# MA261 Spring 2015 Exam 1, 8:00-9:00pm

1. Find $\cos\theta$ where $\theta$ is the angle between $\vec{u} \times \vec{v}$ and $\vec{w}$, where $\vec{u} = \vec{i} + 2\vec{j} + \vec{k}$, $\vec{v} = -\vec{i} + \vec{j} + 2\vec{k}$, and $\vec{w} = -2\vec{i} - \vec{j} + 2\vec{k}$.

A. $4/(3\sqrt{26})$
B. $2/\sqrt{27}$
C. $4/(3\sqrt{3})$
D. ~~$3/(\sqrt{27})$~~
E. $\boxed{1/\sqrt{27}}$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{vmatrix} = 3\vec{i} - 3\vec{j} + 3\vec{k}.$$

$$|\vec{u} \times \vec{v}| = \sqrt{9+9+9} = 3\sqrt{3} \qquad |\vec{w}| = \sqrt{4+4+1} = 3$$

$$(\vec{u} \times \vec{v}) \cdot \vec{w} = (3\vec{i} - 3\vec{j} + 3\vec{k}) \cdot (-2\vec{i} - \vec{j} + 2\vec{k})$$

$$= -6 + 3 + 6 = 3.$$

$$(\vec{u} \times \vec{v}) \cdot \vec{w} = |\vec{u} \times \vec{v}| \cdot |\vec{w}| \cdot \cos\theta = 3$$

$$3\sqrt{3} \cdot 3 \cos\theta = 3 \; ; \quad \boxed{\cos\theta = 1/3\sqrt{3}}$$

2. Find the distance between the origin and the point at which the line

$$x = 2 - 3t, \;\; y = 4 + t, \;\; z = 1 + t$$

intersects the plane x+2y-z=5.

A. $\sqrt{40}$
B. $\sqrt{43}$
C. $\sqrt{48}$
D. $\boxed{\sqrt{61}}$
E. $\sqrt{63}$

The line and the plane intersect when

$$\begin{cases} x = 2 - 3t, \; y = 4 + t; \; z = 1 + t \\ \text{and} \\ x + 2y - z = 5 \end{cases}$$

So: $2 - 3t + 2(4+t) - (1+t) = 5$

$2 + 8 - 1 - 3t + 2t - t = 5$ $\quad \boxed{2t = 4} \quad \boxed{t = 2}$

$x = -4$; $y = 6$, $z = 3$. Point of intersection $(-4, 6, 3)$. Distance to the origin: $\sqrt{16+36+9} = \sqrt{61}$

3. Identify the surface $x^2 - y^2 - z = 0$.

A. hyperboloid of one sheet
B. hyperboloid of two sheets
C. sphere
D. hyperbolic paraboloid
E. elliptic paraboloid

$z = x^2 - y^2$ is a hyperbolic paraboloid

4. Determine whether the planes $3x + 2y + z = 261$ and $-x + 2y - z = 7$ are perpendicular, parallel, or neither.

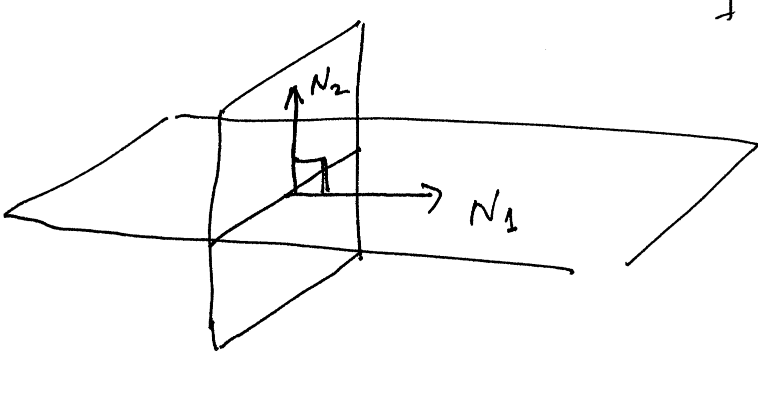
A. perpendicular
B. parallel
C. neither

Normal to the plane $3x + 2y + z = 261$ is $\langle 3, 2, 1 \rangle = N_1$. Normal to ~~the~~ $-x + 2y - z = 7$ is $\langle -1, 2, -1 \rangle = N_2$

$N_1 \cdot N_2 = \langle 3, 2, 1 \rangle \cdot \langle -1, 2, -1 \rangle$

$= -3 + 4 - 1 = 0.$ Normal vectors

Planes are perpendicular



5. Find

$$\left|\int_0^1 \left(t\vec{\imath} + 3t^2\vec{\jmath} + \frac{\pi}{2}\cos\left(\frac{\pi}{2}t\right)\vec{k}\right)\,dt\right|$$

A. 5/2
B. 2/3
$\boxed{\text{C. } 3/2}$
D. 3/4
E. 4/3

$$\int_0^1 \left(t\vec{\imath} + 3t^2\vec{\jmath} + \tfrac{\pi}{2}\cos(\tfrac{\pi}{2}t)\right)dt$$

$= \frac{1}{2}\vec{\imath} + \vec{\jmath} + \vec{k}$. The length of this vector is equal to

$$\sqrt{\tfrac{1}{4}+1+1} = \sqrt{\tfrac{9}{4}} = \tfrac{3}{2}$$

6. Find the length of the curve

$$\vec{r}(t) = \ln(t)\vec{\imath} + \sqrt{2}t\vec{\jmath} + \frac{t^2}{2}\vec{k}$$

for $1 \le t \le 2$.

A. $(3/4) + \ln(2)$
B. $(2/3) + 2\ln(2)$
C. $(1/4) + (1/2)\ln(2)$
$\boxed{\text{D. } (3/2) + \ln(2)}$
E. $(1/3) + 3\ln(2)$

$$\vec{r}\,'(t) = \tfrac{1}{t}\vec{\imath} + \sqrt{2}\vec{\jmath} + t\vec{k}$$

$$|\vec{r}\,'(t)|^2 = \frac{1}{t^2} + 2 + t^2 = \left(t+\tfrac{1}{t}\right)^2.$$

Hence $|\vec{r}\,'(t)| = t + 1/t$

$$\text{Length} = \int_1^2 |\vec{r}\,'(t)| = \int_1^2 \left(t+\tfrac{1}{t}\right)dt$$

$$= \left(\tfrac{t^2}{2} + \ln t\right)\Big|_1^2 = 2 + \ln 2 - \tfrac{1}{2} = \tfrac{3}{2} + \ln 2$$

7. The trajectory of a particle in space is given by $\vec{r}(t)$. Given that its acceleration is $\vec{a}(t) = 2t\vec{i} + 3t^2\vec{j} + 5t^4\vec{k}$, that the velocity at $t = 0$ is $\vec{v}(0) = 2\vec{j} + 4\vec{k}$, and that $\vec{r}(0) = \vec{i} + 2\vec{j} + 3\vec{k}$, then the $\vec{j}$ component of $\vec{r}(2)$ is:

A. 6
B. 8
$\boxed{\text{C. 10}}$
D. 12
E. 16

$$\vec{a}(t) = 2t\vec{i} + 3t^2\vec{j} + 5t^4\vec{k}$$

$$\vec{v}(t) = \int_0^t a(s)\,ds + \vec{v}(0).$$

$$\vec{v}(t) = t^2\vec{i} + t^3\vec{j} + t^5 k + 2\vec{j} + 4\vec{k}$$
$$= t^2\vec{i} + (t^3+2)\vec{j} + (t^5+4)\vec{k}$$

$$\vec{r}(t) = \int_0^t \vec{v}(s)\,ds + \vec{r}(0) = \left(\frac{t^3}{3}+1\right)\vec{i} + \left(\frac{t^4}{4}+2t+2\right)\vec{j} + \left(\frac{t^6}{6}+4t+3\right)\vec{k}$$

The $\vec{j}$ component of $\vec{r}(2)$ is $\frac{16}{4} + 4 + 2 = 10$

8. Consider the following statements:

I The limit $\lim_{(x,y)\to(0,0)}(xy)/(x^2+y^2)$ does not exist. $\boxed{T}$

II The limit $\lim_{(x,y)\to(0,0)}(xy^2)/(x^2+y^2) = 0$. T

III If $f(x,y)$ satisfies $\lim_{x\to 0} f(x,\lambda x) = 0$ for every $\lambda \in \mathbb{R}$, then we can conclude that $\lim_{(x,y)\to(0,0)} f(x,y) = 0$. F

$\boxed{\text{A. I and II are true, but III is false}}$
B. I and III are false, but II is true
C. I is true, but II and III are false
D. I and III are true, but II is false
E. I, II, and III are false

I
$$\lim_{x=y\to 0} \frac{xy}{x^2+y^2} = \lim_{x\to 0}\frac{x^2}{2x^2} = \frac{1}{2} \qquad \text{DNE}$$

$$\lim_{x=2y} \frac{xy}{x^2+y^2} = \lim_{x\to 0} \frac{2y^2}{4y^2+y^2} = \frac{2}{5}.$$

II
$$0 \le \left|\frac{xy^2}{x^2+y^2}\right| \le |y| \cdot \frac{1}{2}\left|\frac{x^2+y^2}{x^2+y^2}\right| \le |y|.$$

$$\text{So} \quad \lim_{(x,y)\to 0} \frac{xy^2}{x^2+y^2} = 0$$

III False
$$\lim_{(x,y)\to 0} \frac{xy^2}{x^2+y^4} \begin{aligned} &= 0 \text{ along } y = \lambda x \text{ for all } \lambda \\ &= 1/2 \text{ along } x = y^2 \end{aligned}$$

9. Let $f(x, y) = \sqrt{xy}$. Use linear approximation to estimate the value of $f(1.06, 0.96)$.

A. 1.01
B. 1.05
C. 0.95
D. 0.99
E. 1.1

Linear approximation:

$$f(x,y) \sim f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b)$$

In this case take $a = 1 = b$. ; $f(x,y) = \sqrt{xy}$

$$f(1,1) = 1 ; \quad \frac{\partial f}{\partial x} = \frac{y}{2\sqrt{xy}} ; \quad \frac{\partial f}{\partial y} = \frac{x}{2\sqrt{xy}}$$

$$\frac{\partial f}{\partial x}(1,1) = \frac{1}{2} ; \quad \frac{\partial f}{\partial y}(1,1) = \frac{1}{2}.$$

take $x = 1.06$, $y = 0.96$.

$$f(1.06, 0.96) = 1 + \frac{1}{2}(1.06 - 1) + \frac{1}{2}(0.96 - 1)$$
$$= 1 + 0.03 - 0.02 = 1.01$$

10. If $f(x, y) = \sin(x^2 y)$, $x = st$, and $y = s^2 + t$, then $\partial f/\partial s$ is:

A. $\cos(s^2t^2(s^2+t))2st^2(s^2+t) + \cos(s^2t^2(s^2+t))2s^3t^2$
B. $\cos(s^2t^2(s^2+t))t + \cos(s^2t^2(s^2+t))2s^3t^2$
C. $\cos(s^2t^2(s^2+t))2st^2(s^2+t) + \cos(s^2t^2(s^2+t))2s^2t^2$
D. $\cos(s^2+t)2st^2(s^2+t) + \cos(s^2t^2(s^2+t))2s^3t^2$
E. $\cos(s^2t^2(s^2+t))2st^2(s^2+t) + \cos(s^2t^2(s^2+t))2s$

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial f}{\partial x} = 2xy \cos(x^2 y) ; \quad \frac{\partial f}{\partial y} = x^2 \cos(x^2 y)$$

substitute $x = st$ ; $y = s^2 + t$.

$$\partial x/\partial s = t ; \quad \partial y/\partial s = 2s$$

$$\frac{\partial f}{\partial s} = 2st(s^2+t) . \cos\left[s^2t^2(s^2+t)\right] . t$$
$$+ s^2t^2 \cos\left[s^2t^2(s^2+t)\right] . 2s$$
$$= 2st^2(s^2+t) \cdot \cos(s^2t^2(s^2+t)) + 2s^3t^2 \cos(s^2t^2(s^2+t))$$

11. Find the directional derivative of the function $F(x, y, z) = \sin(xy) + \cos(yz)$ at the point $(\pi/2, 1, \pi/2)$ in in the direction $\vec{v} = \vec{i} + \vec{j} - \vec{k}$.

A. $(2-\pi)/(4\sqrt{3})$
B. $(2-\pi)/(2\sqrt{3})$
C. $1-(\pi/2)$
D. $(1-\pi/2)/(2\sqrt{3})$
E. None of the above

$D_u F(\pi/2, 1, \pi/2) = \vec{u} \cdot \vec{\nabla} F(\pi/2, 1, \pi/2)$

$\vec{u} = \frac{1}{\sqrt{3}} \langle 1, 1, -1 \rangle$

$\frac{\partial F}{\partial x} = y \cos xy$ [illegible]. $\frac{\partial F}{\partial x}(\pi/2, 1, \pi/2) = 0$

$\partial F/\partial y = x \cos xy - z \sin(yz)$; $\frac{\partial F}{\partial y}(\pi/2, 1, \pi/2) = -\pi/2$

$\partial F/\partial z = -y \sin(yz)$ $\qquad \frac{\partial F}{\partial z}(\pi/2, 1, \pi/2) = -1.$

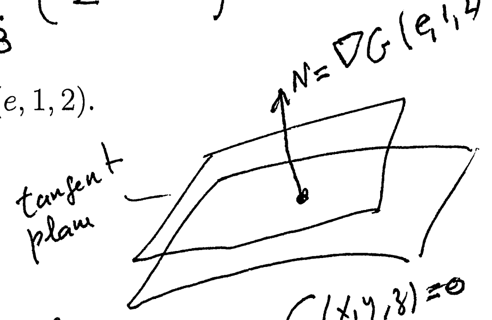
$\vec{\nabla} F(\pi/2, 1, \pi/2) = \langle 0, -\pi/2, -1 \rangle$

$\vec{\nabla} F \cdot \vec{u} = 1/\sqrt{3}\,(1 - \pi/2) = \frac{1}{2\sqrt{3}}(2-\pi)$

12. Find the tangent plane to $f(x, y) = \ln(xy^2) + y$ at the point $(e, 1, 2)$.

A. $(x/e) + 3y - z = 4$
B. $(x/e) + 2y - 2z = 4$
C. $(x/e) + 3y - z = 2$
D. $(x/e) + y + z = 0$
E. $(x/e) + y - z = 0$

$z - f(x,y) = 0.$




$G(x,y,z) = z - \ln(xy^2) - y = 0$

Normal to the plane $N = \nabla G(e,1,2)$

$\nabla G = \langle -1/x \,;\, -1 - 2/y \,;\, 1 \rangle$

$\nabla G(e,1,2) = \langle -1/e \,;\, -3, 1 \rangle = N.$

Plane goes throug $(e,1,2)$ $\quad N \cdot \langle x-e, y-1, z-2 \rangle = 0$

$\langle -1/e, -3, 1 \rangle \cdot \langle x-e, y-1, z-2 \rangle = 0$

$-x/e + 1 - 3y + 3 + z - 2 = 0$ ; $\quad z - 3y - x/e = -2$

$\boxed{x/e + 3y - z = 2}$