MA 265 Final Exam Spring 2007

1. For what values of h and k does the system $A\mathbf{x} = \mathbf{b}$ have infinitely many solutions?

$$A = \begin{bmatrix} 1 & 1 & 4 \\ -3 & -3 & h \\ 1 & 8 & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -2 \\ k \\ 0 \end{bmatrix}.$$

A. $h \neq 12$ and k any number

B. $h = -12$ and k any number

C. $h = -12$ and $k = 6$

D. $h = -11$ and $k = 6$

E. $h =\neq -11$ and $k \neq 6$

$$\left[\begin{array}{ccc|c} 1 & 1 & 4 & -2 \\ -3 & -3 & h & k \\ 1 & 8 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 4 & -2 \\ 0 & 0 & h+12 & k-6 \\ 0 & 7 & -3 & 2 \end{array}\right] \to$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 4 & -2 \\ 0 & 7 & -3 & 2 \\ 0 & 0 & h+12 & k-6 \end{array}\right] \text{ – row echelon form}$$

A is singular $\Leftrightarrow$ $h = -12$

No solutions if $h=-12$, $k \neq 6$

Infinitely many solutions if $h=-12$, $k=6$.

2. The inverse of the matrix

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \quad \text{is} \quad A^{-1} = \begin{bmatrix} a & 1/3 & 1/3 \\ -2/3 & b & 1/3 \\ -2/3 & 1/3 & c \end{bmatrix}.$$

What is $a + b + c$?

A. 0

B. $-1/3$

C. $-2/3$

D. $1/3$

E. $2/3$

$\det(A) = 1\cdot(-1) + 1\cdot(-2) = -3$

(expand in 1st row)

$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A) \Rightarrow$

$a = -\frac{1}{3}\begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} = \frac{1}{3}$

$b = -\frac{1}{3}\begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} = \frac{1}{3}$

$c = -\frac{1}{3}\begin{vmatrix} 1 & -1 \\ 2 & 0 \end{vmatrix} = -\frac{2}{3}$

$\boxed{\begin{array}{l} \operatorname{adj}(A)_{ij} = (-1)^{i+j} M_{ji} \\ \text{where } M_{ji} \text{ is the minor of } A \\ \text{obtained by deleting row } j \text{ and column } i. \end{array}}$

3. Let A, B and C be invertible $n \times n$ matrices. If $A^{-1}B^{-1} = C^{-1}$, then what is A?

A. $A = CB^{-1}$

B. $A = C^{-1}B^{-1}$

C. $A = BC^{-1}$

D. $A = B^{-1}C$

E. $A = BC$

Multiply by A from the left: $B^{-1} = AC^{-1}$

Multiply by C from the right: $B^{-1}C = A$

4. If (x_1, x_2, x_3) is the solution of the following system of equations

$$x_1 + 3x_2 + x_3 = 1$$
$$2x_1 + 4x_2 + 7x_3 = 2$$
$$3x_1 + 10x_2 + 5x_3 = 7$$

then $x_2 =$

A. 29/9

B. 8/9

C. 59/9

D. 9/8

E. 20/9

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & 1 \\ 2 & 4 & 7 & 2 \\ 3 & 10 & 5 & 7 \end{array}\right] \rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & 1 \\ 0 & -2 & 5 & 0 \\ 0 & 1 & 2 & 4 \end{array}\right] \rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & 1 \\ 0 & 1 & 2 & 4 \\ 0 & -2 & 5 & 0 \end{array}\right] \rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & 1 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 9 & 8 \end{array}\right]$$

row echelon form

$x_3 = \frac{8}{9}$, $x_2 + 2 \cdot \frac{8}{9} = 4$

$\Rightarrow x_2 = \frac{20}{9}$

5. Which of the following statements are true?

(i). A linear system of four equations in three unknowns is always inconsistent F

(ii). A linear system with fewer equations than unknowns must have infinitely many solutions F

(iii). If the system $A\mathbf{x} = \mathbf{b}$ has a unique solution, then A must be a square matrix. F

A. all of them

B. (i) and (ii)

C. (ii) and (iii)

D. (iii) only

E. none of them

(i) Any system with RHS = 0 is consistent, as it has solution $\vec{x} = \vec{0}$

(ii) A system with fewer equations than unknowns may have no solutions. For example, $0x_1 + 0x_2 = 1$.

(iii) A system with linearly independent columns may have a unique solution, even if A is not a square matrix. For example, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \vec{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ has a unique solution $\vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

6. If

$$\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \underbrace{\left(\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)}_{= \begin{bmatrix} x \\ y \end{bmatrix}} = \begin{bmatrix} -11 \\ 1 \end{bmatrix}$$

what is $a + b$?

A. -130

B. -50

C. -15

D. 105

E. 83

$$\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -11 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -35 \\ 12 \end{bmatrix}$$

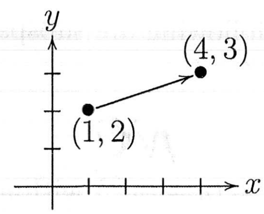
$$\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -35 \\ 12 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -36 \\ 11 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -36 \\ 11 + 3 \cdot 36 \end{bmatrix}$$

$$\Rightarrow a + b = 11 + 2 \cdot 36 = 83$$

7. What vector is represented by the following:



A. $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

B. $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$

C. $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

D. $\begin{bmatrix} -3 \\ -1 \end{bmatrix}$

E. $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$

$$\begin{bmatrix} 4 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

8. Which of the following are subspaces of $\mathcal{P}_3$ (the vector space of all polynomials of degree ≤ 3)?

(I) $\{1+t^2\}$ NO

(II) $\{at+bt^2+(a+b)t^3\}$ with a, b real numbers YES

(III) $\{a+bt+abt^2\}$ with a, b real numbers NO

(IV) $\{$polynomials $p(t)$ with $p(2)=0\}$ YES

A. (II) and (III) only.

B. (I) only.

C. (II) and (IV) only.

D. (I) and (IV) only.

E. (I), (II), and (III) only.

(I) A subspace must contain a zero vector.

(II) This is $\mathrm{span}(t+t^3, t^2+t^3)$

(III) This set contains 1 and t (take $(a,b)=(1,0)$ and $(0,1)$, respectively) but does not contain $1+t$.

(IV) If $p(t)=a_0+a_1t+a_2t^2+a_3t^3$ then $p(2)=0$ means $a_0+2a_1+4a_2+8a_3=0$ which is a homogeneous linear equation on the coefficients.

9. Which of the following sets of vectors in $M_{2\times 2}$ (the vector space of 2×2 matrices) are linearly independent?

(I) $\left\{ \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \right\}$ NO

(II) $\left\{ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 2 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 3 \\ 2 & 1 \end{bmatrix} \right\}$ NO

(III) $\left\{ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \right\}$ NO

(IV) $\left\{ \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 3 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$ YES

A. (III) and (IV) only.

B. (IV) only.

C. (II) and (IV) only.

D. (I) and (II) only.

E. All of them are linearly independent.

(I) $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v}_4 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{v}_3 = \vec{v}_1 + \vec{v}_2$

(II) All matrices have $a_{11} = 0 \Rightarrow$ they all belong to a 3-dimensional subspace $\begin{bmatrix} 0 & a \\ b & c \end{bmatrix}$

(III) All matrices have $a_{11} = a_{22} = 0 \Rightarrow$ they all belong to a two-dimensional subspace.

(IV) $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 3 & 0 \\ 2 & 2 & 0 \\ 3 & 1 & 1 \end{bmatrix}$ has rank 3, $\det\begin{pmatrix} 1 & 3 & 0 \\ 2 & 2 & 0 \\ 3 & 1 & 1 \end{pmatrix} \neq 0$

10. Which of the following span $\mathbb{R}^2$?

(I) $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \end{bmatrix} \right\}$ NO

(II) $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ NO

(III) $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$ YES

(IV) $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\}$ YES

A. (II) only.

B. (I), (III), and (IV) only.

C. (III) only.

D. (I) and (IV) only.

E. (III) and (IV) only.

(I) $\begin{bmatrix} 1 \\ 2 \end{bmatrix} \parallel \begin{bmatrix} 3 \\ 6 \end{bmatrix} \Rightarrow$ thes vectors span a line

(II) need at least two vectors to span $\mathbb{R}^2$

(III), (IV) any two non-parallel vectors span $\mathbb{R}^2$

11. For four vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \in \mathbb{R}^4$, suppose that the 4×4 matrix $A = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \end{bmatrix}$ has its reduced row echelon form

$$rref(A) = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then, which of the following pairs gives a basis for the vector space $Span\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$?

pivot columns

A. $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

B. $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

C. $\{\mathbf{v}_1, \mathbf{v}_3\}$

D. $\{\mathbf{v}_1, \mathbf{v}_2\}$

E. Cannot be determined from the given information.

The column space of A has a basis consisting of the columns of A (not of its row echelon form!) corresponding to pivot columns of the row echelon form.

pivots

12. Suppose that a 4×4 matrix A has its reduced row echelon form

$$rref(A) = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Let r be the rank of the matrix A, and let d be the determinant of the matrix A. Then, what is the value of $r^2 + d^2$?

A. 4

B. 5

C. 6

D. 8

E. 9

rank $r=2$ is equal to the number of non-zero rows (or to the number of pivots) of the row echelon form.

since $r<4$, determinant $d=0$.

Note that det(A) cannot be found from the row echelon form when it is not zero.

13. Consider the following matrices:

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & p \\ 0 & 0 & q \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}.$$

Then, which of the following statement is false?

A. If $q = 0$, then the nullity of the matrix A is 1. T

B. If A is invertible, then the equation $A\mathbf{x} = \mathbf{b}$ has $\mathbf{x} = \begin{bmatrix} -3 & 2 & 0 \end{bmatrix}^T$ as its only solution. T

C. The eigenvalues of the matrix A are 1 and q. T

D. If $A\mathbf{x} = \mathbf{b}$ has more than one solution, then q must be zero. T

E. The rank of the augmented matrix $[A|\mathbf{b}]$ is always 3. F

A) If $q=0$ then rank of A is 2, and nullity + rank = 3

number of columns

B) A invertible $\Rightarrow q \neq 0$
$\Rightarrow x_3 = 0$ from the last equation
$\Rightarrow \vec{x} = \begin{bmatrix} -3 \\ 2 \\ 0 \end{bmatrix}$ by back substitution.

C) A is upper triangular $\Rightarrow$ eigenvalues are the diagonal entries of A.

D) non-unique solution $\Rightarrow$ A is singular $\Rightarrow q = 0$

E) rank is 2 if $q=0$.

14. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$ be two vectors, satisfying the following properties:

(i) $\mathbf{x} \cdot \mathbf{y} = 0$.

(ii) $||\mathbf{x}|| = 2, ||\mathbf{y}|| = 1$.

Then, for real numbers a, b, what is the expression for $||a\mathbf{x} + b\mathbf{y}||^2$?

A. $a^2 + b^2$

B. $2a^2 + b^2$

C. $4a^2 + b^2$

D. $4a^2 + 4ab + b^2$

E. $a^2 + 4ab + 4b^2$

$$||a\vec{x} + b\vec{y}||^2 = (a\vec{x} + b\vec{y}, \; a\vec{x} + b\vec{y}) =$$

$$a^2 \underbrace{||\vec{x}||^2}_{=4} + b^2 \underbrace{||\vec{y}||^2}_{=1} + 2ab \underbrace{(\vec{x}, \vec{y})}_{=0}$$

15. Let W be a subspace of $\mathbb{R}_3$ spanned by $(1,2,3), (2,k,3), (4,5,k+8)$. Determine the values of k so that $W^{\perp}$ has dimension zero.

A. $k \neq 7$

B. $k \neq 7,\ k \neq -1$

C. $k \neq 7$ and $k \neq 1$

D. $k = 7,\ k = 1$

E. $k = 7,\ k = -1$

Matrix $\begin{bmatrix} 1 & 2 & 3 \\ 2 & k & 3 \\ 4 & 5 & k+8 \end{bmatrix}$ should be nonsingular

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & k & 3 \\ 4 & 5 & k+8 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & k-4 & -3 \\ 0 & -3 & k-4 \end{bmatrix}$$

$$\Rightarrow \det(A) = (k-4)^2 - 9 = (k-1)(k-7)$$

16. Let A be the standard matrix representing the linear transformation $L : \mathbb{R}_3 \to \mathbb{R}_3$. Let $\mathbf{v}_1 = (2, 1, 4), \mathbf{v}_2 = (0, 5, 2), \mathbf{v}_3 = (0, 0, 1)$ be eigenvectors of the matrix A associated with eigenvalues $\lambda_1 = 1, \lambda_2 = -3, \lambda_3 = -2$ respectively. Find $L(\mathbf{v}_1 - \mathbf{v}_2 + 3\mathbf{v}_3)$.

A. $(2, -4, 5)$

B. $(2, -14, -8)$

C. $(2, 16, 16)$

D. $(2, 16, 4)$

E. $(2, -14, 4)$

$$L(\vec{v}_1 - \vec{v}_2 + 3\vec{v}_3) =$$

$$L(\vec{v}_1) - L(\vec{v}_2) + 3L(\vec{v}_3) =$$

$$\vec{v}_1 - (-3)\vec{v}_2 + 3\cdot(-2)\vec{v}_3 =$$

$$\begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} + \begin{bmatrix} 0 \\ 15 \\ 6 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \\ 16 \\ 4 \end{bmatrix}$$

17. Let W be the subspace of $\mathbb{R}_3$ with basis $\{(1,1,0),(0,1,-1)\}$, and let $\mathbf{v} = (2,0,-4)$. Find the vector $\mathbf{w}$ in W closest to $\mathbf{v}$.

A. $(1,3,-2)$
B. $(0,2,-2)$
C. $(2,-2,-2)$
D. $(1,-3,-2)$
E. $(1,2,1)$

$\begin{bmatrix}1\\1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\\-1\end{bmatrix}$ are <u>not</u> orthogonal

$\begin{bmatrix}1\\1\\0\end{bmatrix}$ and $\begin{bmatrix}1/2\\-1/2\\1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\\1\\0\end{bmatrix} - \begin{bmatrix}0\\1\\-1\end{bmatrix}$

are orthogonal and span W

$$\vec{w} = \frac{\vec{v}\cdot\begin{bmatrix}1\\1\\0\end{bmatrix}}{\begin{bmatrix}1\\1\\0\end{bmatrix}\cdot\begin{bmatrix}1\\1\\0\end{bmatrix}}\begin{bmatrix}1\\1\\0\end{bmatrix} + \frac{\vec{v}\cdot\begin{bmatrix}1/2\\-1/2\\1\end{bmatrix}}{\begin{bmatrix}1/2\\-1/2\\1\end{bmatrix}\cdot\begin{bmatrix}1/2\\-1/2\\1\end{bmatrix}}\begin{bmatrix}1/2\\-1/2\\1\end{bmatrix}$$

$\parallel$

proj$_{\vec{W}}\vec{v}$

$$= \frac{2}{2}\begin{bmatrix}1\\1\\0\end{bmatrix} + \frac{-3}{3/2}\begin{bmatrix}1/2\\-1/2\\1\end{bmatrix} = \begin{bmatrix}1\\1\\0\end{bmatrix} - \begin{bmatrix}1\\-1\\2\end{bmatrix}$$

<u>Alternatively</u>, $\begin{bmatrix}1\\1\\0\end{bmatrix}\times\begin{bmatrix}0\\1\\-1\end{bmatrix} = \begin{bmatrix}-1\\1\\1\end{bmatrix}$

spans $W^{\perp}$ $\Rightarrow$ $\vec{v} = \vec{u} + \vec{w}$ where

$$\vec{u} = \frac{\vec{v}\cdot\begin{bmatrix}-1\\1\\1\end{bmatrix}}{\begin{bmatrix}-1\\1\\1\end{bmatrix}\cdot\begin{bmatrix}-1\\1\\1\end{bmatrix}}\begin{bmatrix}-1\\1\\1\end{bmatrix} = \frac{-6}{3}\begin{bmatrix}-1\\1\\1\end{bmatrix} = \begin{bmatrix}2\\-2\\-2\end{bmatrix}$$

$\parallel$

proj$_{\vec{W}^{\perp}}\vec{v}$

18. If A and B are $n \times n$-matrices, which statement is false?

A. $\det(AB) = \det(A)\det(B)$ T

B. $\det(A^{\mathrm{T}}) = \det(A)$ T

C. If k is a nonzero scalar, then $\det(kA) = k\det(A)$. F

D. If A is nonsingular, then $\det(A^{-1}) = 1/\det(A)$. T

E. If A and B are similar matrices, then $\det(A) = \det(B)$. T

c) $\det(kA) = k^n \det(A)$

because all n rows are multiplied by k.

19. Compute the det(A).

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

A. 5
B. 16
C. 0
D. -5
E. 11

Expand in 1st row:

$$\det(A) = 2\begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix} - \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix} =$$

$$2\left(2\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix}\right) - \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} =$$

$$3\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} - 2\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 3 \cdot 3 - 2 \cdot 2 = 5$$

Alternatively

$$A \to \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \\ 2 & 1 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & -3 & -2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 4 & 3 \end{bmatrix} \to$$

(3 row exchanges)

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -5 \end{bmatrix}$$

20. Find the values of α for which A is singular.

$$A = \begin{bmatrix} 2 & 1 & 3\alpha & 4 \\ 0 & \alpha - 1 & 4 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & \alpha & 4 \end{bmatrix}$$

A. $\alpha = 0$

B. $\alpha = 1$

C. $\alpha = 2$ and $\alpha = 3$

D. $\alpha = 1$ and $\alpha = 8$

E. $\alpha = 0$ and $\alpha = 1$

$$\det(A) = \begin{vmatrix} 2 & 1 \\ 0 & \alpha-1 \end{vmatrix} \cdot \begin{vmatrix} 2 & 1 \\ \alpha & 4 \end{vmatrix}$$

$$= 2(\alpha-1)(8-\alpha)$$

<u>Alternatively</u>

$$A \to \begin{bmatrix} 2 & 1 & 3\alpha & 4 \\ 0 & \alpha-1 & 4 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 4-\frac{\alpha}{2} \end{bmatrix}$$

21. What is the coefficient of the x^3 term in the polynomial

$$q(x) = \begin{vmatrix} 3x & 5 & 7 & 1 \\ 2x^2 & 5x & 6 & 2 \\ 1 & x & 0 & 3 \\ 2 & 1 & 4 & 7 \end{vmatrix}$$

A. 17

B. -17

C. 90

D. -90

E. 0

Expand in 1st column

$$q(x) = 3x \begin{vmatrix} 5x & 6 & 2 \\ x & 0 & 3 \\ 1 & 4 & 7 \end{vmatrix} - 2x^2 \begin{vmatrix} 5 & 7 & 1 \\ x & 0 & 3 \\ 1 & 4 & 7 \end{vmatrix}$$

$$+ \begin{vmatrix} 5 & 7 & 1 \\ 5x & 6 & 2 \\ 1 & 4 & 7 \end{vmatrix} - 2 \begin{vmatrix} 5 & 7 & 1 \\ 5x & 6 & 2 \\ x & 0 & 3 \end{vmatrix}$$

Only $-2x^2 \begin{vmatrix} 5 & 7 & 1 \\ x & 0 & 3 \\ 1 & 4 & 7 \end{vmatrix}$ contains x^3 term,

which is $2x^3 \begin{vmatrix} 7 & 1 \\ 4 & 7 \end{vmatrix} = 90x^3$

22. Let A^{-1} be the inverse of the following matrix A.

$$A = \begin{bmatrix} 1+i & -1 \\ 1 & i \end{bmatrix}$$

What is

$$A^{-1} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}?$$

A. $\begin{bmatrix} 1+i & 1 \\ 1 & 1-i \end{bmatrix}$

B. $\begin{bmatrix} 1-i & -1 \\ -1 & 1+i \end{bmatrix}$

C. $\begin{bmatrix} 2 & -i \\ i & 2-i \end{bmatrix}$

D. $\begin{bmatrix} 1-i & 1 \\ 1 & 1+i \end{bmatrix}$

E. $\begin{bmatrix} 4 & 1-i \\ i & 2-i \end{bmatrix}$

$$\det(A) = i(1+i) + 1 = i$$

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A) = -i \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

$$= -i \begin{bmatrix} i & 1 \\ -1 & 1+i \end{bmatrix} = \begin{bmatrix} 1 & -i \\ i & 1-i \end{bmatrix}$$

Alternatively

$$\left[\begin{array}{cc|cc} 1+i & -1 & 1 & 0 \\ 1 & i & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & i & 0 & 1 \\ 1+i & -1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & i & 0 & 1 \\ 0 & -i & 1 & -1-i \end{array}\right]$$

$$\to \left[\begin{array}{cc|cc} 1 & i & 0 & 1 \\ 0 & 1 & i & 1-i \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & 1 & -i \\ 0 & 1 & i & 1-i \end{array}\right]$$

23. The matrix A is

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 2 & 1 & 0 \end{bmatrix}.$$

The eigenvalues of A are

A. $0, 1, 2$

B. $0, -1, 2$

C. $0, 1, -2$

D. $0, -1, -2$

E. $-1, 0, 1$

$$A - \lambda I = \begin{bmatrix} -\lambda & 0 & 0 \\ 0 & 1-\lambda & 2 \\ 2 & 1 & -\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = -\lambda \begin{vmatrix} 1-\lambda & 2 \\ 1 & -\lambda \end{vmatrix} =$$

$$-\lambda\left((-\lambda)\cdot(1-\lambda) - 1\cdot 2\right) = -\lambda(\lambda^2 - \lambda - 2) =$$

$$-\lambda(\lambda+1)(\lambda-2)$$

24. Let matrix A be the following 3×3 matrix.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Which matrix P below gives us the following result?

$$P^T A P = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix},$$

where P^T is the transpose of matrix P.

A. $P = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} & 0 \end{bmatrix}$

B. $P = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ -\frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \end{bmatrix}$

C. $P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ -2 & 0 & 1 \end{bmatrix}$

D. $P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ -2 & 1 & 0 \end{bmatrix}$

E. $P = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \end{bmatrix}$

Eigenvalues 0, 0, 3

Unit eigenvector with eigenvalue 3

$\pm \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$

must be the third column of P

The other two columns of P are orthogonal unit vectors spanning null space of A (= eigenspace with eigenvalue 0)

$\Rightarrow$ P is ortogonal, $P^{-1} = P^T$

25. The eigenvectors of $\begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix}$ are $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ with eigenvalues 1 and 4 respectively. If $x_1(t)$ and $x_2(t)$ is the solution of the initial value problem

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix},$$

$$x_1(0) = 90, \quad x_2(0) = 150,$$

then

$$x_1(1) + x_2(1) \quad \text{is equal to}$$

(a) $240e$
(b) $200e$
(c) $230e$
(d) $60e$
(e) $360e$

$$\vec{x}(t) = a\begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{t} + b\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{4t}$$

where

$$\begin{cases} x_1(0) = a + b = 90 \\ x_2(0) = 2a - b = 150 \end{cases}$$

$$\Rightarrow a = 80, \quad b = 10$$

$$\vec{x}(1) = \begin{bmatrix} 80 \\ 160 \end{bmatrix} e + \begin{bmatrix} 10 \\ -10 \end{bmatrix} e^4$$

$$\Rightarrow x_1(1) + x_2(1) = 240e$$