NAME Solutions

10-DIGIT PUID

RECITATION INSTRUCTOR

RECITATION TIME

LECTURER

INSTRUCTIONS

1. There are 11 different test pages (including this cover page). Make sure you have a complete test.
2. Fill in the above items in print. Also write your name at the top of pages 2–11.
3. Do any necessary work for each problem on the space provided or on the back of the pages of this test booklet. Circle your answers in this test booklet. No partial credit will be given, but if you show your work on the test booklet, it may be used in borderline cases.
4. No books, notes, calculators, or any electronic devices may be used on this exam.
5. Each problem is worth 8 points. The maximum possible score is 200 points.
6. Using a #2 pencil, fill in each of the following items on your answer sheet:
   (a) On the top left side, write your name (last name, first name), and fill in the little circles.
   (b) On the bottom left side, under SECTION, write in your division and section number and fill in the little circles. (For example, for division 9 section 1, write 0901. For example, for division 38 section 2, write 3802).
   (c) On the bottom, under STUDENT IDENTIFICATION NUMBER, write in your 10-digit PUID, and fill in the little circles.
   (d) Using a #2 pencil, put your answers to questions 1–25 on your answer sheet by filling in the circle of the letter of your response. Double check that you have filled in the circles you intended. If more than one circle is filled in for any question, your response will be considered incorrect. Use a #2 pencil.
7. After you have finished the exam, hand in your answer sheet and your test booklet to your recitation instructor.

1. For what value of c is the vector $2\vec{i} - \vec{j} + c\vec{k}$ perpendicular to the vector $c\vec{i} + \vec{j} + \vec{k}$?

$(2\vec{i} - \vec{j} + c\vec{k}) \cdot (c\vec{i} + \vec{j} + \vec{k}) = 0$

$2c - 1 + c = 0$

$c = \frac{1}{3}$

A. $\frac{2}{3}$

B. -1

C. -2

D. $\frac{1}{5}$

E. $\frac{1}{3}$

2. Which of the following statements are true for any three-dimensional vectors $\vec{a}$ and $\vec{b}$?

(I) $(\vec{a} \times \vec{b}) \cdot \vec{a} = 0$ True: $\vec{a} \times \vec{b}$ is $\perp$ to $\vec{a}$ and $\vec{b}$

(II) $\vec{a} \times \vec{b} = \vec{b} \times \vec{a}$ Not true: $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

(III) $|\vec{a} \cdot \vec{b}| \leq |\vec{a}||\vec{b}|$ True: $|a \cdot b| = |\vec{a}||\vec{b}||\cos\theta|$

(IV) $\vec{a} \times (3\vec{a}) = \vec{0}$ True $\vec{a}$ and $3\vec{a}$ are parallel.

A. (I) and (IV) only

B. (I), (III) and (IV) only

C. (II) and (III) only

D. (III) and (IV) only

E. All

3. Find the value of k for which the graph of $x^2 + y^2 + z^2 - 4y + 6z = k$ is a sphere of radius 7.

$x^2 + y^2 - 4y + 4 + z^2 + 6z + 9 = k + 4 + 9$

$x^2 + (y-2)^2 + (z+3)^2 = k + 13$

$k + 13 = 7^2$

$k = 49 - 13 = 36$

A. 30

B. 16

C. 36

D. 25

E. 54

4. $\int_0^{\frac{\pi}{2}} x \cos x dx = x \sin x \Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin x dx$

$u = x \quad dv = \cos x dx$

$du = dx \quad v = \sin x$

$= \frac{\pi}{2} + [\cos x]_0^{\frac{\pi}{2}}$

$= \frac{\pi}{2} - 1$

A. π

B. $\frac{\pi}{2} - 1$

C. 1

D. $\frac{\pi}{2}$

E. $\frac{\pi}{2} + 1$

5. $\int_0^{\frac{\pi}{2}} \sin^2 x \cos^3 x\, dx =$ $\int_0^{\frac{\pi}{2}} \sin^2 x\,(1-\sin^2 x)\cos x\, dx$

$= \int_0^{\frac{\pi}{2}} (\sin^2 x - \sin^4 x)\cos x\, dx =$

$u = \sin x \quad du = \cos x\, dx$

$x = 0 \to u = 0$

$x = \frac{\pi}{2} \to u = 1$

$= \int_0^1 (u^2 - u^4)\, du = \left[\frac{u^3}{3} - \frac{u^5}{5}\right]_0^1 = \frac{1}{3} - \frac{1}{5} = \frac{2}{15}$

A. $\frac{1}{15}$

B. $\frac{1}{3}$

C. $\frac{2}{15}$

D. 0

E. $\frac{1}{5}$

6. $\int_0^{\frac{\pi}{4}} \tan x \sec^4 x\, dx =$ $\int_0^{\frac{\pi}{4}} \sec^3 x\,(\sec x \tan x)\, dx =$

$u = \sec x \quad du = \sec x \tan x\, dx$

$x = 0 \to u = 1$

$x = \frac{\pi}{4} \to u = \sqrt{2}$

$= \int_1^{\sqrt{2}} u^3\, du = \frac{u^4}{4}\Big|_1^{\sqrt{2}}$

$= \frac{(\sqrt{2})^4}{4} - \frac{1}{4} = 1 - \frac{1}{4} = \frac{3}{4}$

A. $\frac{4\sqrt{2}}{5}$

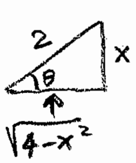
B. $\frac{\sqrt{2}}{5}$

C. $\frac{3}{4}$

D. $\frac{1}{5}$

D. $\frac{5\pi}{4}$

7. For the integral $\int \frac{dx}{x\sqrt{4-x^2}}$, (i) choose a trigonometric substitution to simplify the integral and (ii) give the resulting integral.



$x = 2\sin\theta$

$dx = 2\cos\theta\, d\theta$

$\sqrt{4-x^2} = 2\cos\theta$

$\int \frac{dx}{x\sqrt{4-x^2}} = \int \frac{2\cos\theta\, d\theta}{2\sin\theta\; 2\cos\theta} =$

$= \int \frac{1}{2\sin\theta}\, d\theta$

A. (i) $x = 2\sec\theta$, (ii) $\int \frac{1}{2} d\theta$

B. (i) $x = 2\tan\theta$, (ii) $\int \frac{\sec\theta}{2\tan\theta} d\theta$

C. (i) $x = 2\sin\theta$, (ii) $\int \frac{1}{2\sin\theta} d\theta$

D. (i) $x = 2\sin\theta$, (ii) $\int \frac{1}{4\sin\theta\cos\theta} d\theta$

E. (i) $x = 2\cos\theta$, (ii) $-\int \frac{1}{\cos\theta} d\theta$

8. $\displaystyle\int \frac{2x+5}{x^3+x}\,dx$ is of the form (where a, b, c are constants):

$$\frac{2x+5}{x^3+x} = \frac{2x+5}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$\int \frac{2x+5}{x^3+x}\,dx = \int \frac{A}{x}\,dx + \int \frac{Bx}{x^2+1}\,dx + \int \frac{C}{x^2+1}\,dx$$

$$= a\ln|x| + b\ln(x^2+1) + c\tan^{-1}x + C$$

A. $a\ln(x^2+1) + b\ln|x| + C$

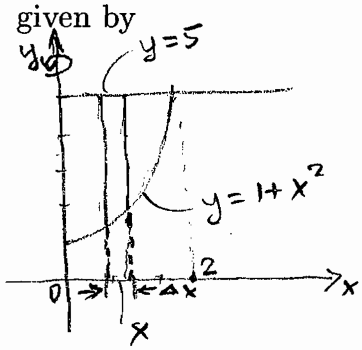
B. $a\ln|x| + b\ln|x+1| + c\ln|x-1| + C$

C. $a\tan^{-1}x + b\ln|x| + C$

D. $a\ln|x^3+x| + C$

E. $a\ln|x| + b\ln(x^2+1) + c\tan^{-1}x + C$

9. The region in the first quadrant bounded by the graph of $y = 1+x^2$, the line $y=5$, and the y-axis is rotated about the y-axis to form a solid. The volume of that solid is given by



$$1+x^2=5$$
$$x^2=4$$
$$x=\pm 2$$

A. $\displaystyle\int_0^2 2\pi[5^2-(1-x^2)^2]dx$

B. $\displaystyle\int_0^2 2\pi x(1+x^2)dx$

C. $\displaystyle\int_1^5 2\pi[5x^2-(1+x^2)^2]dx$

D. $\displaystyle\int_0^2 2\pi x(4-x^2)dx$

E. $\displaystyle\int_1^5 \pi(1+x^2)^2dx$

Volume of typical shell:

$$\Delta V = 2\pi x\left[5-(1+x^2)\right]\Delta x$$

$$V = \int_0^2 2\pi x\left[4-x^2\right]dx$$

10. A solid sphere of radius 1 is divided into two parts by a plane perpendicular to a diameter, mid-way between the center and a tip of the diameter. Find the volume of the smaller part.

A. $\frac{1}{4}\pi$

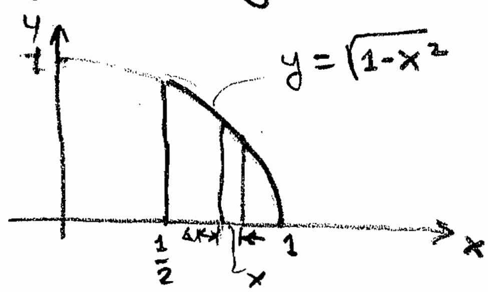
B. $\frac{7}{32}\pi$

C. $\frac{2}{3}\pi$

D. $\frac{3}{16}\pi$

E. $\frac{5}{24}\pi$

The smaller part of the solid sphere is the solid of revolution generated by rotating about the x-axis the region



Volume of typical disk: $\Delta V = \pi\left(\sqrt{1-x^2}\right)^2 \Delta x$

$$V = \int_{1/2}^{1} \pi\left(\sqrt{1-x^2}\right)^2 dx = \pi \int_{\frac{1}{2}}^{1} (1-x^2)\,dx$$

$$= \pi\left[x - \frac{x^3}{3}\right]_{\frac{1}{2}}^{1} = \pi\left(1-\frac{1}{3}\right) - \pi\left(\frac{1}{2} - \frac{\left(\frac{1}{2}\right)^3}{3}\right)$$

$$= \frac{2\pi}{3} - \pi\left(\frac{1}{2} - \frac{1}{24}\right) = \pi\left(\frac{2}{3} - \frac{11}{24}\right) = \frac{5\pi}{24}$$

11. Consider the lamina bounded by the graph of $y = x^2$, the x-axis, and the line $x = 3$, and with density $\rho = 1$. The x–coordinate $\bar{x}$ of the center of mass of the lamina is

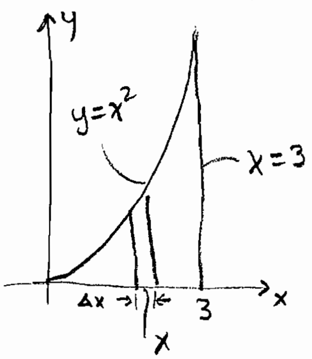
A. $\frac{9}{4}$

B. $\frac{3}{2}$

C. 2

D. $\frac{8}{3}$

E. $\frac{7}{3}$



$$m\bar{x} = M_y$$

$$m = \int_0^3 1\, x^2 dx = \frac{x^3}{3}\Big|_0^3 = 9$$

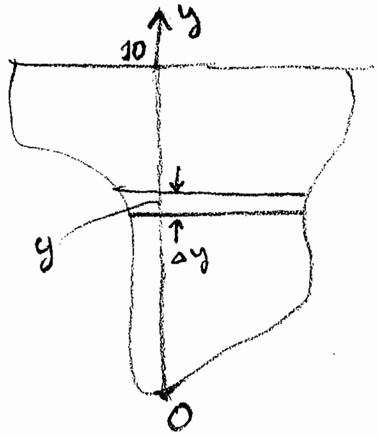
$$M_y = \int_0^3 x \cdot 1 \cdot x^2 dx$$

$$= \frac{x^4}{4}\Big|_0^3 = \frac{81}{4}$$

$$9\bar{x} = \frac{81}{4}$$

$$\bar{x} = \frac{9}{4}$$

12. A tank is 10 ft. high and filled with water weighing 62.5 lbs/ft^3. The cross-sectional area of the tank at y ft. above its bottom is $A(y)$. The work required to pump all the water to the top of the tank is



A. $62.5 \int_0^{10} (10-y)A(y)\,dy$

B. $62.5 \int_0^{10} \pi[A(y)]^2(10-y)\,dy$

C. $62.5 \int_0^{10} (10-y)[10-A(y)]\,dy$

D. $62.5 \int_0^{10} (10-y)\{10^2-[A(y)]^2\}\,dy$

E. $62.5 \int_0^{10} (10-y)^2 A(y)\,dy$

Volume of typical layer $A(y)\Delta y$

$$\Delta W = (10-y)\underbrace{(62.5)A(y)\,\Delta y}_{\text{weight}}$$

$$W = \int_0^{10} (10-y)(62.5)A(y)\,dy$$

13. Which of these improper integrals converge?

(I) $\int_0^{\infty} \cos x\,dx$ (II) $\int_0^{\infty} \frac{x}{1+x^2}\,dx$ (III) $\int_0^{1} \frac{1}{x}\,dx$

A. Only (I)

B. Only (II)

C. Only (III)

D. All of them

E. None of them

(I) $\int_0^{\infty} \cos x\,dx = \lim_{t\to\infty} \int_0^{t} \cos x\,dx = \lim_{t\to\infty} [\sin x]_0^t$
$= \lim_{t\to\infty} \sin t$ DNE
$\therefore$ DIV

(II) $\int_0^{\infty} \frac{x}{1+x^2}\,dx = \lim_{t\to\infty} \int_0^{t} \frac{x}{1+x^2}\,dx$
$= \lim_{t\to\infty} \left[\frac{1}{2}\ln(1+x^2)\right]_0^t = \lim_{t\to\infty} \frac{1}{2}\ln(1+t^2) = \infty$
$\therefore$ DIV

(III) $\int_0^{1} \frac{1}{x}\,dx = \lim_{t\to 0^+} \int_t^{1} \frac{1}{x}\,dx = \lim_{t\to 0^+} [\ln x]_t^1 = \lim_{t\to 0^+} (-\ln t) = \infty$
$\therefore$ DIV

14. Suppose that $\sum_{n=1}^{\infty} a_n = 5$ and $s_n = a_1 + a_2 + \cdots + a_n$. Which one of these statements is true?

$\sum_{n=1}^{\infty} a_n = 5$ means that the limit of the sequence of partial sums s_n as $n \to \infty$ is 5:

$\lim_{n\to\infty} s_n = 5$

We also know that if $\sum_{n=1}^{\infty} a_n$ is convergent $\Rightarrow \lim_{n\to\infty} a_n = 0$

A. $\lim_{n\to\infty} a_n = 5$ and $\lim_{n\to\infty} s_n = 0$

B. $\lim_{n\to\infty} a_n = 0$ and $\lim_{n\to\infty} s_n = 0$

C. $\lim_{n\to\infty} a_n = 5$ and $\lim_{n\to\infty} s_n = 5$

D. $\lim_{n\to\infty} a_n = 0$ and $\lim_{n\to\infty} s_n = 5$

E. $\lim_{n\to\infty} s_n = 5$ but $\lim_{n\to\infty} a_n$ cannot be determined

15. Which of these series converge?

(I) $\sum_{n=1}^{\infty} \frac{1}{n+5}$, $= \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots$ Div. p-series $p=1$

(II) $\sum_{n=1}^{\infty} \frac{n}{(1.01)^n}$, Ratio test: $\frac{n+1}{(1.01)^{n+1}} \frac{(1.01)^n}{n} \to \frac{1}{1.01} < 1$, as $n\to\infty$ ∴ Conv.

(III) $\sum_{n=1}^{\infty} (-1)^n \frac{n^2}{8n^2+5n+9}$ $\lim_{n\to\infty} a_n$ DNE ∴ Div

A. None

B. Only (II)

C. (II) and (III)

D. Only (III)

E. All

16. Which of these series converge absolutely?

(I) $\sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{\sqrt{3}}\right)^n$, (II) $\sum_{n=1}^{\infty}(-1)^n\frac{1}{\sqrt{3n}}$, (III) $\sum_{n=2}^{\infty}(-1)^n\frac{1}{\sqrt{\ln n}}$

(I) $\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{3}}\right)^n$ geom. ser. $r=\frac{1}{\sqrt{3}}<1$ ∴ conv abs.

(II) $\sum_{n=1}^{\infty}\frac{1}{\sqrt{3}\,n^{1/2}}$ div. : p-ser. $p=\frac{1}{2}<1$

(III) $\sum_{n=2}^{\infty}\frac{1}{\sqrt{\ln n}}$ div.: compare with $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ which div. p-series $p=\frac{1}{2}<1$

A. Only (I)
B. All
C. (I) and (II)
D. (II) and (III)
E. (I) and (III)

17. Find the interval of convergence of the power series $\sum_{n=1}^{\infty}\frac{x^n}{(n+1)2^n}$

Ratio test:

$$\left|\frac{x^{n+1}}{(n+2)\,2^{n+1}}\,\frac{(n+1)2^n}{x^n}\right| = \frac{1}{2}\,\frac{n+1}{n+2}|x| \to \frac{1}{2}|x| \text{ as } n\to\infty$$

∴ conv if $\frac{1}{2}|x|<1$ or $-2<x<2$

When $x=-2$; $\sum_{n=1}^{\infty}\frac{(-1)^n}{n+1}$ conv. by Alt. ser. test

When $x=2$: $\sum_{n=1}^{\infty}\frac{1}{n+1}$ div. p-ser. $p=1$.

A. $[-2,2)$
B. $(-\infty,\infty)$
C. $(-2,2)$
D. $(-\frac{1}{2},\frac{1}{2}]$
E. $[-\frac{1}{2},\frac{1}{2})$

18. The radius of convergence of the power series $\sum_{n=1}^{\infty}n!x^n$ is

Ratio test

$$\left|\frac{(n+1)!\,x^{n+1}}{n!\,x^n}\right| = (n+1)|x| \longrightarrow \infty \text{ as } n\to\infty \text{ for } x\neq 0$$

∴ conv only at $x=0$

$R=0$

A. ∞
B. 1
C. 2
D. e
E. 0

19. Match the functions with their Maclaurin series.

(1)	e^x	(b)	(a)	$\sum_{n=0}^{\infty} x^n, \ -1 < x < 1$
(2)	$\frac{1}{1-x}$	(a)	(b)	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots, \ -\infty < x < \infty$
(3)	$\sin x$	(d)	(c)	$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots, \ -\infty < x < \infty$
(4)	$\cos x$	(c)	(d)	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots, \ -\infty < x < \infty$
(5)	$\frac{1}{1+x^2}$	(e)	(e)	$1 - x^2 + x^4 - x^6 + \ldots, \ -1 < x < 1$

A. 1a,2b,3d,4c,5e

B. 1b,2a,3d,4c,5e

C. 1b,2e,3c,4d,5a

D. 1b,2c,3a,4e,5d

E. 1a,2e,3d,4c,5b

20. $1 - \frac{1}{2!}\left(\frac{\pi}{2}\right)^2 + \frac{1}{4!}\left(\frac{\pi}{2}\right)^4 - \frac{1}{6!}\left(\frac{\pi}{2}\right)^6 + \cdots =$ $\cos\left(\frac{\pi}{2}\right) = 0$

A. $\frac{4}{4-\pi^2}$

B. $e^{-\frac{\pi^2}{4}}$

C. $e^{-\frac{\pi}{2}}$

D. 0

E. $\frac{2}{\pi - 2}$

21. In the Taylor series of $f(x) = \frac{1}{x}$ centered at $a = 1$, the coefficient of $(x-1)^3$ is

The coefficient of $(x-1)^3$ is $\frac{f^{(3)}(1)}{3!}$

$f(x) = x^{-1}$

$f^{(1)}(x) = -x^{-2}$

$f^{(2)}(x) = 2x^{-3}$

$f^{(3)}(x) = -2 \cdot 3 x^{-4}$

$f^{(3)}(1) = -3!$

$\frac{f^{(3)}(1)}{3!} = \frac{-3!}{3!} = -1$

A. $\frac{1}{3}$

B. $\frac{1}{3!}$

C. 3

D. 1

E. -1

22. The length of the ellipse parametrized by $x = 2\cos t$, $y = \sin t$, for $0 \le t \le 2\pi$ is given by

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{(-2\sin t)^2 + (\cos t)^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{4\sin^2 t + \cos^2 t}\, dt$$

$$= \int_0^{2\pi} \sqrt{3\sin^2 t + 1}\, dt$$

A. $\int_0^{2\pi} \sqrt{3\sin^2 t + 1}\, dt$

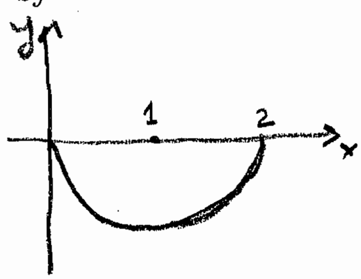
B. $\int_0^{2\pi} \sqrt{3\cos^2 t + 1}\, dt$

C. $\int_0^{2\pi} (\sqrt{3}\cos t + 1) dt$

D. $\int_0^{2\pi} (\sqrt{3}\sin t + 1) dt$

E. $2\int_0^{\frac{\pi}{2}} \sqrt{3\sin^2 t + 1}\, dt$

23. The lower half ($y \le 0$) of the circle $(x-1)^2 + y^2 = 1$ is described in polar coordinates by



A. $r = 2\sin\theta$, for $0 \le \theta \le \frac{\pi}{2}$

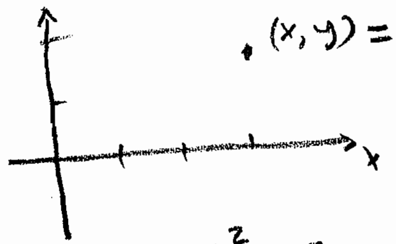
B. $r = 2\cos\theta$, for $-\frac{\pi}{2} \le \theta \le 0$

C. $r = 2\cos\theta$, for $-\pi \le \theta \le 0$

D. $r = \cos\theta$, for $-\frac{\pi}{2} \le \theta \le 0$

E. $r = \sin\theta$, for $0 \le \theta \le \pi$

24. A point P has Cartesian coordinates $(x, y) = (3, \sqrt{3})$. Polar coordinates (r, θ) for P are



$(x,y) = (3, \sqrt{3})$

A. $(\sqrt{6}, \frac{\pi}{6})$

B. $(\sqrt{3}, \frac{\pi}{2})$

C. $(2\sqrt{3}, \frac{\pi}{3})$

D. $(2\sqrt{3}, -\frac{\pi}{6})$

E. $(2\sqrt{3}, \frac{\pi}{6})$

$r^2 = 9 + 3 = 12 \quad r = \pm 2\sqrt{3}$

$\tan\theta = \frac{\sqrt{3}}{3} \longrightarrow \theta = \frac{\pi}{6}, \frac{2\pi}{6}, \ldots$

$(r, \theta) = (2\sqrt{3}, \frac{\pi}{6})$

25. The complex conjugate of the number $\dfrac{1+4i}{3+2i}$ is

$$\frac{1+4i}{3+2i} = \frac{1+4i}{3+2i}\,\frac{3-2i}{3-2i} =$$

$$= \frac{3-2i+12i+8}{9+4}$$

$$= \frac{11+10i}{13}$$

$$= \frac{11}{13} + \frac{10}{13}i$$

$$\overline{\frac{1+4i}{3+2i}} = \overline{\frac{11}{13} + \frac{10}{13}i} = \frac{11}{13} - \frac{10}{13}i$$

A. $\dfrac{10}{13} + \dfrac{11}{13}i$

B. $\dfrac{11}{13} - \dfrac{10}{13}i$

C. $\dfrac{10}{12} - \dfrac{11}{12}i$

D. $\dfrac{11}{10} - \dfrac{12}{10}i$

E. $\dfrac{11}{13} + \dfrac{10}{13}i$