

(I) $\int_1^3 \frac{1}{x-2} \, dx = \int_1^2 \frac{1}{x-2} \, dx + \int_2^3 \frac{1}{x-2} \, dx$

$\int_1^2 \frac{1}{x-2} \, dx = \lim_{t \to 2^-} \int_1^t \frac{1}{x-2} \, dx = \lim_{t \to 2^-} \ln|x-2| \Big|_1^t$

$= \lim_{t \to 2^-} [\ln|t-2| - \ln 1] = -\infty$

$\therefore \int$ diverges

(II) $\int_1^\infty \frac{1}{x^2} \, dx = \lim_{t \to \infty} \int_1^t \frac{1}{x^2} \, dx = \lim_{t \to \infty} \left(-\frac{1}{x}\right)\Big|_1^t = \lim_{t \to \infty} \left(-\frac{1}{t} + 1\right) = 1$

$\therefore \int$ converges

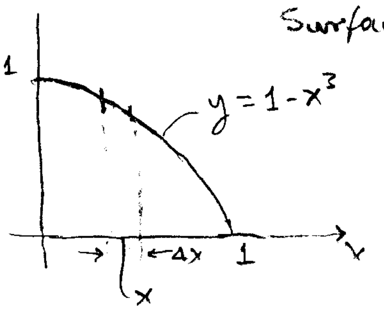
A. (I) converges, (II) converges

B. (I) converges, (II) diverges

C. (I) diverges, (II) converges

D. (I) diverges, (II) diverges

12. The area of the surface obtained by rotating the curve $y = 1 - x^3$, $0 \le x \le 1$, about the x-axis is given by



Surface area of typical belt

$\Delta S = 2\pi(1-x^3)\sqrt{1+(-3x^2)^2}\,\Delta x$

$$S = \int_0^1 2\pi(1-x^3)\sqrt{1+9x^4} \, dx$$

A. $\int_0^1 \sqrt{1+9x^4} \, dx$

B. $\int_0^1 2\pi(1-x^3)\sqrt{1+9x^4} \, dx$

C. $\int_0^1 \pi\sqrt{1+(1-x^3)^2} \, dx$

D. $\int_0^1 2\pi x\sqrt{1+9x^4} \, dx$

E. $\int_0^1 \pi(1-x^3)^2 dx$

13. Decide which of the following sequences converge.

(I) $\left\{\frac{(-1)^n n}{n+1}\right\}$ (II) $a_n = \frac{n^2}{2^n}$ (III) $b_n = (-1)^n\sqrt{n}$

(I) $-\frac{1}{2}, \frac{2}{3}, -\frac{3}{4}, \frac{4}{5}, - \cdots$ diverges

(II) $\lim_{n\to\infty} \frac{n^2}{2^n} = \lim_{x\to\infty} \frac{x^2}{2^x} = \lim_{x\to\infty} \frac{x^2}{e^{x\ln 2}}$

$\overset{L'H}{=} \lim_{x\to\infty} \frac{2x}{\ln 2\, e^{x\ln 2}} \overset{L'H}{=} \lim_{x\to\infty} \frac{2}{(\ln 2)^2 e^{x\ln 2}} = 0$ $\frac{\infty}{\infty}$ $\frac{\infty}{\infty}$

(III) $-1, \sqrt{2}, -\sqrt{3}, \sqrt{4}, \cdots$ ~~converges~~ diverges

A. (I) only
B. (II) only
C. (III) only
D. (I) and (II) only
E. (I) and (III) only

14. $\sum_{n=1}^{\infty} 3\left(\frac{4}{5}\right)^n = \sum_{n=1}^{\infty} 3\cdot\frac{4}{5}\left(\frac{4}{5}\right)^{n-1}$

$= \frac{12}{5}\ \frac{1}{1-\frac{4}{5}} = 12$

A. 15
B. $\frac{12}{5}$
C. 10
D. 5
E. 12

15. Which of the following statements is always true for any series $\sum_{n=1}^{\infty} a_n$ with positive terms?

(I) If $\lim_{n\to\infty} a_n\sqrt{n} = 1$, then $\sum_{n=1}^{\infty} a_n$ converges.

(II) If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{3}{2}$, then $\sum_{n=1}^{\infty} a_n$ diverges.

(III) If $\lim_{n\to\infty} \sqrt[n]{a_n} = \frac{1}{3}$, then $\sum_{n=1}^{\infty} a_n$ converges.

A. (I) only
B. (I) and (II) only
C. (II) and (III) only
D. (III) only
E. (II) only

(I) Not true. Compare $\sum_{n=1}^{\infty} a_n$ with $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ which diverges, and use limit comparison test.

(II) True by ratio test

(III) True by root test

16. If $f(x) = \dfrac{x^2}{1+x^2}$, $1 \le x < \infty$, decide which of the following series converge.

(I) $\sum_{n=1}^{\infty} f(n)$ (II) $\sum_{n=1}^{\infty} f\left(\frac{1}{n}\right)$ (III) $\sum_{n=1}^{\infty} f\left(\frac{1}{\sqrt{n}}\right)$

A. (I) only
B. (III) only
C. (I) and (III) only
D. (II) only
E. (II) and (III) only

(I) $\sum_{n=1}^{\infty} \frac{n^2}{1+n^2}$ diverges because $\lim_{n\to\infty} \frac{n^2}{1+n^2} = 1 \neq 0$

(II) $\sum_{n=1}^{\infty} \frac{\left(\frac{1}{n}\right)^2}{1+\left(\frac{1}{n}\right)^2} = \sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges

Compare with $\sum_{n=1}^{\infty} \frac{1}{n^2}$ which converges and use comparison test

(III) $\sum_{n=1}^{\infty} \frac{\frac{1}{n}}{1+\frac{1}{n}} = \sum_{n=1}^{\infty} \frac{1}{n+1}$ diverges (p-series, p=1)

17. The interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{1}{n}(x-1)^n$ is

A. $[-1, 1]$
B. $[0, 1]$
C. $(0, 2)$
D. $[0, 2)$
E. $(-2, 2)$

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\frac{1}{n+1}(x-1)^{n+1}}{\frac{1}{n}(x-1)^n} \right| = \lim_{n\to\infty} \frac{n}{n-1} |x-1| = |x-1|$

$\therefore$ series converges when $|x-1| < 1$

or $0 < x < 2$

When $x = 0$: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges by alternating series test

When $x = 2$: $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

18. The coefficient of x^2 in the Maclaurin series for $f(x) = \dfrac{e^x}{1-x}$ is

A. 0
B. $\frac{3}{2}$
C. $\frac{5}{2}$
D. 1
E. -1

$\frac{e^x}{1-x} = e^x \frac{1}{1-x} = \left(1 + x + \frac{x^2}{2!} + \cdots\right)\left(1 + x + x^2 + \cdots\right)$

$= 1 + (x + x) + \left(x^2 + x^2 + \frac{x^2}{2}\right) + \cdots$

$= 1 + 2x + \frac{5}{2}x^2 + \cdots$

Or $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$

$f'(x) = \frac{(1-x)e^x - e^x(-1)}{(1-x)^2} = \frac{(2-x)e^x}{(1-x)^2}$

$f''(x) = \frac{(1-x)^2(2e^x - xe^x - e^x) - (2-x)e^x\, 2(1-x)(-1)}{(1-x)^4}$, $f''(0) = \frac{1+4}{1}$, $\frac{f''(0)}{2!} = \frac{5}{2}$

19. If $f^{(7)}(x) = x^2 \sin x$, the coefficient of $\left(x - \frac{\pi}{2}\right)^8$ in the Taylor series for f centered at $a = \frac{\pi}{2}$, is

Taylor series for f centered at $a = \frac{\pi}{2}$:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(\frac{\pi}{2})}{n!}\left(x-\frac{\pi}{2}\right)^n$$

$$f^{(7)}(x) = x^2 \sin x$$

$$f^{(8)}(x) = x^2 \cos x + 2x \sin x$$

$$f^{(8)}\left(\frac{\pi}{2}\right) = \pi \quad , \quad \frac{f^{(8)}(\frac{\pi}{2})}{8!} = \frac{\pi}{8!}$$

A. $\frac{\pi}{8!}$

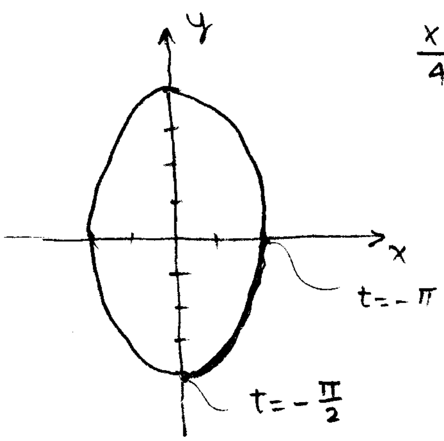
B. 0

C. $\frac{1}{8!}$

D. $\frac{\pi}{7!}$

E. $\frac{1}{4!}$

20. The graph of the parametric curve $x = -2\cos t$, $y = 4 \sin t$; $-\pi \le t \le -\frac{\pi}{2}$ is



$$\frac{x^2}{4} + \frac{y^2}{16} = 1$$

When $t=-\pi$: $x=2$, $y=0$

when $t=-\frac{\pi}{2}$: $x=0$, $y=-4$

When $-\pi \le t \le -\frac{\pi}{2}$:

$0 \le x \le 2$ and $-4 \le y \le 0$

A. A circle

B. An ellipse

C. A quarter of an ellipse

D. A quarter of a circle

E. The lower half of an ellipse

21. If $x = t^2 + 1$ and $y = e^t$, then the value of $\frac{d^2y}{dx^2}$ when $t = 1$ is

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{e^t}{2t}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{2te^t - e^t 2}{4t^2}}{2t}$$

When $t=1$: $\frac{d^2y}{dx^2} = \frac{2e-2e}{8} = 0$

A. 0

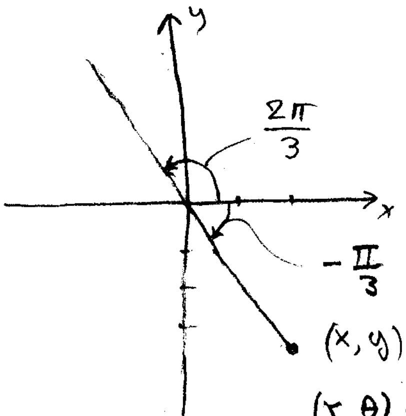
B. $\frac{1}{2}$

C. e

D. $-e$

E. 1

22. The Cartesian coordinates of a point are $(x, y) = (2, -2\sqrt{3})$. Polar coordinates (r, θ) of the point are



$r^2 = x^2 + y^2 = 4 + 4 \cdot 3 = 16$

$\tan\theta = \frac{y}{x} = \frac{-2\sqrt{3}}{2} = -\sqrt{3}$

$(x, y) = (2, -2\sqrt{3})$

$(r, \theta) = \left(-4, \frac{2\pi}{3}\right)$

A. $\left(4, \frac{\pi}{3}\right)$

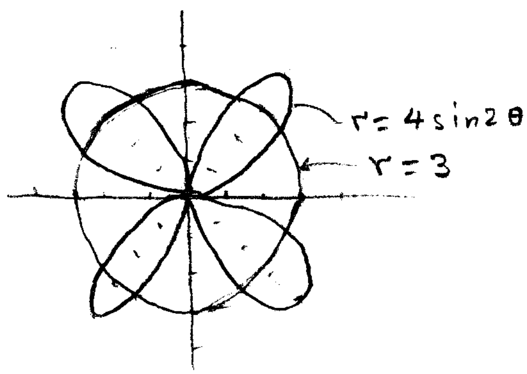
B. $\left(4, -\frac{\pi}{6}\right)$

C. $\left(-4, \frac{\pi}{3}\right)$

D. $\left(4, -\frac{\pi}{2}\right)$

E. $\left(-4, \frac{2\pi}{3}\right)$

23. The polar curves $r = 3$ and $r = 4\sin 2\theta$ intersect at n points where $n =$



A. 4

B. 3

C. 2

D. 8

E. 1

24. Convert the polar equation $r = -2\sin\theta$ to a Cartesian equation

$r = -2\sin\theta$

$r^2 = -2r\sin\theta$

$x^2 + y^2 = -2y$

$x^2 + y^2 + 2y = 0$

$x^2 + y^2 + 2y + 1 = 1$

$x^2 + (y+1)^2 = 1$

A. $x^2 + (y-1)^2 = 1$

B. $(x+1)^2 + y^2 = 1$

C. $(x-1)^2 + y^2 = 1$

D. $x^2 + (y+1)^2 = 1$

E. $x^2 + y^2 = 2$

25. The area enclosed by one loop of the rose curve $r = \cos 3\theta$ is

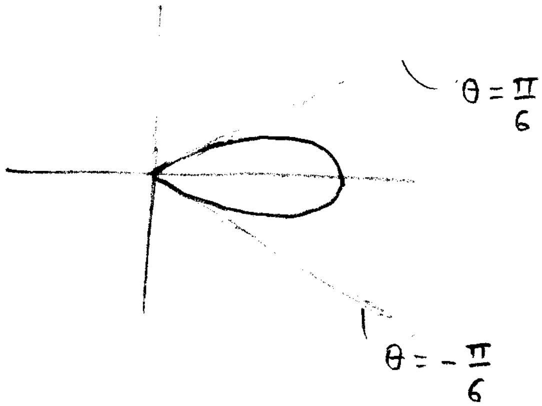
A. $\frac{\pi}{2}$

B. π

C. $\frac{\pi}{6}$

D. $\frac{\pi}{12}$

E. 2



$$A = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{1}{2} r^2 \, d\theta = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{1}{2} \cos^2 3\theta \, d\theta$$

$$= \int_0^{\frac{\pi}{6}} \cos^2 3\theta \, d\theta = \int_0^{\frac{\pi}{6}} \frac{1 + \cos 6\theta}{2} \, d\theta$$

$$= \left[\frac{1}{2}\theta + \frac{\sin 6\theta}{12} \right]_0^{\frac{\pi}{6}} = \frac{\pi}{12}$$