NAME GRADING KEY AND SOLUTIONS

STUDENT ID # ______________________

RECITATION INSTRUCTOR ______________________

RECITATION TIME ______________________

LECTURER ______________________

INSTRUCTIONS

1. There are 10 different test pages (including this cover page). Make sure you have a complete test.
2. Fill in the above items in print. I.D.# is your 9 digit ID (probably your social security number). Also write your name at the top of pages 2–10.
3. Do any necessary work for each problem on the space provided or on the back of the pages of this test booklet. Circle your answers in this test booklet. No partial credit will be given, but if you show your work on the test booklet, it may be used in borderline cases.
4. No books, notes or calculators may be used on this exam.
5. Each problem is worth 8 points. The maximum possible score is 200 points.
6. Using a #2 pencil, fill in each of the following items on your answer sheet:
   (a) On the top left side, write your name (last name, first name), and fill in the little circles.
   (b) On the bottom left side, under SECTION, write in your division and section number and fill in the little circles. (For example, for division 9 section 1, write 0901. For example, for division 38 section 2, write 3802).
   (c) On the bottom, under STUDENT IDENTIFICATION NUMBER, write in your student ID number, and fill in the little circles.
   (d) Using a #2 pencil, put your answers to questions 1–25 on your answer sheet by filling in the circle of the letter of your response. Double check that you have filled in the circles you intended. If more than one circle is filled in for any question, your response will be considered incorrect. Use a #2 pencil.
7. After you have finished the exam, hand in your answer sheet and your test booklet to your recitation instructor.

1. If θ is the angle between the vectors $\vec{a} = 2\vec{i} + 2\vec{j} - \vec{k}$ and $\vec{b} = 5\vec{i} - 3\vec{j} + 2\vec{k}$, then $\cos\theta =$

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

$$2 \cdot 5 + 2(-3) + (-1)2 = \sqrt{4+4+1}\,\sqrt{25+9+4}\,\cos\theta$$

$$2 = \sqrt{9}\,\sqrt{38}\,\cos\theta$$

$$\cos\theta = \frac{2}{3\sqrt{38}}$$

A. 2

B. $\frac{3\sqrt{38}}{2}$

C. $\frac{2}{3\sqrt{38}}$

D. $\frac{1}{2}$

E. $\frac{1}{3\sqrt{38}}$

2. Let $\vec{a} = \vec{i} + 4\vec{j} - 7\vec{k}$, $\vec{b} = 2\vec{i} - \vec{j} + 4\vec{k}$ and $\vec{c} = -9\vec{j} + 18\vec{k}$ then $\vec{a} \cdot (\vec{b} \times \vec{c}) =$

$$\vec{b} \times \vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 4 \\ 0 & -9 & 18 \end{vmatrix} = 18\vec{i} - 36\vec{j} - 18\vec{k}$$

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = 1(18) + 4(-36) + (-7)(-18)$$

$$= 18 - 144 + 126$$

$$= 144 - 144 = 0$$

A. $18\vec{i} - 36\vec{j} - 18\vec{k}$

B. $-18\vec{j} + 9\vec{k}$

C. 2

D. 1

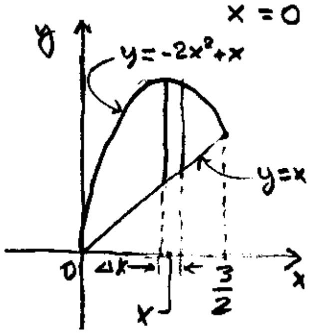
E. 0

3. The volume of the solid obtained by rotating about the x-axis the region in the first quadrant bounded by the graphs of $y = -2x^2 + 4x$ and $y = x$ is given by

Intersection points: $-2x^2 + 4x = x$

$-2x^2 + 3x = 0 \rightarrow x(-2x+3) = 0$

$x = 0,\ y = 0$ and $x = \frac{3}{2},\ y = \frac{3}{2}$



Volume of typical washer:

$$\Delta V = \pi\left[(-2x^2+4x)^2 - x^2\right]\Delta x$$

$$V = \int_0^{\frac{3}{2}} \pi\left[(-2x^2+4x)^2 - x^2\right]dx$$

A. $\int_0^{\frac{3}{2}} \pi[(-2x^2+4x)^2 - x^2]dx$

B. $\int_0^{\frac{3}{2}} 2\pi x[(-2x^2+4x) - x]dx$

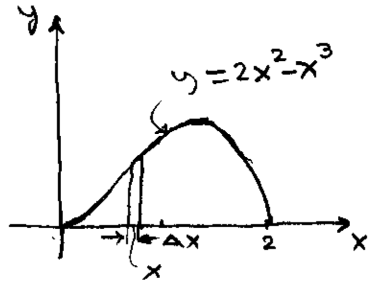
C. $\int_0^{2} \pi[(-2x^2+4x)^2 - x^2]dx$

D. $\int_0^{\frac{3}{2}} \pi[(-2x^2+4x) - x]^2 dx$

E. $\int_0^{\frac{3}{2}} [(-2x^2+4x) - x]dx$

4. Let R be the region bounded by the curves $y = 2x^2 - x^3$ and $y = 0$. Using the method of cylindrical shells, the volume of the solid generated by rotating R about the y-axis is given by

Intersection points : $2x^2 - x^3 = 0 \to x^2(2-x) = 0$, $x = 0$ and $x = 2$.



Volume of typical shell:

$\Delta V = 2\pi x(2x^2 - x^3)\Delta x$

$V = \int_0^2 2\pi x(2x^2 - x^3)\,dx$

A. $\int_0^2 (2y^2 - y^3)dy$

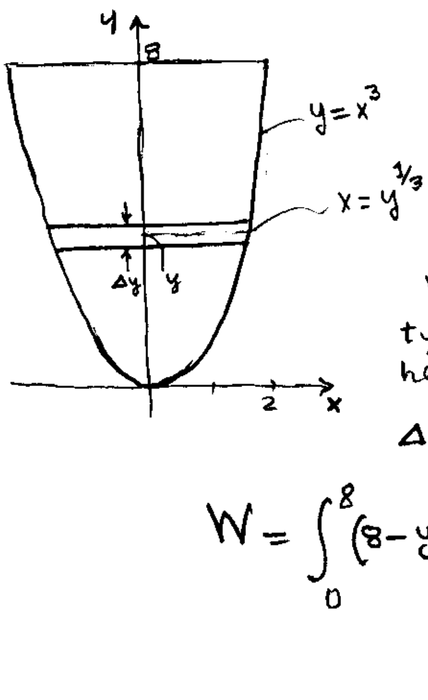
B. $\int_0^2 \pi(2x^2 - x^3)^2 dx$

C. $\int_0^2 2\pi x(2x^2 - x^3)dx$

D. $\int_0^{\frac{4}{3}} 2\pi x(2x^2 - x^3)dx$

E. $\int_0^2 \sqrt{1 + (4x - 3x^2)^2}dx$

5. A tank has the shape of the surface generated by rotating the curve $y = x^3$, $0 \le x \le 2$, about the y-axis. If the tank is full of water at 62.5 lbs/ft^3, and the dimensions of the tank are measured in feet, the work required to pump all the water to the top of the tank is given by



Weight of typical layer of water :

$62.5(\pi x^2)\Delta y =$
$= 62.5\pi\, y^{2/3}\Delta y$

Work required to lift typical layer from height y to height 8 :

$\Delta W = (8 - y)\,62.5\pi\, y^{2/3}\Delta y$

$W = \int_0^8 (8 - y)\,62.5\pi\, y^{2/3}\,dy$

A. $62.5\pi \int_0^8 (8 - y)y^{\frac{2}{3}}dy$

B. $62.5\pi \int_0^2 (8 - y)y^{\frac{2}{3}}\,dy$

C. $62.5\pi \int_0^8 (2 - y)y^{\frac{1}{3}}dy$

D. $62.5\pi \int_0^2 (2 - y)y^{\frac{2}{3}}dy$

E. $62.5\pi \int_0^8 (2 - y)y^3 dy$

6. $\int_1^2 x \ln x \, dx =$ $\frac{x^2}{2}\ln x\Big|_1^2 - \int_1^2 \frac{x^2}{2}\frac{1}{x}dx =$

$u = \ln x \quad dv = x\,dx$

$du = \frac{1}{x}dx \quad v = \frac{x^2}{2}$

$= 2\ln 2 - \left[\frac{x^2}{4}\right]_1^2$

$= 2\ln 2 - 1 + \frac{1}{4}$

$= 2\ln 2 - \frac{3}{4}$

A. $2\ln 2 - 1$

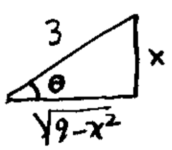
B. $\frac{1}{2}(\ln 2)^2$

C. $2\ln 2$

D. $2\ln 2 - \frac{3}{4}$

E. $1 - \ln x$

7. For the integral $\int \frac{dx}{x\sqrt{9-x^2}}$, (i) choose a trigonometric substitution to simplify the integral and (ii) give the resulting integral.



$x = 3\sin\theta \quad dx = 3\cos\theta\, d\theta$

$\sqrt{9-x^2} = 3\cos\theta$

$\int \frac{dx}{x\sqrt{9-x^2}} =$

$= \int \frac{3\cos\theta\, d\theta}{3\sin\theta \; 3\cos\theta}$

$= \int \frac{1}{3\sin\theta}d\theta$

A. (i) $x = 3\sec\theta$, (ii) $\int \frac{1}{3}d\theta$

B. (i) $x = 3\tan\theta$, (ii) $\int \frac{\sec\theta}{3\tan\theta}d\theta$

C. (i) $x = 3\sin\theta$, (ii) $\int \frac{1}{3\sin\theta}d\theta$

D. (i) $x = 3\sin\theta$, (ii) $\int \frac{1}{9\sin\theta\cos\theta}d\theta$

E. (i) $x = \cos\theta$, (ii) $-\int \frac{1}{\cos\theta}d\theta$

8. $\int_2^3 \frac{3}{x^2+x-2}dx =$ $\int_2^3 \left(\frac{1}{x-1} - \frac{1}{x+2}\right)dx =$

$\frac{3}{x^2+x-2} = \frac{3}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$

$3 = A(x+2) + B(x-1)$

$3 = (A+B)x + 2A - B$

$A + B = 0$

$2A - B = 3$

$A = 1, \; B = -1$

$= \left[\ln(x-1) - \ln(x+2)\right]_2^3$

$= \left[\ln\frac{x-1}{x+2}\right]_2^3$

$= \ln\frac{2}{5} - \ln\frac{1}{4}$

$= \ln\frac{\frac{2}{5}}{\frac{1}{4}} = \ln\frac{8}{5}$

A. $\ln\frac{5}{4}$

B. $\ln\frac{3}{2}$

C. $\frac{2}{3}$

D. $\ln 6$

E. $\ln\frac{8}{5}$

9. $\int_0^{\frac{\pi}{4}} (\sin x + \cos x)^2\, dx = \int_0^{\frac{\pi}{4}} (\sin^2 x + \cos^2 x + 2\sin x \cos x)\,dx$

$= \int_0^{\frac{\pi}{4}} (1 + 2\sin x \cos x)\,dx$

$= \left[x + \sin^2 x\right]_0^{\frac{\pi}{4}}$

$= \frac{\pi}{4} + \sin^2\left(\frac{\pi}{4}\right) = \frac{\pi}{4} + \frac{1}{2}$

A. $\frac{\pi}{4}$

B. $\frac{\pi}{4} + \frac{1}{2}$

C. $\frac{\pi}{8}$

D. $\frac{\pi}{4} - \frac{1}{2}$

E. $\frac{1}{2}$

10. Evaluate the improper integral $\int_0^\infty x^2 e^{-x^3}\, dx$.

$\int_0^\infty x^2 e^{-x^3}\,dx = \lim_{t\to\infty} \int_0^t x^2 e^{-x^3}\,dx$

$= \lim_{t\to\infty} \left[-\frac{1}{3}e^{-x^3}\right]_0^t$ *

$= \lim_{t\to\infty} \left[-\frac{1}{3}e^{-t^3} + \frac{1}{3}\right] = \frac{1}{3}$

A. $\frac{1}{3}$

B. divergent

C. 1

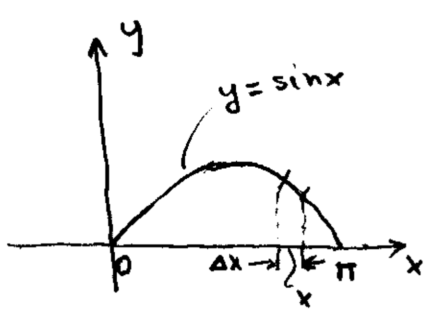
D. $\frac{3}{2}$

E. 2

* $\int x^2 e^{-x^3}\,dx = \int -\frac{1}{3}e^u\,du = -\frac{1}{3}e^u + C = -\frac{1}{3}e^{-x^3} + C$

$u = -x^3 \quad du = -3x^2\,dx$

11. The area of the surface obtained by rotating the curve $y = \sin x$, $0 \le x \le \pi$ about the y-axis is given by



Area of typical belt:

$dA = 2\pi x\, ds = 2\pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$

$= 2\pi x\sqrt{1 + \cos^2 x}\,dx$

$A = \int_0^\pi 2\pi x\sqrt{1 + \cos^2 x}\,dx$

A. $\int_0^\pi 2\pi x\sqrt{1 + \cos^2 x}\, dx$

B. $\int_0^\pi 2\pi \sin x\sqrt{1 + \cos^2 x}\, dx$

C. $\int_0^\pi \pi \sin^2 x\sqrt{1 + \cos^2 x}\, dx$

D. $\int_{-\pi}^\pi \pi \cos^2 x\, dx$

E. $\int_{-\pi}^\pi 2\pi \sin x\sqrt{1 + \cos^2 x}\, dx$

12. $\lim_{n\to\infty} n \sin\left(\frac{1}{\pi n}\right) = \lim_{n\to\infty} \frac{\sin\frac{1}{\pi n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{\pi} \frac{\sin\frac{1}{\pi n}}{\frac{1}{\pi n}}$

$= \lim_{x\to\infty} \frac{1}{\pi} \frac{\sin\frac{1}{\pi x}}{\frac{1}{\pi x}} = \frac{1}{\pi} \lim_{y\to 0} \frac{\sin y}{y}$

$y = \frac{1}{\pi x}$

$= \frac{1}{\pi}\, 1 = \frac{1}{\pi}$

A. 0

B. $\frac{1}{\pi}$

C. π

D. 1

E. ∞

13. $\sum_{n=1}^{\infty} \frac{2^n + (-1)^n 3^n}{5^n} =$

$= \sum_{n=1}^{\infty} \left(\frac{2}{5}\right)^n + \sum_{n=1}^{\infty} \left(-\frac{3}{5}\right)^n$

$= \frac{2}{5} \sum_{n=1}^{\infty} \left(\frac{2}{5}\right)^{n-1} + \left(-\frac{3}{5}\right) \sum_{n=1}^{\infty} \left(-\frac{3}{5}\right)^{n-1}$

$= \frac{2}{5} \frac{1}{1-\frac{2}{5}} - \frac{3}{5} \frac{1}{1-\left(-\frac{3}{5}\right)}$

$= \frac{2}{3} - \frac{3}{5} \frac{1}{1+\frac{3}{5}} = \frac{2}{3} - \frac{3}{8} = \frac{7}{24}$

A. $\frac{1}{5}$

B. $\frac{5}{16}$

C. $\frac{7}{24}$

D. $\frac{3}{22}$

E. $\frac{8}{15}$

14. Which of the following series converge?

(I) $\sum_{n=1}^{\infty} \frac{1}{n+\ln n}$ (II) $\sum_{n=1}^{\infty} \frac{\tan^{-1}(n)}{n^2+1}$ (III) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{n}}$

A. (I) only

B. (II) only

C. (II) and (III) only

D. (III) only

E. all

(I) diverges: compare with $\sum_{n=1}^{n} \frac{1}{n}$ and use limit comparison test

$\left(\lim_{n\to\infty} \frac{\frac{1}{n+\ln n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n+\ln n} = \lim_{n\to\infty} \frac{1}{1+\frac{\ln n}{n}} = 1 \neq 0\right)$

(II) converges: Note that $\tan^{-1}(n) < \frac{\pi}{2}$, and compare with $\sum_{n=1}^{\infty} \frac{\frac{\pi}{2}}{n^2}$

(III) diverges: $\lim_{n\to\infty} \frac{1}{\sqrt[n]{n}} = 1 \neq 0$

15. Which of the following series are conditionally convergent?

(I) $\sum_{n=1}^{\infty}(-1)^n e^n$ (II) $\sum_{n=1}^{\infty}\frac{n!}{(2n)!}$ (III) $\sum_{n=1}^{\infty}(-1)^n\frac{1}{\sqrt{n}}$

A. (I) only

B. (II) only

C. (III) only

D. none

E. all

conditionally convergent: convergent but not abs. conv.

(I) divergent: $\lim_{n\to\infty}(-1)^n e^n$ DNE

(II) This is a series with positive terms and cannot be conditionally conv. (convergent and abs. convergent are equivalent)

(III) abs. conv.? No: $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ is divergent.

conv.? Yes: by alternating series test

16. Which of the following statements are always true for any series $\sum_{n=1}^{\infty}a_n$ with positive terms?

(I) If $a_{n+1}=\frac{n+1}{2n+1}a_n$ for all n, then $\sum_{n=1}^{\infty}a_n$ converges.

(II) If $a_{n+1}<a_n$ for all n, then $\sum_{n=1}^{\infty}a_n$ converges.

(III) If $\lim_{n\to\infty}\frac{\sqrt[n]{a_n}}{2}=1$, then $\sum_{n=1}^{\infty}a_n$ diverges.

A. (I) only

B. all

C. (II) and (III) only

D. (I) and (III) only

E. (I) and (II) only

(I) $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\frac{n+1}{2n+1}=\frac{1}{2}$ $\therefore \sum_{n=1}^{\infty}a_n$ conv. by ratio test $\to$ true

(II) Not true: ex. $\sum_{n=1}^{\infty}\frac{1}{n}$

(III) $\lim_{n\to\infty}\sqrt[n]{|a_n|}=2>1$ $\therefore \sum_{n=1}^{\infty}a_n$ diverges by root test $\to$ true

17. The interval of convergence of the power series $\sum_{n=1}^{\infty}\frac{3^n}{n!}(x-1)^n$ is

A. $\left[\frac{2}{3},\frac{4}{3}\right]$

B. $\left(\frac{2}{3},\frac{4}{3}\right)$

C. $\left(-\frac{1}{3},\frac{1}{3}\right)$

D. $\left[-\frac{1}{3},\frac{1}{3}\right)$

E. $(-\infty,\infty)$

Ratio test:

$$\left|\frac{a_{n+1}}{a_n}\right|=\left|\frac{\frac{3^{n+1}(x-1)^{n+1}}{(n+1)!}}{\frac{3^n(x-1)^n}{n!}}\right|$$

$$=\left|\frac{3^{n+1}(x-1)^{n+1}\,n!}{(n+1)!\,3^n(x-1)^n}\right|$$

$$=\frac{3}{n+1}|x-1|\xrightarrow[n\to\infty]{}0<1 \text{ for any } x$$

$\therefore$ series converges for all x

18. Match the functions with their Maclaurin series.

(1) e^{-x} (a) $\sum_{n=0}^{\infty}(-1)^n x^n$, $-1 < x < 1$

(2) $\frac{1}{1+x}$ (b) $x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} - \frac{x^8}{7!} + \dots$, $-\infty < x < \infty$

(3) $x \sin x$ (c) $1 - \frac{3^2x^2}{2!} + \frac{3^4x^4}{4!} - \frac{3^6x^6}{6!} + \dots$, $-\infty < x < \infty$

(4) $\cos 3x$ (d) $\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{n!}$, $-\infty < x < \infty$

A. 1d, 2a, 3b, 4c

B. 1a, 2d, 3b, 4c

C. 1d, 2b, 3c, 4a

D. 1c, 2b, 3d, 4a

E. 1b, 2c, 3a, 4d

1 – d (replace x with $-x$ in $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$)

2 – a (replace x with $-x$ in $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$)

3 – b $x \sin x = x\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)$

4 – c (replace x with $3x$ in $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$)

19. Let $f(x) = xg(x)$ and suppose that $g(1) = 2$, $g'(1) = 3$ and $g''(1) = 4$. The coefficient of $(x-1)^2$ of the Taylor series for f centered at $a = 1$ is

A. 3

B. 5

C. -7

D. $\frac{3}{2}$

E. $\frac{7}{2}$

$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n$

The coefficient of $(x-1)^2$ is $\frac{f^{(2)}(1)}{2!}$

$f(x) = x\,g(x)$

$f'(x) = xg'(x) + g(x)$

$f''(x) = xg''(x) + g'(x) + g'(x)$

$f''(1) = 1 \cdot 4 + 3 + 3 = 10$ $\therefore$ $\frac{f^{(2)}(1)}{2!} = \frac{10}{2} = 5$

20. An equation of the tangent line to the parametric curve $x = t^2 - t^3 + 1$, $y = 2t^5 + t$, at the point corresponding to $t = 1$ is

A. $11x + 11y - 3 = 0$

B. $11x - 3y + 5 = 0$

C. $2x + 11y - 3 = 0$

D. $11x + y - 14 = 0$

E. $7x - 11y + 8 = 0$

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{10t^4 + 1}{2t - 3t^2}$

when $t = 1$: $x = 1 - 1 + 1 = 1$

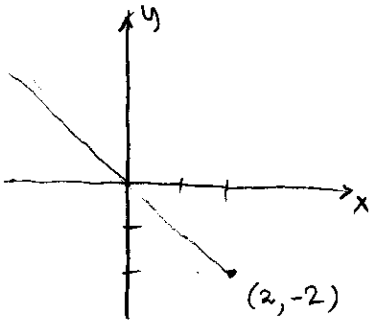
$y = 2 \cdot 1 + 1 = 3$

$\frac{dy}{dx} = \frac{11}{-1} = -11$

$y - 3 = -11(x - 1) \rightarrow 11x + y - 3 - 11 = 0$

$11x + y - 14 = 0$

21. The Cartesian coordinates of a point are $(x, y) = (2, -2)$. Polar coordinates (r, θ) of the point are



$r^2 = 4 + 4 = 8$

$r = \pm 2\sqrt{2}$

If $\theta = \frac{3\pi}{4} \rightarrow r = -2\sqrt{2}$

A. $(2\sqrt{2}, \pi)$

B. $\left(2\sqrt{2}, \frac{3\pi}{4}\right)$

C. $\left(2\sqrt{2}, \frac{\pi}{4}\right)$

D. $\left(-2\sqrt{2}, -\frac{\pi}{4}\right)$

E. $\left(-2\sqrt{2}, \frac{3\pi}{4}\right)$

22. The graph of the polar equation $r = -4\cos\theta$ is

$r^2 = -4r\cos\theta$

$x^2 + y^2 = -4x$

$x^2 + 4x + 4 + y^2 = 4$

$(x+2)^2 + y^2 = 4$

A. a horizontal straight line

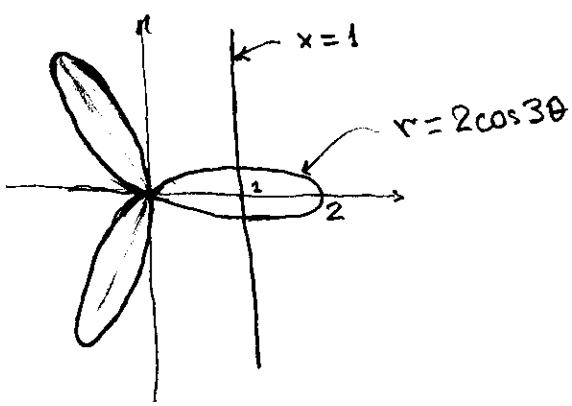
B. a vertical straight line

C. a circle of radius 2 and center $(0, -2)$

D. a circle of radius 2 and center $(-2, 0)$

E. a cardioid

23. The polar curve $r = 2\cos 3\theta$ intersects the vertical line $x = 1$ at n points where $n =$



A. 0

B. 1

C. 2

D. 3

E. 4

24. The polar form of the complex number $\dfrac{1+i}{1-i}$ with argument between 0 and 2π is

$$\frac{1+i}{1-i} = \frac{1+i}{1-i}\,\frac{1+i}{1+i} = \frac{2i}{2} = i$$

$$i = 1\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$

<u>or</u> $1+i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$1-i = \sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$

$$\frac{1+i}{1-i} = \frac{\sqrt{2}}{\sqrt{2}}\left[\cos\left(\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right) + i\sin\left(\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right)\right]$$

$$= 1\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$

A. $\sqrt{2}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

B. $\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$

C. $\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$

D. $\cos\pi + i\sin\pi$

E. $\frac{1}{\sqrt{2}}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

25. $e^{1-i\frac{\pi}{4}} = e^1\left[\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right]$

$$= e\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)$$

$$= \frac{e}{\sqrt{2}}(1-i)$$

A. $\frac{1}{\sqrt{2}}(1+i)$

B. $\frac{e}{\sqrt{2}}(1+i)$

C. e

D. $\frac{e}{\sqrt{2}}(1-i)$

E. $e+i$