Problem: Let the sequence \( \{x_n\} \) of integers (modulo 11) be defined by the recurrence relation \( x_{n+3} \equiv \frac{1}{3}(x_{n+2} + x_{n+1} + x_n) \pmod{11} \) for \( n = 1, 2, \ldots \). Show that every such sequence \( \{x_n\} \) is either constant or periodic with period 10.

Solution (by the Panel)

The general solution of a 3-term recurrence relation is a linear combination of 3 linearly independent solutions. Three linearly independent solutions (which can be arrived at by the use of the characteristic equation \( 3r^3 - r^2 - r - 1 \equiv 0 \)) are:

\[
\begin{align*}
x_n &\equiv 1^n, \\
x_n &\equiv (-3)^n, \\
x_n &\equiv (-5)^n \quad \pmod{11}.
\end{align*}
\]

So the general solution is

\[
x_n \equiv A + B(-3)^n + C(-5)^n \quad \pmod{11}.
\]

Now \((-3)^{10} \equiv 1, \) while \((-3)^2 \not\equiv 1, \) \((-3)^5 \not\equiv 1, \)
and also \((-5)^{10} \equiv 1, \) while \((-5)^2 \not\equiv 1, \) \((-5)^5 \not\equiv 1.\)

So \( x_n \equiv A \) if \( B = C = 0, \) while \( \{x_n\} \) is constant or \( \{x_n\} = \{A + B(-3)^n + C(-5)^n\} \) if \( BC \not\equiv 0, \) which has period 10.

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