PROBLEM OF THE WEEK

Solution of Problem No. 6 (Fall 2001 Series)

Problem: Suppose $\alpha, \beta, \gamma, \delta$ are real numbers and $e^{i\alpha}+e^{i\beta} = e^{i\gamma}+e^{i\delta}$. Show that, modulo 2π, either

(a) $\{\alpha, \beta\} = \{\gamma, \delta\}$, or

(b) $\alpha = \beta + \pi$ and $\gamma = \delta + \pi$.

Solution (by the Panel)

Suppose $e^{i\alpha} + e^{i\beta} = 0$, then also $e^{i\gamma} + e^{i\delta} = 0$, and we have

$$e^{i(\alpha-\beta)} = -1, \quad \alpha - \beta \equiv \pi \pmod{2\pi}, \text{ also } \gamma - \delta \equiv \pi \pmod{2\pi}.$$

Assume $e^{i\alpha} + e^{i\beta} \neq 0$, so that $e^{i\gamma} + e^{i\delta} \neq 0$. $e^{i\alpha}, e^{i\beta}$ are represented by vectors from the center O to the perimeter of the unit circle C and, by assumption, the angle between them is $< \pi$; $\frac{1}{2}(e^{i\alpha} + e^{i\beta})$ is the vector from O to the midpoint of the segment from $e^{i\alpha}$ to $e^{i\beta}$. $e^{i\gamma}$ is a unit vector from O to the perimeter of C, and if this vector is to the left (right) of $e^{i\alpha}$ then $e^{i\delta}$ is to the right (left) of $e^{i\beta}$ because the vector $\frac{1}{2}(e^{i\alpha} + e^{i\beta})$ has the same direction as $\frac{1}{2}(e^{i\gamma} + e^{i\delta})$. But then the magnitudes of $\frac{1}{2}(e^{i\alpha} + e^{i\beta})$ and $\frac{1}{2}(e^{i\gamma} + e^{i\delta})$ are not the same. Hence $e^{i\gamma}$ must coincide with $e^{i\alpha}$ or $e^{i\beta}$, $\{\alpha, \beta\} \equiv \{\gamma, \delta\} \pmod{2\pi}$.

Also solved by:

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Three unacceptable solutions were received.