Problem: Two players $A, B$, engage in a game. A move consists in each showing simultaneously an open (O) or closed (C) hand. If two O’s show, $A$ wins $3$; if two C’s show, $A$ wins $1$; if an O and a C show, $B$ wins $2$.

a) Is there a winning strategy for $A$? for $B$?

b) If there is one, is it unique?

Solution (by Rob Pratt, Grad. at U. of North Carolina)

Assume that the loser pays the winner the prize money. Let $p$ be the probability that $A$ shows O, and let $q$ be the probability that $B$ shows O. Then $A$ wants to choose $p$ so that, no matter which action $B$ takes, the expected payoff to $A$ (under $A$’s randomized strategy) will be positive. That is,

$$\min\{3p - 2(1 - p), -2p + 1(1 - p)\} > 0.$$  

But this condition implies that $p > 2/5$ and $p < 1/3$, an impossibility. So $A$ has no winning strategy. Similarly, $B$ wants to choose $q$ so that

$$\min\{-3q + 2(1 - q), 2q - 1(1 - q)\} > 0,$$

which implies that $1/3 < q < 2/5$. Any such $q$ defines a winning strategy for $B$, so the winning strategy is not unique. But we now show that $q = 3/8$ is optimal in the sense that it maximizes the worst-case expected payoff to $B$. Since the minimum of two linear functions with slopes of opposite sign has a unique maximum at the intersection point of the two lines, we have

$$\max_{0 \leq q \leq 1} \min\{3p - 2(1 - p), -2p + 1(1 - p)\} = \max_{0 \leq q \leq 1} \{-3q + 2(1 - q), 2q - 1(1 - q)\} = \min\{-5(3/8) + 2, 3(3/8) - 1\} = \min\{1/8, 1/8\} = 1/8.$$

Hence, $q = 3/8$ achieves the maximum worst-case expected payoff to $B$ of 12.5 cents.

Also solved by:

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Two incorrect solutions were received.