NAME SOLUTIONS

10-DIGIT PUID ______

REC. INSTR. ______ REC. TIME ______

LECTURER ______

INSTRUCTIONS:

1. There are 14 different test pages (including this cover page). Make sure you have a complete test.
2. Fill in the above items in print. Also write your name at the top of pages 2–14.
3. Do any necessary work for each problem on the space provided or on the back of the pages of this test booklet. Circle your answers in this test booklet. No partial credit will be given, but if you show your work on the test booklet, it may be used in borderline cases.
4. No books, notes, calculators or any electronic devices may be used on this exam.
5. Each problem is worth 8 points. The maximum possible score is 200 points.
6. Using a #2 pencil, fill in each of the following items on your answer sheet:
   (a) On the top left side, write your name (last name, first name), and fill in the little circles.
   (b) On the bottom left side, under SECTION NUMBER, put 0 in the first column and then enter the 3-digit section number. For example, for section 016 write 0016. Fill in the little circles.
   (c) On the bottom, under TEST/QUIZ NUMBER, write 01 and fill in the little circles.
   (d) On the bottom, under STUDENT IDENTIFICATION NUMBER, write in your 10–digit PUID, and fill in the little circles.
   (e) Using a #2 pencil, put your answers to questions 1–25 on your answer sheet by filling in the circle of the letter of your response. Double check that you have filled in the circles you intended. If more than one circle is filled in for any question, your response will be considered incorrect. Use a #2 pencil.
7. After you have finished the exam, hand in your answer sheet and your test booklet to your recitation instructor.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!},\ |x| < \infty \qquad \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\, x^{2n+1},\ |x| < \infty$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\, x^{2n},\ |x| < \infty \qquad \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n,\ |x| < 1$$

1. Let $P(3,2,4)$ and $Q(5,-4,7)$ be points in $\mathbb{R}^3$. Find a vector that has direction opposite of $\overrightarrow{PQ}$ and has length 2.

$\overrightarrow{PQ} = \langle 2,-6,3 \rangle$

$|\overrightarrow{PQ}| = \sqrt{4+36+9} = \sqrt{49} = 7$

Unit vector in the direction of $\overrightarrow{PQ}$

$\frac{1}{7}\langle 2,-6,3\rangle$

Unit vector in the direction opposite to $\overrightarrow{PQ}$

$-\frac{1}{7}\langle 2,-6,3\rangle$

Desired vector

$-\frac{2}{7}\langle 2,-6,3\rangle = \langle -\frac{4}{7}, \frac{12}{7}, -\frac{6}{7}\rangle$

A. $\langle -2,6,-3\rangle$

B. $\langle -\frac{2}{5}, \frac{6}{5}, -\frac{3}{5}\rangle$

C. $\langle \frac{2}{6}, -1, \frac{3}{6}\rangle$

D. $\langle -\frac{2}{3}, 2, -\frac{1}{2}\rangle$

E. $\langle -\frac{4}{7}, \frac{12}{7}, -\frac{6}{7}\rangle$

2. For what values of b are the vectors $\vec{i}+3\vec{j}+b\vec{k}$ and $b\vec{i}-b\vec{j}+2b\vec{k}$ orthogonal?

$(\vec{i}+3\vec{j}+b\vec{k})\cdot(b\vec{i}-b\vec{j}+2b\vec{k}) = 0$

$b-3b+2b^2=0$

$2(b^2-b)=0 \rightarrow b(b-1)=0$

$b=0,1$

A. $b=2$

B. $b=1,2$

C. $b=2,0$

D. $b=1,0$

E. $b=1$

3. If $\vec{a} = \langle 1, 2, -1\rangle$ and $\vec{b} = \langle 2, 1, -1\rangle$, find the vector projection of $\vec{b}$ onto $\vec{a}$, $\text{proj}_{\vec{a}}\, \vec{b}$.

$$\text{proj}_{\vec{a}} \vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|} \frac{\vec{a}}{|\vec{a}|}$$

$$\vec{a}\cdot\vec{b} = 2+2+1 = 5$$

$$|\vec{a}| = \sqrt{1+4+1} = \sqrt{6}$$

$$\text{proj}_{\vec{a}} \vec{b} = \frac{5}{6}\vec{a} = \frac{5}{6}\langle 1,2,-1\rangle$$

A. $\frac{5}{6}\langle 1, 2, -1\rangle$

B. $\frac{5}{6}\langle 2, 1, -1\rangle$

C. $\frac{1}{6}\langle 2, 1, -1\rangle$

D. $\frac{1}{6}\langle 1, 2, -1\rangle$

E. $\frac{1}{3}\langle 1, 2, -1\rangle$

4. The area of the parallelogram determined by the vectors $3\vec{\imath} + \vec{\jmath}$ and $\vec{\jmath} + 2\vec{k}$ is

Let $\vec{a} = 3\vec{i} + \vec{j}$ and $\vec{b} = \vec{j} + 2\vec{k}$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 1 & 0 \\ 0 & 1 & 2 \end{vmatrix}$$

$$= 2\vec{i} - 6\vec{j} + 3\vec{k}$$

$$A = |\vec{a}\times\vec{b}| = \sqrt{4+36+9}$$

$$= \sqrt{49}$$

A. $\sqrt{11}$

B. $\sqrt{40}$

C. $\sqrt{45}$

D. $\sqrt{49}$

E. $\sqrt{50}$

5. Find the area of the region bounded by $y = x^2$ and $y = 4x - x^2$.

Points of intersection of the curves:

$x^2 = 4x - x^2$

$2x^2 - 4x = 0$

$x(x-2) = 0 \Rightarrow x = 0, 2$

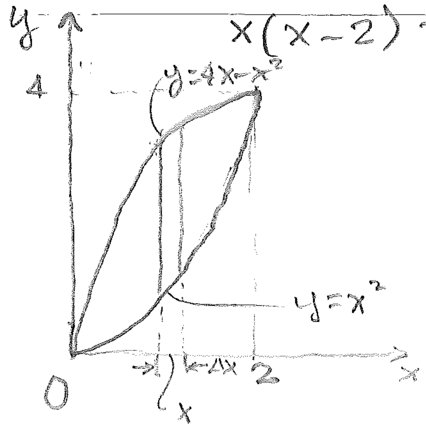
A. $\frac{8}{3}$

B. $\frac{4}{3}$

C. $\frac{16}{3}$

D. 8

E. 2



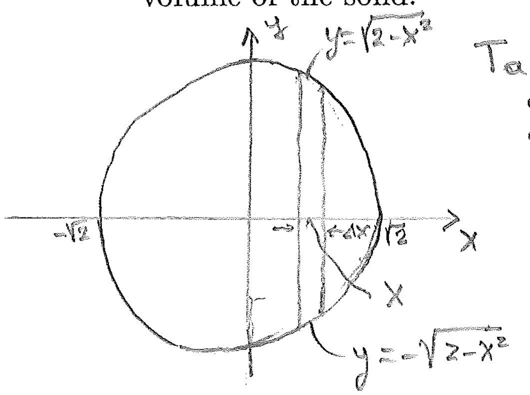
Area of typical approximating rectangle:

$$\Delta A = [(4x-x^2) - x^2]\Delta x$$

$$A = \int_0^2 (4x - x^2 - x^2)\,dx = \int_0^2 (4x - 2x^2)\,dx$$

$$= \left(2x^2 - \frac{2}{3}x^3\right)\Big|_0^2 = 8 - \frac{2}{3}\cdot 8 = \frac{8}{3}$$

6. The base of a 3–dimensional solid is in the xy–plane and is bounded by the circle $x^2 + y^2 = 2$. Parallel cross sections perpendicular to the base are squares. Find the volume of the solid.



Take the parallel cross sections perpendicular to the x-axis

A. $\frac{64}{3}\sqrt{2}$

B. $\frac{48}{3}\sqrt{2}$

C. $\frac{16}{3}\sqrt{2}$

D. $\frac{32}{3}\sqrt{2}$

E. $8\sqrt{2}$

Volume of typical approximating slice:

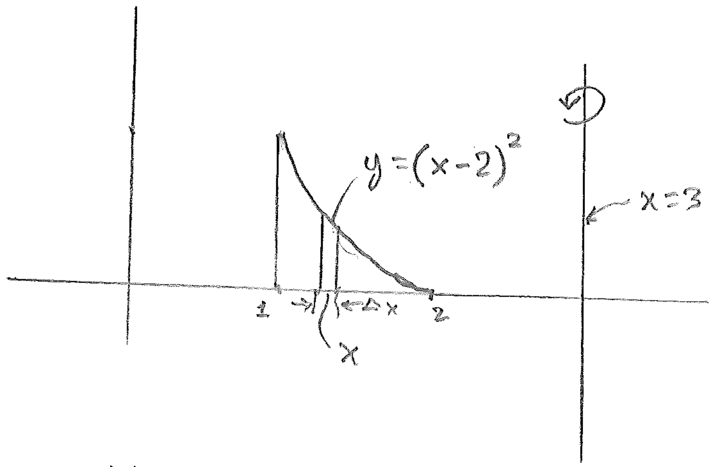
$$\Delta V = (2\sqrt{2-x^2})^2 \Delta x$$

$$V = \int_{-\sqrt{2}}^{\sqrt{2}} 4(2-x^2)\,dx = 2\int_0^{\sqrt{2}} 4(2-x^2)\,dx$$

$$= 8\left[2x - \frac{x^3}{3}\right]_0^{\sqrt{2}} = 8\left[2\sqrt{2} - \frac{2}{3}\sqrt{2}\right]$$

$$= \frac{32}{3}\sqrt{2}$$

7. Let R be the region bounded by the curves $y = (x-2)^2,\ 1 \le x \le 2;\ y = 0$ and $x = 1$. Using the method of cylindrical shells, the volume of the solid generated by rotating R about the line $x = 3$ is given by



A. $\int_0^1 2\pi(x-3)(x-2)^2\,dx$

B. $\int_0^1 2\pi(x-2)^2\,dx$

C. $\int_1^2 2\pi(2-x)(x-2)^2\,dx$

D. $\int_1^2 2\pi x(x-2)^2\,dx$

E. $\int_1^2 2\pi(3-x)(x-2)^2\,dx$

Volume of typical approximating shell:

$$\Delta V = 2\pi(3-x)(x-2)^2\Delta x$$

$$V = \int_1^2 2\pi(3-x)(x-2)^2\,dx$$

8. Evaluate $\int_e^{2e} \ln x dx$

A. $2\ln e$

B. 1

C. 3

D. $2e[\ln(2e) - 1]$

E. $2e(\ln 2 - 1)$

Integration by parts

$$\int_e^{2e} \ln x\,dx = x\ln x\Big|_e^{2e} - \int_e^{2e} x\frac{1}{x}\,dx =$$

$u = \ln x \quad dv = dx$

$du = \frac{1}{x}dx \quad v = x$

$$= \Big[x\ln x - x\Big]_e^{2e}$$

$$= 2e\ln(2e) - 2e - [e\ln e - e]$$

$$= 2e[\ln(2e) - 1] - [e\cdot 1 - e]$$

$$= 2e[\ln(2e) - 1]$$

9. Evaluate $\displaystyle\int_1^2 \frac{1}{\sqrt{2x-x^2}}\,dx$

A. -2

B. 2π

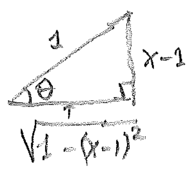
C. $\dfrac{\pi}{2}$

D. π

E. $\dfrac{3\pi}{2}$

$$2x - x^2 = -(x^2 - 2x + 1) + 1$$
$$= 1 - (x-1)^2$$

$$\int_1^2 \frac{1}{\sqrt{2x-x^2}}\,dx = \int_1^2 \frac{1}{\sqrt{1-(x-1)^2}}\,dx =$$



$x - 1 = \sin\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$dx = \cos\theta\,d\theta$

$\sqrt{1-(x-1)^2} = \cos\theta$

$x = 1 \rightarrow \theta = 0$

$x = 2 \rightarrow \theta = \frac{\pi}{2}$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{\cos\theta}\cos\theta\,d\theta = \theta\Big|_0^{\frac{\pi}{2}} = \frac{\pi}{2}$$

Or Use substitution $u = x - 1$

10. Evaluate $\displaystyle\int_0^2 \frac{2}{x^2+4x+3}\,dx$

A. $2\ln 3$

B. $2(\ln 19 - \ln 3)$

C. $5\ln 3 - 2\ln 5$

D. $2\ln 3 - \ln 5$

E. $\ln 3 - \ln 5$

Partial fractions:

$$\frac{2}{x^2+4x+3} = \frac{2}{(x+1)(x+3)} = \frac{A}{x+1} + \frac{B}{x+3}$$

$$2 = Ax + 3A + Bx + B$$

$$\left.\begin{array}{l} A + B = 0 \\ 3A + B = 2 \end{array}\right] \rightarrow A = 1 \quad B = -1$$

$$\int_0^2 \frac{2}{x^2+4x+3}\,dx = \int_0^2 \left(\frac{1}{x+1} - \frac{1}{x+3}\right)dx$$

$$= \Big[\ln(x+1) - \ln(x+3)\Big]_0^2$$
$$= \ln 3 - \ln 5 - (\ln 1 - \ln 3)$$
$$= 2\ln 3 - \ln 5$$

11. The length of the curve $x = \frac{2}{3}(y-1)^{3/2}$ from $(0,1)$ to $(\frac{16}{3}, 5)$ is

$$L = \int_1^5 \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$
$$= \int_1^5 \sqrt{1 + (y-1)}\, dy$$
$$= \int_1^5 \sqrt{y}\, dy$$
$$= \frac{2}{3} y^{3/2}\Big|_1^5 = \frac{2}{3}\left(5^{3/2} - 1\right)$$

A. $\frac{2}{3}(\frac{16}{3})^{2/3} - 1$

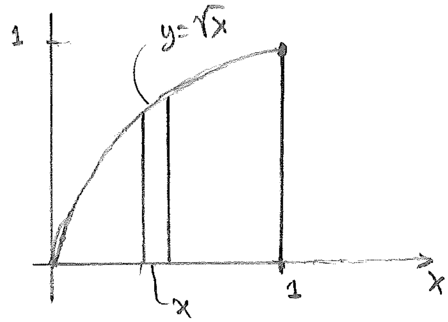
B. $\frac{4}{3}\, 5^{3/2}$

C. $\frac{2}{3}\, 5^{3/2}$

D. $\frac{4}{3}(5^{3/2} - 1)$

E. $\frac{2}{3}(5^{3/2} - 1)$

12. Consider the lamina bounded by the graph of $y = \sqrt{x}$, $0 \le x \le 1$; the x–axis, and the line $x = 1$, and with density $\rho = 1$. The x–coordinate $\overline{x}$ of the center of mass of the lamina is



$$m\overline{x} = M_y$$
$$m = \int_0^1 1\sqrt{x}\, dx = \frac{2}{3} x^{3/2}\Big|_0^1 = \frac{2}{3}$$
$$M_y = \int_0^1 x\, 1\sqrt{x}\, dx = \frac{2}{5} x^{5/2}\Big|_0^1 = \frac{2}{5}$$
$$\frac{2}{3}\overline{x} = \frac{2}{5} \longrightarrow \overline{x} = \frac{3}{5}$$

A. $\overline{x} = \frac{3}{8}$

B. $\overline{x} = \frac{3}{5}$

C. $\overline{x} = \frac{1}{2}$

D. $\overline{x} = \frac{4}{7}$

E. $\overline{x} = \frac{5}{8}$

13. Which of these improper integrals converge?

I. $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx$

II. $\displaystyle\int_0^3 \frac{1}{x-2}\,dx$

III. $\displaystyle\int_0^\infty \frac{1}{1+x^2}\,dx$

A. Only I

B. Only II

C. Only III

D. I and III

E. II and III

I. $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{t\to 0^+}\int_t^1 \frac{1}{\sqrt{x}}\,dx = \lim_{t\to 0^+} 2\sqrt{x}\Big|_t^1 = 2$ ∴ conv.

II. $\displaystyle\int_0^3 \frac{1}{x-2}\,dx = \int_0^2 \frac{1}{x-2}\,dx + \int_2^3 \frac{1}{x-2}\,dx$?

$\displaystyle\int_0^2 \frac{1}{x-2}\,dx = \lim_{t\to 2^-}\int_0^t \frac{1}{x-2}\,dx = \lim_{t\to 2^-}\Big[\ln|x-2|\Big]_0^t = \lim_{t\to 2^-}\big[\ln|t-2|-\ln 2\big]$

$= -\infty$ ∴ div.

III $\displaystyle\int_0^\infty \frac{1}{1+x^2}\,dx = \lim_{t\to\infty}\int_0^t \frac{1}{1+x^2}\,dx = \lim_{t\to\infty}\Big[\tan^{-1}x\Big]_0^t = \lim_{t\to\infty}\tan^{-1}t = \frac{\pi}{2}$

∴ conv

14. $\displaystyle\lim_{n\to\infty}\frac{\ln(2n)}{\ln(3n)} =$

A. 1

B. 0

C. $\frac{2}{3}$

D. $\ln\frac{2}{3}$

E. Does not exist

$\displaystyle = \lim_{x\to\infty}\frac{\ln(2x)}{\ln(3x)} \overset{L'H}{=} \lim_{x\to\infty}\frac{\frac{1}{x}}{\frac{1}{x}}$

$\frac{\infty}{\infty}$

$\displaystyle = \lim_{x\to\infty} 1 = 1$

15. $-\frac{4}{5^2}+\frac{4^2}{5^3}-\frac{4^3}{5^4}+\frac{4^4}{5^5}-\frac{4^5}{5^6}+\ldots=$

$$=\frac{1}{5}\left[-\frac{4}{5}+\left(\frac{4}{5}\right)^2-\left(\frac{4}{5}\right)^3+\left(\frac{4}{5}\right)^4-\left(\frac{4}{5}\right)^5+\cdots\right]$$

$$=-\frac{4}{25}\left[1-\frac{4}{5}+\left(\frac{4}{5}\right)^2-\left(\frac{4}{5}\right)^3+\left(\frac{4}{5}\right)^4-\cdots\right]$$

$$=-\frac{4}{25}\left[1+\left(-\frac{4}{5}\right)+\left(-\frac{4}{5}\right)^2+\left(-\frac{4}{5}\right)^3+\left(-\frac{4}{5}\right)^4+\cdots\right]$$

geometric series

$r=-\frac{4}{5}$, $\left|-\frac{4}{5}\right|<1$

$$=-\frac{4}{25}\,\frac{1}{1-\left(-\frac{4}{5}\right)}=-\frac{4}{25}\,\frac{1}{1+\frac{4}{5}}$$

$$=-\frac{4}{5}\,\frac{1}{9}=-\frac{4}{45}$$

A. $-\frac{4}{5}$

B. $\frac{4}{5}$

C. $\frac{4}{45}$

D. $-\frac{4}{45}$

E. $\frac{9}{5}$

16. Which of the following series converge?

I. $\sum_{n=1}^{\infty}\frac{n!}{5^n}$

II. $\sum_{n=1}^{\infty}\frac{1}{n\sqrt{n}}$

III. $\sum_{n=1}^{\infty}\frac{1}{e^n+1}$

IV. $\sum_{n=1}^{\infty}(-1)^n\sqrt{n}$

A. I and II only

B. II and III only

C. II, III and IV only

D. III and IV only

E. IV only

I. Ratio test: $\left|\frac{a_{n+1}}{a_n}\right|=\frac{(n+1)!}{5^{n+1}}\,\frac{5^n}{n!}=\frac{n+1}{5}\to\infty$ as $n\to\infty$, $\therefore$ div.

II. $\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$, p-series $p=\frac{3}{2}>1$ $\therefore$ conv,

III compare with $\sum_{n=1}^{\infty}\frac{1}{e^n}=\sum_{n=1}^{\infty}\left(\frac{1}{e}\right)^n$ which is conv (geom. series $r=\frac{1}{e}$, $\frac{1}{e}<1$)

$\frac{1}{e^n+1}<\frac{1}{e^n}$ for all $n\geq 1$ $\therefore$ conv,

IV $\lim_{n\to\infty}(-1)^n\sqrt{n}$ DNE $\therefore$ div by the test for div.

17. Find the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{e^5}{n^2}(x-3)^n$

Ratio test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{e^5(x-3)^{n+1}}{(n+1)^2} \frac{n^2}{e^5(x-3)^n}\right|$$

$$= \left(\frac{n}{n+1}\right)^2 |x-3| \to |x-3| \text{ as } n\to\infty$$

$\therefore$ series conv if $|x-3|<1$

or $2<x<4$

when $x=2$: $e^5\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$ conv. by AST

when $x=4$: $e^5\sum_{n=1}^{\infty}\frac{1}{n^2}$ conv p-series $p=2>1$

$\therefore$ interval of convergence is $[2,4]$

A. $[2, 4]$

B. $[2, 4)$

C. $(3-e, 3+e)$

D. $[3-e, 3+e]$

E. $(-\infty, \infty)$

18. Use a Maclaurin series and the Estimation Theorem for alternating series to approximate e^{-1} using the fewest number of terms necessary so that the error is less than 0.01.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

$$e^{-1} = 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} +$$

error $\frac{1}{120} < 0.01$

$$\approx \frac{1}{2} - \frac{1}{6} + \frac{1}{24}$$

$$= \frac{12-4+1}{24} = \frac{9}{24} = \frac{3}{8}$$

A. $e^{-1} \approx \frac{1}{3}$

B. $e^{-1} \approx \frac{11}{18}$

C. $e^{-1} \approx \frac{3}{8}$

D. $e^{-1} \approx \frac{2}{5}$

E. $e^{-1} \approx \frac{3}{7}$

19. Find the coefficient of x^9 in the Maclaurin series for $f(x) = \dfrac{1}{(1+x)^2}$.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

The coefficient of x^n is $\dfrac{f^{(n)}(0)}{n!}$

$f(x) = (1+x)^{-2}$ $\quad f(0) = 1$

$f^{(1)}(x) = -2(1+x)^{-3}$ $\quad f^{(1)}(0) = -2 = -2!$

$f^{(2)}(x) = 2\cdot 3(1+x)^{-4}$ $\quad f^{(2)}(0) = 2\cdot 3 = 3!$

$f^{(3)}(x) = -2\cdot 3\cdot 4(1+x)^{-5}$ $\quad f^{(3)}(0) = -2\cdot 3\cdot 4 = -4!$

$f^{(4)}(x) = 2\cdot 3\cdot 4\cdot 5(1+x)^{-6}$ $\quad f^{(4)}(0) = 2\cdot 3\cdot 4\cdot 5 = 5!$

$\vdots$

$f^{(n)}(0) = (-1)^n (n+1)!$

$f^{(9)}(0) = -(10)!$ $\quad \dfrac{f^{(9)}(0)}{9!} = -\dfrac{10!}{9!} = -10$

Or $\dfrac{1}{(1+x)^2} = -\dfrac{d}{dx}\left(\dfrac{1}{1+x}\right) = -\dfrac{d}{dx}\left[1 - x + x^2 - x^3 + \cdots + x^{10} - \cdots\right]$

A. -8

B. 9

C. -10

D. -9

E. 0

20. Find the coefficient of x^{13} in the power series for $\displaystyle\int \frac{1}{1+x^6}\,dx$

$$\int \frac{1}{1+x^6}\,dx = \int (1 - x^6 + x^{12} - x^{18} + \cdots)\,dx$$

$$= C + \left[x - \frac{x^7}{7} + \underbrace{\frac{x^{13}}{13}}_{\frac{1}{13}} - \frac{x^{19}}{19} + \cdots\right]$$

A. 0

B. -13

C. $-\dfrac{1}{12}$

D. 12

E. $\dfrac{1}{13}$

21. Evaluate $\lim_{x\to 0} \dfrac{x\sin x - x^2 + \dfrac{x^4}{6}}{\cos x - 1 + \dfrac{x^2}{2} - \dfrac{x^4}{24}}$

A. -6

B. -3

C. 0

D. 3

E. 6

$$= \lim_{x\to 0} \frac{x\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots\right) - x^2 + \frac{x^4}{6}}{\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots\right) - 1 + \frac{x^2}{2} - \frac{x^4}{4!}}$$

$$= \lim_{x\to 0} \frac{\frac{x^6}{5!} - \frac{x^8}{7!} + \dots}{-\frac{x^6}{6!} + \frac{x^8}{8!} - \dots}$$

$$= \lim_{x\to 0} \frac{\frac{1}{5!} - \frac{x^2}{7!} + \dots}{-\frac{1}{6!} + \frac{x^2}{8!} - \dots} = \frac{\frac{1}{5!}}{-\frac{1}{6!}} = -\frac{6!}{5!} = -6$$

22. Find an equation of the tangent line to the parametric curve $x = t^5 + 1$, $y = t^3 - 2t$ at the point corresponding to $t = 1$.

A. $x - 5y - 7 = 0$

B. $x + y - 3 = 0$

C. $x + 5y + 3 = 0$

D. $x - y + 3 = 0$

E. $5x - y - 11 = 0$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t^2 - 2}{5t^4}$$

When $t = 1$: $x = 2$, $y = -1$, $\frac{dy}{dx} = \frac{1}{5}$

eq. of tangent line:

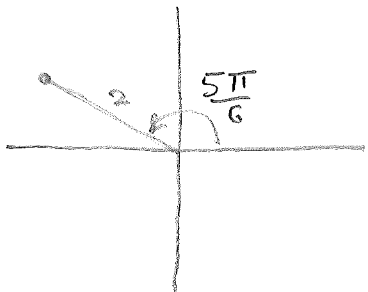
$$y + 1 = \frac{1}{5}(x - 2)$$

$$5y + 5 = x - 2$$

$$x - 5y - 7 = 0$$

23. The point P has polar coordinates $(2, \frac{5\pi}{6})$. Which of the following are also polar coordinates of P?

I. $(2, -\frac{\pi}{6})$ II. $(-2, -\frac{\pi}{6})$ III. $(2, -\frac{7\pi}{6})$ IV. $(-2, \frac{7\pi}{6})$



I $(2, -\frac{\pi}{6})$ no

II $(-2, -\frac{\pi}{6})$ yes

III $(2, -\frac{7\pi}{6})$ yes

IV $(-2, \frac{7\pi}{6})$ no

A. I and II only

B. II and III only

C. II and IV only

D. I and III only

E. III and IV only

24. The graph of the polar equation $r = -4\sin\theta$ is a circle. The Cartesian coordinates of the center of the circle are

$r = -4\sin\theta$

$r^2 = -4r\sin\theta$

$x^2 + y^2 = -4y$

$x^2 + y^2 + 4y + 4 = 4$

$x^2 + (y+2)^2 = 2^2$

center $(0, -2)$

A. $(-2, 0)$

B. $(2, 0)$

C. $(0, -2)$

D. $(0, 1)$

E. $(0, -1)$

25. The polar form of the complex number $\dfrac{1-\frac{1}{\sqrt{3}}\,i}{1+\frac{1}{\sqrt{3}}\,i}$ with argument between 0 and 2π is

A. $\cos\dfrac{\pi}{3}+i\sin\dfrac{\pi}{3}$

B. $\sqrt{3}(\cos\dfrac{\pi}{3}+i\sin\dfrac{\pi}{3})$

C. $\dfrac{1}{2}(\cos\dfrac{3\pi}{4}+i\sin\dfrac{3\pi}{4})$

D. $\cos\dfrac{5\pi}{6}+i\sin\dfrac{5\pi}{6}$

E. $\cos\dfrac{5\pi}{3}+i\sin\dfrac{5\pi}{3}$

Polar form of $1-\frac{1}{\sqrt{3}}i$

$r^2 = 1+\frac{1}{3}=\frac{4}{3} \quad r=\frac{2}{\sqrt{3}}$

$\tan\theta = \dfrac{-\frac{1}{\sqrt{3}}}{1} = -\frac{\sqrt{3}}{3} \to \theta = -\frac{\pi}{6}$

$\therefore\ 1-\frac{1}{\sqrt{3}}i = \frac{2}{\sqrt{3}}\left[\cos(-\frac{\pi}{6})+i\sin(-\frac{\pi}{6})\right]$

Polar form of $1+\frac{1}{\sqrt{3}}i$

$r^2 = 1+\frac{1}{3}=\frac{4}{3} \quad r=\frac{2}{\sqrt{3}}$

$\tan\theta = \dfrac{\frac{1}{\sqrt{3}}}{1} = \frac{\sqrt{3}}{3} \to \theta = \frac{\pi}{6}$

$\therefore\ 1+\frac{1}{\sqrt{3}}i = \frac{2}{\sqrt{3}}\left[\cos(\frac{\pi}{6})+i\sin(\frac{\pi}{6})\right]$

$$\therefore\ \frac{1-\frac{1}{\sqrt{3}}i}{1+\frac{1}{\sqrt{3}}i} = \cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right) = \cos\left(\frac{5\pi}{3}\right)+i\sin\left(\frac{5\pi}{3}\right)$$

Or

$$\frac{1-\frac{1}{\sqrt{3}}i}{1+\frac{1}{\sqrt{3}}i} = \frac{\left(1-\frac{1}{\sqrt{3}}i\right)\left(1-\frac{1}{\sqrt{3}}i\right)}{\left(1+\frac{1}{\sqrt{3}}i\right)\left(1-\frac{1}{\sqrt{3}}i\right)}$$

$$= \frac{1-\frac{1}{\sqrt{3}}i-\frac{1}{\sqrt{3}}i-\frac{1}{3}}{\frac{4}{3}} = \frac{\frac{2}{3}-\frac{2}{\sqrt{3}}i}{\frac{4}{3}}$$

$$= \frac{1}{2}-\frac{\sqrt{3}}{2}i$$

$r^2 = \frac{1}{4}+\frac{3}{4} = 1 \quad r=1$

$\tan\theta = -\sqrt{3} \to \theta = -\frac{\pi}{3}$

$$\therefore\ \frac{1-\frac{1}{\sqrt{3}}i}{1+\frac{1}{\sqrt{3}}i} = \cos(-\frac{\pi}{3})+i\sin(-\frac{\pi}{3}) = \cos(\frac{5\pi}{3})+i\sin(\frac{5\pi}{3})$$