Problem: Suppose $x$ and $y$ are rational numbers satisfying the equation $y^2 = x^3 + ax + b$, where $a, b$ are integers. Show that there are integers $r, s, t$ with $s, r$ and $t, r$ relatively prime such that $x = \frac{s}{r^2}, \ y = \frac{t}{r^3}$.

Solution (by Jason Andersson, Fr. Math)

$x$ and $y$ are rational, so write $x = \frac{s}{p}$ and $y = \frac{t}{q}$ where $GCD(s, p) = GCD(t, q) = 1$. Also may assume $p > 0, q > 0$. Then $t^2p^3 = s^3q^2 + asp^2q^2 + bp^3q^2$. All numbers here are integers. $q^2$ divides the right hand side, so $q^2$ must also divide $t^2p^3$. Since $GCD(t, q) = 1$, $q^2|p^3$. Similarly, $p^3$ must divide $s^3q^2 + asp^2q^2 = (s^3 + asp^2)q^2$. $GCD(s^3 + asp^2, p) = 1$, since if the prime $u$ divides both $s^3 + asp^2$ and $p$, then $u$ divides $p^2$ and so $u$ divides $s^3$ and hence $s$, which contradicts the fact that $GCD(s, p) = 1$. Consequently, $p^3$ divides $q^2$.

Thus the numbers $p^3$ and $q^2$ divide each other and therefore $p^3 = q^2$. Suppose there is a prime $r$ which occurs an odd number of times in the prime factorization of $p$. Then $r$ divides $p^3$ an odd number of times and so it divides $q^2$ an odd number of times. But this is impossible. Hence every prime divides $p$ an even number of times, and it is deduced that $p = r^2$ for some integer $r$. Then $q^2 = p^3 = r^6$, so $q = r^3$, which proves the assertion.

Also solved by:

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Four unacceptable solutions were received. Three late solutions of problem 3 were received, two incorrect and one incomplete.