Problem: Suppose \( f(x) \) is a polynomial with integer coefficients and degree \( n \geq 2 \), and suppose \( |f(x_i)| \) is prime for at least \( 2n + 1 \) integers \( x_i \). Show that:

a) \( f(x) \) is irreducible, that is, \( f(x) \) is not the product of two polynomials of degree \( \geq 1 \) with integer coefficients.

b) for at least one value of \( n \), (a) does not hold if \( 2n + 1 \) is replaced by \( 2n \).

Solution (by Eric Tkaczyk, Sr. EE/MA)

Proof:

a) Assume, conversely, that \( f(x) = g(x)h(x) \), where \( g(x) \), \( h(x) \) are polynomials of degree \( m \) and \( k \) respectively \( \geq 1 \), with integer coefficients and \( m + k = n \). Now, the polynomials \( g(x) + 1 \), \( g(x) - 1 \), \( h(x) + 1 \), and \( h(x) - 1 \) can have at most \( m \), \( m \), \( k \), and \( k \) distinct integer roots, respectively. So there are at most \( m + m + k + k = 2n \) \( x_i \)'s for which \( |g(x_i)| \) or \( |h(x_i)| = 1 \). Thus, if \( f(x) \) is reducible, \( |f(x)| \) will be prime for at most \( 2n \) integers \( x_i \). This proves (a).

b) As a counterexample for the case \( n = 2 \), consider \( f(x) = (2x + 1)(x - 2) \). Clearly, \( f(x) \) is reducible, and \( |f(x)| \) is prime for \( x \) in \( \{-1, 0, 1, 3\} \). So (a) does not hold if \( 2n + 1 \) is replaced by \( 2n \).

Also solved by:

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J.L.C. (Fishers, IN) submitted a correct solution of Problem 5 which, though late, we have credited to him.