PROBLEM OF THE WEEK

Solution of Problem No. 3 (Fall 2002 Series)

Problem: Two players A, B, engage in a game. A move consists in each showing simultaneously an open (O) or closed (C) hand. If two O's show, A wins \$3; if two C's show, A wins \$1; if an O and a C show, B wins \$2.

a) Is there a winning strategy for A? for B?

b) If there is one, is it unique?

Solution (by Rob Pratt, Grad. at U. of North Carolina)

Assume that the loser pays the winner the prize money. Let p be the probability that A shows O, and let q be the probability that B shows O. Then A wants to choose p so that, no matter which action B takes, the expected payoff to A (under A's randomized strategy) will be positive. That is,

$$\min\{3p - 2(1-p), -2p + 1(1-p)\} > 0.$$

But this condition implies that $p > 2/5$ and $p < 1/3$, an impossibility. So A has no winning strategy. Similarly, B wants to choose q so that

$$\min\{-3q + 2(1-q), 2q - 1(1-q)\} > 0,$$

which implies that $1/3 < q < 2/5$. Any such q defines a winning strategy for B, so the winning strategy is not unique. But we now show that $q = 3/8$ is optimal in the sense that it maximizes the worst-case expected payoff to B. Since the minimum of two linear functions with slopes of opposite sign has a unique maximum at the intersection point of the two lines, we have

$$\begin{aligned} &\max_{0\le q\le 1} \min\{-3q + 2(1-q), 2q - 1(1-q)\} \\ =&\max_{0\le q\le 1} \min\{-5q + 2, 3q - 1\} \\ =&\min\{-5(3/8) + 2, 3(3/8) - 1\} \\ =&\min\{1/8, 1/8\} \\ =&1/8. \end{aligned}$$

Hence, $q = 3/8$ achieves the maximum worst-case expected payoff to B of 12.5 cents.

Also solved by:

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Two incorrect solutions were received.